Descent property of (co)sheaves on manifolds via Thurston's fragmentation

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Let $F : (\mathsf{Mfld}_n^{\partial})^{op} \to \mathsf{S}$ be a presheaf from the category of smooth n-manifolds (possibly with nonempty boundary) with smooth embeddings as morphisms to a convenient category of spaces S. For our purpose, it is enough to consider the category of simplicial sets or compactly generated Hausdorff spaces. Let F^h be the homotopy sheafification of F with respect to 1-good covers meaning contractible open sets whose nontrivial intersections are also contractible. One can describe the value of $F^h(M)$ as the space of sections of the bundle $\mathrm{Fr}(M)\times_{\mathrm{GL}_n(\mathbb{R})} F(\mathbb{R}^n) \to M$, where $\mathrm{Fr}(M)$ is the frame bundle of M. We say F satisfies an h-principle if the natural map from the functor to its homotopy sheafification

$$j : F(M) \to F^h(M),$$

induces a weak equivalence and we say it satisfies c-principle if the above map is a homology isomorphism. Some important examples of such presheaf in the manifold topology are the space of generalized Morse functions ([Igu84]), space of framed functions ([Igu87]), space of smooth functions on M^n that avoid singularities of codimension $n+2$ (this is in general a c-principle, see [Vas92]), space of configuration of points with labels in a connected space ([McD75]), etc.

Given a fixed element of $s_0 \in F(M)$, one could also consider the compactly supported versions (precosheaf) $F_c(M, s_0)$ of these examples and still the natural map between F_c and F_c^h satisfies h-principle or c-principle. Proving that geometrically defined functorx of interest have nice homotopical properties (being homotopy (co)sheaf) is usually hard and it is the main step in proving h-principle theorems. Different techniques were developed ([Gro86], [EM02]) to prove homotopical properties for certain geometric functors. But in the above examples, the known proofs are not "local to global" argument. In particular, they do not approach it by proving that F_c and F_c^h have descent property with respect to certain covers.

One common feature of the above examples is that $F(\mathbb{R}^n)$ is at least $(n-1)$-connected. So the fiber of the bundle whose compactly supported section space recovers $F_c^h(M)$ is at least $(n-1)$-connected. For such section spaces, there is a descent property known as *non-abelian Poincare duality* ([Lur16, Theorem 5.5.6.6]). So it is expected that if $F(\mathbb{R}^n)$ is at least $(n-1)$-connected, proving h(c)-principle is equivalent to a descent property for $F_c(M)$. Inspired by Thurston's work in foliation theory, we introduce the notion of fragmentation for F_c as a way to prove a descent property for geometrically defined cosheaves. We talk about how fragmentation implies the known version of the non-abelian Poincare duality for space of sections and how it can be generalized when the connectivity of the hypothesis is relaxed.

0.1. **Non-abelian Poincare duality via fragmentation.** To state fragmentation property for the space of sections, let $\pi : E \to M$ be a Serre fibration over the manifold M. Let s_0 be a base section. By the support of a section s, we mean the closure of the points on which s differs from the base section s_0. Let $\mathrm{Sect}_c(\pi)$ be the space of compactly supported sections of the fiber bundle $\pi : E \to M$ equipped with the compact-open topology. Let $\mathrm{Sect}_\epsilon(\pi)$ denote the subspace of sections s such that the support of s can be covered by k geodesically convex balls of radius $2^{-k}\epsilon$ for some positive integer k.

Theorem 0.1 (Fragmentation property)**.** *If the fiber of π is at least $(n-1)$-connected, the inclusion*

$$\mathrm{Sect}_\epsilon(\pi) \hookrightarrow \mathrm{Sect}_c(\pi),$$

is a weak homotopy equivalence.

Remark 0.2. Thurston proved this property with the hypothesis that the fiber of π is at least n-connected.

One can improve on the same ideas to relax the connectivity hypothesis even more. For example, if the fiber of π is at least $(n-2)$-connected, one can show that

$$\mathrm{Sect}_\epsilon^{\mathrm{graph}}(\pi) \hookrightarrow \mathrm{Sect}_c(\pi),$$

is a weak homotopy equivalence where $\mathrm{Sect}_\epsilon^{\mathrm{graph}}(\pi)$ is the subspace of sections whose support is in a $2^{-k}\epsilon$-neighborhood of a graph with k vertices. Using Thurston's ideas in the foliation theory, one could prove the following c-principle theorem.

Definition 0.3. We say F is *good*, if it satisfies

- The subspace of elements with empty support in $F(M)$ is contractible.
- Let U and V be open disks. All embeddings $U \hookrightarrow V$ induces a homology isomorphism between $F_c(U)$ and $F_c(V)$.
- For an open subset U of a manifold M, the inclusion $F_c(U) \to F_c(M)$ is an open embedding.
- Let ∂_1 be the northern-hemisphere boundary of D^n. Let $F(D^n, \partial_1)$ be the subspace of $F(D^n)$ that restricts to the base element in a germ of ∂_1 inside D^n. We assume $F(D^n, \partial_1)$ is contractible.

Theorem 0.4 (N)**.** *Let F be a good presheaf on manifolds such that*

- *$F(\mathbb{R}^n)$ is at least $(n-1)$-connected.*
- *It has the fragmentation property.*

Then F satisfies the c-principle.

Proving fragmentation property instead of descent property with respect to good covers for geometrically defined functors F_c is approachable using Thurston's ideas in foliation theory. For example one could prove Vassiliev c-principle theorem ([Vas92]) for space of smooth functions not having certain singularity via fragmentation technique.

0.2. **Relating two c-principle theorems in foliation theory.** Let $\mathrm{Vect}(M)$ denote the Lie algebra of smooth vector fields on a manifold M with its C^∞-topology and let $C^*_{GF}(\mathrm{Vect}(M))$ denote the Gelfand-Fuks cochains (continuous Chevalley-Eilenberg cochains). Bott and Segal showed that $C^*_{GF}(\mathrm{Vect}(-))$ has a descent property and used a local to global argument to find a zig-zag of quasi-isomorphism between $C^*_{GF}(\mathrm{Vect}(M))$ and real cochains of the space of sections of $\mathrm{Fr}(M) \times_{\mathrm{GL}_n(\mathbb{R})} F \to M$ where F is a $2n$-connected $\mathrm{GL}_n(\mathbb{R})$-space whose real cohomology $H^*(F;\mathbb{R})$ is isomorphic to the cohomology of $C^*_{GF}(\mathrm{Vect}(\mathbb{R}^n))$.

On the other hand, Thurston studied space of foliated trivial M-bundles ([Thu74]) and proved a c-principle for such a functor. More formally, one can represent this functor using the Lie algebra of vector fields as follows. Let

$$\mathsf{MC}_\bullet(\mathrm{Vect}(M)) \coloneqq \mathsf{MC}(\Omega_{\mathrm{dR}}(\Delta^\bullet) \otimes \mathrm{Vect}(M)),$$

be the simplicial set given by smooth Maurer-Cartan elements of dgla $\Omega_{\mathrm{dR}}(\Delta^\bullet) \otimes \mathrm{Vect}(M)$.

He showed that $|\mathsf{MC}_\bullet(\mathrm{Vect}(M))|$ has the fragmentation property and proved it is homology isomorphic to a section space. For simplicity, suppose that M is parallelizable. (this assumption is to express the section space as a mapping space). Then the Thurston theorem states that there is a map

$$|\mathsf{MC}_\bullet(\mathrm{Vect}(M))| \to \mathrm{Map}(M, |\mathsf{MC}_\bullet(\mathrm{Vect}_0(\mathbb{R}^n))|), \tag{1}$$

where $\mathrm{Vect}_0(\mathbb{R}^n)$ is the formal vector fields on $\mathbb{R}^n$ (i.e. germs of vector fields at the origin). Thurston's theorem implies that the above map is a homology isomorphism. Inspired by rational homotopy theory, the mapping space $\mathrm{Map}(W, |\mathsf{MC}_\bullet(\mathrm{Vect}(\mathbb{R}^n))|$ can be modeled by the Maurer-Cartan element of the dgla $\Omega_{\mathrm{dR}}(M) \otimes \mathrm{Vect}_0(\mathbb{R}^n)$.

Our goal is to enhance Thurston's theorem to a statement about the comparison between $\mathsf{MC}_\bullet(\Omega_{\mathrm{dR}}(W) \otimes \mathrm{Vect}_0(\mathbb{R}^n))$ and $\mathsf{MC}_\bullet(\mathrm{Vect}(W))$ that implies homology isomorphism after realization. This is inspired by the work of Haefliger on differential cohomology ([Hae10]) to relate these two theorems locally.

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