On a comparison of symplectomorphisms with finite dimensional Lie groups

Sam Nariman
University of Copenhagen

May 31, 2020

Surface diffeomorphism groups made discrete

Surface symplectomorphisms made discrete

Inspiring work of Milnor on Lie groups

Let G be a Lie group and let G^δ denote the same group with the discrete topology. The natural homomorphism from G^δ to G induces a continuous mapping $\eta : BG^\delta \to BG$.

Inspiring work of Milnor on Lie groups

Let G be a Lie group and let G^δ denote the same group with the discrete topology. The natural homomorphism from G^δ to G induces a continuous mapping $\eta : BG^\delta \to BG$.

Conjecture (Isomorphism conjecture)

For a any Lie group G with finitely many connected components, the map $\eta : BG^\delta \to BG$ induces isomorphisms in homology and cohomology with mod p coefficients.

Inspiring work of Milnor on Lie groups

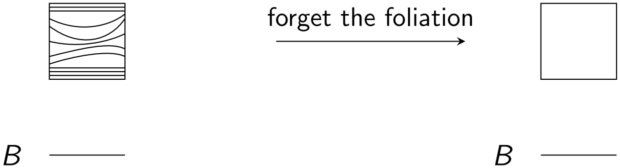
Let G be a Lie group and let G^δ denote the same group with the discrete topology. The natural homomorphism from G^δ to G induces a continuous mapping $\eta : BG^\delta \to BG$.

Conjecture (Isomorphism conjecture)

For a any Lie group G with finitely many connected components, the map $\eta : BG^\delta \to BG$ induces isomorphisms in homology and cohomology with mod p coefficients.

A moduli theoretic interpretation of η

$$\left\{ \begin{array}{c} \text{isomorphism classes of } \textbf{flat} \text{ principal} \\ G\text{-bundles over a manifold } B \end{array} \right\} \to \left\{ \begin{array}{c} \text{isomorphism classes of principal} \\ G\text{-bundles over the manifold } B \end{array} \right\}$$



Inspiring work of Milnor on Lie groups

Let G be a Lie group and let G^δ denote the same group with the discrete topology. The natural homomorphism from G^δ to G induces a continuous mapping $\eta : BG^\delta \to BG$.

Conjecture (Isomorphism conjecture)

For a any Lie group G with finitely many connected components, the map $\eta : BG^\delta \to BG$ induces isomorphisms in homology and cohomology with mod p coefficients.

- Milnor proved the isomorphism conjecture for solvable Lie groups.
- We can ask a similar question for other topological group $G = \mathrm{Diff}(M), Homeo(M), Symp(M,\omega), Ham(M,\omega), \ldots$.

Theorem (Milnor)

For any Lie group with finitely many connected components, the induced maps

$$H^*(BG; \mathbb{F}_p) \to H^*(BG^\delta; \mathbb{F}_p),$$

$$H^*(BG; \mathbb{Z}) \to H^*(BG^\delta; \mathbb{Z}),$$

are injective.

Idea of the proof.

Becker-Gottlieb transfer for the map $BN \to BG$ where N is the normalizer of the maximal torus. □

Theorem (Milnor)

For any Lie group with finitely many connected components, the induced maps

$$H^*(BG;\mathbb{F}_p) \to H^*(BG^\delta;\mathbb{F}_p),$$

$$H^*(BG;\mathbb{Z}) \to H^*(BG^\delta;\mathbb{Z}),$$

are injective.

Idea of the proof.

Becker-Gottlieb transfer for the map $BN \to BG$ where N is the normalizer of the maximal torus. □

With $\mathbb{R}$-coefficients, this map in many cases is not interesting because of the Chern-Weil theory!

Stable result for Lie groups

Suslin proved the isomorphism conjecture for $GL_n(\mathbb{C})$ where , in the stable range:

Theorem (Suslin)

The natural map

$$BGL_n(\mathbb{C})^\delta \to BGL_n(\mathbb{C})$$

induces isomorphisms

$$H_i(BGL_n(\mathbb{C})^\delta; \mathbb{F}_p) \xrightarrow{\simeq} H_i(BGL_n(\mathbb{C}); \mathbb{F}_p).$$

for $i \leq n$.

Stable result for Lie groups

Suslin proved the isomorphism conjecture for $GL_n(\mathbb{C})$ where , in the stable range:

Theorem (Suslin)

The natural map

$$BGL_n(\mathbb{C})^{\delta} \to BGL_n(\mathbb{C})$$

induces isomorphisms

$$H_i(BGL_n(\mathbb{C})^{\delta}; \mathbb{F}_p) \xrightarrow{\simeq} H_i(BGL_n(\mathbb{C}); \mathbb{F}_p).$$

for $i \leq n$.

Question

Does the map $\eta : \mathrm{BDiff}^{\delta}(M) \to \mathrm{BDiff}(M)$ induce an injective map in cohomology? Is there a "range" depending on M that η induces an injective map on cohomology or surjective map in homology?

The peculiar case of the low regularity!

Theorem (Thurston 1975)

The map $\eta : \mathrm{BHomeo}^{\delta}(M) \to \mathrm{BHomeo}(M)$ *induces a homology isomorphism for any manifold* M.

The peculiar case of the low regularity!

Theorem (Thurston 1975)

The map $\eta : \mathrm{BHomeo}^{\delta}(M) \to \mathrm{BHomeo}(M)$ *induces a homology isomorphism for any manifold* M*.*

Theorem (Tsuboi 1985)

The map $\eta : \mathrm{BDiff}^{1^{\delta}}(M) \to \mathrm{BDiff}^{1}(M)$ *induces a homology isomorphism for any manifold* M*.*

A geometric enhancement of these peculiar cases

Theorem (Freedman 2020)

Let N be a 3-manifold. Any fiber bundle $M \to E \to N$ whose structure group is $\mathrm{Homeo}_0(M)$ is semi-s-cobordant to a bundle $M \to E' \to N'$ which is flat.

A geometric enhancement of these peculiar cases

Theorem (Freedman 2020)

Let N be a 3-manifold. Any fiber bundle $M \to E \to N$ whose structure group is $\mathrm{Homeo}_0(M)$ is semi-s-cobordant to a bundle $M \to E' \to N'$ which is flat.

semi-s-cobordism W between N and N' means that there is a simple deformation retraction $W \to N$.

A geometric enhancement of these peculiar cases

Theorem (Freedman 2020)

Let N be a 3-manifold. Any fiber bundle $M \to E \to N$ whose structure group is $\mathrm{Homeo}_0(M)$ is semi-s-cobordant to a bundle $M \to E' \to N'$ which is flat.

semi-s-cobordism W between N and N' means that there is a simple deformation retraction $W \to N$.

Conjecture (Freedman)

Consider the Hopf fibration $S^3 \to S^7 \to S^4$. There is a homology 4-sphere H and a degree one map $f : H \to S^4$ such that the pull back of the Hopf fibration along f gives a flat bundle.

What is so different about $G = \mathrm{Diff}(M)$?

For a manifold M, the group of diffeomorphisms $\mathrm{Diff}^{\delta}(M)$ contains "the information" of two very different groups

$$1 \to \mathrm{Diff}_0(M) \to \mathrm{Diff}(M) \to \mathsf{MCG}(M) \to 1$$

- The group $\mathrm{Diff}_0(M)$ is an interesting object from dynamical system and foliation point of view.
- The group $\mathsf{MCG}(M)$ is an interesting object from geometric topology point of view.

$M = S^1$

$$\eta : \mathrm{BDiff}^{\delta}(S^1) \to \mathrm{BDiff}(S^1) \simeq \mathbb{C}P^{\infty}$$

$M = S^1$

$$\eta : \mathrm{BDiff}^{\delta}(S^1) \to \mathrm{BDiff}(S^1) \simeq \mathbb{C}P^{\infty}$$

Theorem (Herman, '71)

$H_1(\mathrm{Diff}^{\delta}(S^1); \mathbb{Z}) = 0$

$M = S^1$

$$\eta : \mathrm{BDiff}^{\delta}(S^1) \to \mathrm{BDiff}(S^1) \simeq \mathbb{C}P^{\infty}$$

Theorem (Herman, '71)

$H_1(\mathrm{Diff}^{\delta}(S^1); \mathbb{Z}) = 0$

Theorem (Thurston, '72)

There is a surjective map

$$H_2(\mathrm{Diff}^{\delta}(S^1); \mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{R}.$$

$M = S^1$

$$\eta : \mathrm{BDiff}^{\delta}(S^1) \to \mathrm{BDiff}(S^1) \simeq \mathbb{C}P^{\infty}$$

Theorem (Herman, '71)

$H_1(\mathrm{Diff}^{\delta}(S^1); \mathbb{Z}) = 0$

Theorem (Thurston, '72)

There is a surjective map

$$H_2(\mathrm{Diff}^{\delta}(S^1); \mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{R}.$$

Theorem (Morita, '85)

For all $k \geq 1$, there is a surjective map

$$H_{2k}(\mathrm{Diff}^{\delta}(S^1); \mathbb{Z}) \to \mathbb{Z} \oplus S^k_{\mathbb{Q}}\mathbb{R}.$$

$M = S^1$

$$\eta : \mathrm{BDiff}^{\delta}(S^1) \to \mathrm{BDiff}(S^1) \simeq \mathbb{C}P^{\infty}$$

Theorem (Herman, '71)

$H_1(\mathrm{Diff}^{\delta}(S^1); \mathbb{Z}) = 0$

Theorem (Thurston, '72)

There is a surjective map

$$H_2(\mathrm{Diff}^{\delta}(S^1); \mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{R}.$$

Question

Does there exist family of nontrivial central extensions of $\mathrm{Diff}(M)$ *when* $\dim(M) > 1$*?*

A lost theorem of Thurston

Let $\mathrm{Diff}^{\omega}(S^1)$ denote the analytic diffeomorphisms of the circle. Thurston claimed that for all flat analytic S^1-bundle on 6-manifolds the cube of the Euler class is zero. This means that the map

$$H^6(\mathbb{C}P^{\infty};\mathbb{Q}) \to H^6(\mathrm{BDiff}^{\omega,\delta}(S^1);\mathbb{Q})$$

is zero. However, I observed that

$$H^6(\mathbb{C}P^{\infty};\mathbb{Z}) \to H^6(\mathrm{BDiff}^{\omega,\delta}(S^1);\mathbb{Z})$$

is not zero!

$M = \text{surface}$

Let $\Sigma_{g,k}$ denote a surface of genus g and k boundary components. We consider the following cases:

$$B\mathrm{Diff}^\delta(\Sigma_{g,k}, \partial) \to B\mathrm{Diff}(\Sigma_{g,k}, \partial)$$

$$B\mathrm{Symp}^\delta(\Sigma_{g,k}, \partial) \to B\mathrm{Symp}(\Sigma_{g,k}, \partial)$$

$$B\mathrm{Diff}^\delta(\mathbb{D}^2 - n \text{ points}, \partial) \to B\mathrm{Diff}(\mathbb{D}^2 - n \text{ points}, \partial)$$

M = surface

Let $\Sigma_{g,k}$ denote a surface of genus g and k boundary components. We consider the following cases:

$$B\mathrm{Diff}^{\delta}(\Sigma_{g,k},\partial) \to B\mathrm{Diff}(\Sigma_{g,k},\partial)$$
$$B\mathrm{Symp}^{\delta}(\Sigma_{g,k},\partial) \to B\mathrm{Symp}(\Sigma_{g,k},\partial)$$
$$B\mathrm{Diff}^{\delta}(\mathbb{D}^2 - n \text{ points},\partial) \to B\mathrm{Diff}(\mathbb{D}^2 - n \text{ points},\partial)$$

Remark

$$B\mathrm{Diff}(\Sigma_{g,k},\partial) \simeq B\mathrm{MCG}(\Sigma_{g,k}) \text{ Earle-Eells Theorem}$$
$$B\mathrm{Diff}(\Sigma_{g,k},\partial) \simeq B\mathrm{Symp}(\Sigma_{g,k},\partial) \text{ Moser's Theorem}$$
$$B\mathrm{Diff}(\mathbb{D}^2 - n \text{ points},\partial) \simeq B\mathrm{Br}_n$$

.

Main theorems

Theorem (N)

The induced maps on cohomology

$$H^*(\mathrm{BDiff}(\Sigma_{g,k},\partial);\mathbb{F}_p)\to H^*(\mathrm{BDiff}^\delta(\Sigma_{g,k},\partial);\mathbb{F}_p)$$

$$H^*(\mathrm{BSymp}(\Sigma_{g,k},\partial);\mathbb{F}_p)\to H^*(\mathrm{BSymp}^\delta(\Sigma_{g,k},\partial);\mathbb{F}_p)$$

are injective for $* \leq (2g-2)/3$.

Main theorems

Theorem (N)

The induced maps on cohomology

$$H^*(\mathrm{BDiff}(\Sigma_{g,k},\partial);\mathbb{F}_p) \to H^*(\mathrm{BDiff}^\delta(\Sigma_{g,k},\partial);\mathbb{F}_p)$$

$$H^*(\mathrm{BSymp}(\Sigma_{g,k},\partial);\mathbb{F}_p) \to H^*(\mathrm{BSymp}^\delta(\Sigma_{g,k},\partial);\mathbb{F}_p)$$

are injective for $* \leq (2g-2)/3$.

Theorem (N)

In the case of punctured disk, we have

$$H^*(B\mathrm{Diff}(\mathbb{D}^2 - n \text{ points}, \partial);\mathbb{Z}) \to H^*(B\mathrm{Diff}^\delta(\mathbb{D}^2 - n \text{ points}, \partial);\mathbb{Z})$$

is injective in all degrees.

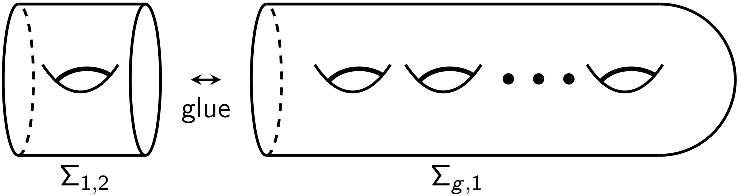
The idea: $\coprod_n B\mathrm{Br}_n$ is a free E_2-algebra!

Idea of the proof

Step 1: We know that $B\mathrm{Diff}(\Sigma_{g,k}, \partial)$ exhibits homological stability (Harer). So we showed that $B\mathrm{Diff}^{\delta}(\Sigma_{g,k}, \partial)$ is also homologically stable (Morita's problem).

Idea of the proof

Step 1: We know that $B\mathrm{Diff}(\Sigma_{g,k}, \partial)$ exhibits homological stability (Harer). So we showed that $B\mathrm{Diff}^{\delta}(\Sigma_{g,k}, \partial)$ is also homologically stable (Morita's problem).



$$s : \mathrm{BDiff}^{\delta}(\Sigma_{g,1}, \partial) \to \mathrm{BDiff}^{\delta}(\Sigma_{g+1,1}, \partial),$$

induce a homology isomorphism in the "stable" range.

Idea of the proof

Step 1: We know that $B\mathrm{Diff}(\Sigma_{g,k}, \partial)$ exhibits homological stability (Harer). So we showed that $B\mathrm{Diff}^{\delta}(\Sigma_{g,k}, \partial)$ is also homologically stable (Morita's problem).
Step 2: Let $G = \mathrm{Diff}(\Sigma_{\infty,k}, \partial)$,

$$\begin{array}{ccc} BG^{\delta} & \xrightarrow{\eta} & BG \\ \downarrow & & \downarrow \\ ? & \longrightarrow & \Omega_0^{\infty}\mathrm{MTSO}(2) \end{array}$$

where the righthand vertical map is a homology isomorphism (Madsen-Weiss Theorem).

Idea of the proof

Step 1: We know that $B\mathrm{Diff}(\Sigma_{g,k}, \partial)$ exhibits homological stability (Harer). So we showed that $B\mathrm{Diff}^{\delta}(\Sigma_{g,k}, \partial)$ is also homologically stable (Morita's problem).
Step 2: Let $G = \mathrm{Diff}(\Sigma_{\infty,k}, \partial)$,

$$\begin{array}{ccc} BG^{\delta} & \xrightarrow{\quad \eta \quad} & BG \\ \downarrow & & \downarrow \\ ? & \longrightarrow & \Omega^{\infty}_{0}\mathrm{MTSO}(2) \end{array}$$

where the righthand vertical map is a homology isomorphism (Madsen-Weiss Theorem). The Madsen-Tillmann spectrum MTSO(2) is the Thom spectrum of of the virtual bundle $-\gamma$, where γ is the tautological bundle over $BGL_2^+(\mathbb{R})$.

Haefliger spaces

Definition

Let Γ_2 denote the topological groupoid whose **objects are** $\mathbb{R}^2$ and whose **morphisms are germs** of orientation preserving diffeomorphisms (with sheaf topology). The classifying space of this groupoid is the Haefliger space of **oriented codimension two foliations**.

There is a map

$$\nu : B\Gamma_2 \to BGL_2^+(\mathbb{R}).$$

Haefliger spaces

Definition

Let Γ_2 denote the topological groupoid whose **objects are** $\mathbb{R}^2$ and whose **morphisms are germs** of orientation preserving diffeomorphisms (with sheaf topology). The classifying space of this groupoid is the Haefliger space of **oriented codimension two foliations**.

There is a map

$$\nu : B\Gamma_2 \to BGL_2^+(\mathbb{R}).$$

Definition

Let $\mathrm{MT}\nu$ be the Thom spectrum of the virtual bundle $\nu^*(-\gamma)$.

Idea of the proof continued

Theorem (N)

There is a map

$$BG^{\delta} \to \Omega_0^{\infty} MT\nu$$

which induces a homology isomorphism.

So we have

$$\begin{array}{ccc} BG^{\delta} & \xrightarrow{\eta} & BG \\ \Big\downarrow{\scriptstyle H_* - iso} & & \Big\downarrow{\scriptstyle H_* - iso} \\ \Omega_0^{\infty}\mathrm{MT}\nu & \longrightarrow & \Omega_0^{\infty}\mathrm{MTSO}(2) \end{array}$$

Step 3: One can use maps of groupoids

$${S^1}^{\delta} \to \Gamma_2 \to GL_2^{+}(\mathbb{R}),$$

We showed that the natural map

$$\Omega_0^{\infty}\mathrm{MT}\nu \to \Omega_0^{\infty}\mathrm{MTSO}(2)$$

has a section after p-completion.

MMM-classes

One can use the main theorem to show that the map

$$\mathbb{Z}[\kappa_1, \kappa_2, \dots] \to H^*(BG^\delta; \mathbb{Z})$$

is injective. However, Morita showed that κ_i in $H^*(BG^\delta; \mathbb{Q})$ is zero for $i > 2$. This implies that the natural map

$$H^*(BG^\delta; \mathbb{Z}) \otimes \mathbb{Q} \to H^*(BG^\delta; \mathbb{Q})$$

has a huge kernel. Also for $i > 2$ one can use Cheeger-Simons theory to define MMM-characters

$$\hat{\kappa}_i : H_{2i-1}(BG^\delta; \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$$

which maps to κ_i via the Bockstein map.

Some geometric consequences

- It is an **open problem** whether every surface bundle over a surface is flat. Kotschick-Morita (2005) proved that all surface bundles over surfaces are cobordant to a flat surface bundle.

Theorem (N)

For $g > 5$, every Σ_g-bundle over a three manifold is cobordant to a flat surface bundle.

Some geometric consequences

- It is an **open problem** whether every surface bundle over a surface is flat. Kotschick-Morita (2005) proved that all surface bundles over surfaces are cobordant to a flat surface bundle.

Theorem (N)

For $g > 5$, every Σ_g-bundle over a three manifold is cobordant to a flat surface bundle.

- Jonathan Bowden (2012) showed that $H_3(BG^\delta; \mathbb{Q})$ is a vector space of uncountable dimension.

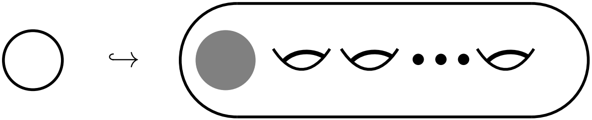
Some geometric consequences

- It is an **open problem** whether every surface bundle over a surface is flat. Kotschick-Morita (2005) proved that all surface bundles over surfaces are cobordant to a flat surface bundle.

Theorem (N)

For $g > 5$, every Σ_g-bundle over a three manifold is cobordant to a flat surface bundle.

- Jonathan Bowden (2012) showed that $H_3(BG^\delta; \mathbb{Q})$ is a vector space of uncountable dimension.



Some geometric consequences

- It is an **open problem** whether every surface bundle over a surface is flat. Kotschick-Morita (2005) proved that all surface bundles over surfaces are cobordant to a flat surface bundle.

Theorem (N)

For $g > 5$, every Σ_g-bundle over a three manifold is cobordant to a flat surface bundle.

- Jonathan Bowden (2012) showed that $H_3(BG^\delta; \mathbb{Q})$ is a vector space of uncountable dimension.

Theorem (N)

The map induced by embedding of the disk

$$H_3(\mathrm{BDiff}^\delta(D^2, \partial); \mathbb{Q}) \to H_3(BG^\delta; \mathbb{Q})$$

is surjective.

Fundamental diagram for principal G-bundles

$$
\begin{array}{ccc}
H^*(\mathfrak{g}) & \longrightarrow & H^*(\overline{BG};\mathbb{R}) \\
\uparrow & & \uparrow \\
H^*(\mathfrak{g},\mathfrak{k}) & \longrightarrow & H^*(BG^\delta;\mathbb{R}) \\
\uparrow & & \uparrow \\
I^*(G) & \longrightarrow & H^*(BG;\mathbb{R})
\end{array}
$$

- If G is semi-simple, the map $H^*(\mathfrak{g},\mathfrak{k}) \to H^*(BG^\delta;\mathbb{R})$ is injective (Borel-Harish-Chandra).
- Morita ('83) showed that similarly $H^*(\mathrm{Vect}(S^1), so(2)) \to H^*(\mathrm{BDiff}^\delta(S^1);\mathbb{R})$ is injective.

Flux homomorphism

Recall that $\mathit{Flux} : \mathrm{Symp}_0(\Sigma_g) \to H^1(\Sigma_g; \mathbb{R})$ for $g > 1$ is defined by $\mathit{Flux}(\psi) = \int_0^1 \iota_{\dot{\psi}_t} \omega dt$.

- Kotschick and Morita (2005) extended this definition to a crossed homomorphism $\widetilde{\mathit{Flux}} : \mathrm{Symp}(\Sigma_g) \to H^1(\Sigma_g; \mathbb{R})$.

$$\widetilde{\mathrm{Ham}}^{\delta}(\Sigma_g) \to \mathrm{Symp}^{\delta}(\Sigma_g) \to H^1(\Sigma_g; \mathbb{R}).$$

Flux homomorphism

Recall that $\mathit{Flux} : \mathrm{Symp}_0(\Sigma_g) \to H^1(\Sigma_g; \mathbb{R})$ for $g > 1$ is defined by $\mathit{Flux}(\psi) = \int_0^1 \iota_{\dot{\psi}_t}\omega dt$.

- Kotschick and Morita (2005) extended this definition to a crossed homomorphism $\widetilde{\mathit{Flux}} : \mathrm{Symp}(\Sigma_g) \to H^1(\Sigma_g; \mathbb{R})$.

$$\widetilde{\mathrm{Ham}}^{\delta}(\Sigma_g) \to \mathrm{Symp}^{\delta}(\Sigma_g) \to H^1(\Sigma_g; \mathbb{R}).$$

- $\Omega^k_{dR}(\mathrm{Symp}(\Sigma_g)/\widetilde{\mathrm{Ham}}(\Sigma_g))^{\mathsf{MCG}(\Sigma_g)} \cong (\bigwedge^k H^1(\Sigma_g; \mathbb{R}))^{\mathsf{MCG}(\Sigma_g)}$.

Flux homomorphism

Recall that *Flux* : $\mathrm{Symp}_0(\Sigma_g) \to H^1(\Sigma_g;\mathbb{R})$ for $g > 1$ is defined by $\mathit{Flux}(\psi) = \int_0^1 \iota_{\dot{\psi}_t}\omega dt$.

- Kotschick and Morita (2005) extended this definition to a crossed homomorphism $\widetilde{\mathit{Flux}} : \mathrm{Symp}(\Sigma_g) \to H^1(\Sigma_g;\mathbb{R})$.

$$\widetilde{\mathrm{Ham}}^{\delta}(\Sigma_g) \to \mathrm{Symp}^{\delta}(\Sigma_g) \to H^1(\Sigma_g;\mathbb{R}).$$

- $\Omega^k_{dR}(\mathrm{Symp}(\Sigma_g)/\widetilde{\mathrm{Ham}}(\Sigma_g))^{\mathsf{MCG}(\Sigma_g)} \cong (\bigwedge^k H^1(\Sigma_g;\mathbb{R}))^{\mathsf{MCG}(\Sigma_g)}$.
- ω^g is non-zero in $\bigwedge^{2g} H^1(\Sigma_g;\mathbb{R})$ and is invariant under $\mathsf{MCG}(\Sigma_g)$.

Flux homomorphism

Recall that *Flux* : $\mathrm{Symp}_0(\Sigma_g) \to H^1(\Sigma_g;\mathbb{R})$ for $g > 1$ is defined by $\mathit{Flux}(\psi) = \int_0^1 \iota_{\dot{\psi}_t}\omega dt$.

- Kotschick and Morita (2005) extended this definition to a crossed homomorphism $\widetilde{\mathit{Flux}} : \mathrm{Symp}(\Sigma_g) \to H^1(\Sigma_g;\mathbb{R})$.

$$\widetilde{\mathrm{Ham}}^\delta(\Sigma_g) \to \mathrm{Symp}^\delta(\Sigma_g) \to H^1(\Sigma_g;\mathbb{R}).$$

- $\Omega^k_{dR}(\mathrm{Symp}(\Sigma_g)/\widetilde{\mathrm{Ham}}(\Sigma_g))^{\mathsf{MCG}(\Sigma_g)} \cong (\bigwedge^k H^1(\Sigma_g;\mathbb{R}))^{\mathsf{MCG}(\Sigma_g)}$.
- ω^g is non-zero in $\bigwedge^{2g} H^1(\Sigma_g;\mathbb{R})$ and is invariant under $\mathsf{MCG}(\Sigma_g)$.

Question

Is the image of ω^g *non-zero in* $H^{2g}(\mathrm{BSymp}^\delta(\Sigma_g);\mathbb{R})$*?*

Flux homomorphism

Recall that *Flux* : $\mathrm{Symp}_0(\Sigma_g) \to H^1(\Sigma_g; \mathbb{R})$ for $g > 1$ is defined by $\mathit{Flux}(\psi) = \int_0^1 \iota_{\dot{\psi}_t} \omega dt$.

- Kotschick and Morita (2005) extended this definition to a crossed homomorphism $\widetilde{\mathit{Flux}} : \mathrm{Symp}(\Sigma_g) \to H^1(\Sigma_g; \mathbb{R})$.

$$\widetilde{\mathrm{Ham}}^{\delta}(\Sigma_g) \to \mathrm{Symp}^{\delta}(\Sigma_g) \to H^1(\Sigma_g; \mathbb{R}).$$

- $\Omega^k_{dR}(\mathrm{Symp}(\Sigma_g)/\widetilde{\mathrm{Ham}}(\Sigma_g))^{\mathsf{MCG}(\Sigma_g)} \cong (\bigwedge^k H^1(\Sigma_g; \mathbb{R}))^{\mathsf{MCG}(\Sigma_g)}$.
- ω^g is non-zero in $\bigwedge^{2g} H^1(\Sigma_g; \mathbb{R})$ and is invariant under $\mathsf{MCG}(\Sigma_g)$.

Question

Is the image of ω^g non-zero in $H^{2g}(\mathrm{BSymp}^{\delta}(\Sigma_g); \mathbb{R})$?

- Kotschick and Morita proved that the image of ω^k is non-zero if $3k \leq g$.

Kotschick-Morita classes

- Morita and Kotschick (2007) used the extended flux

$$[\widetilde{Flux}] \in H^1(\mathrm{BSymp}^\delta(\Sigma_g); H^1(\Sigma_g; \mathbb{R}))$$

to show that there is a surjective map

$$H_{2k}(\mathrm{Symp}^\delta(\Sigma_g); \mathbb{Q}) \to \mathbb{Q} \oplus S^2\mathbb{R} \oplus \cdots \oplus S^k(S^2\mathbb{R}).$$

for $g \geq 3k$.

Stable results

Let Γ_2^{vol} be the Haefliger groupoid of germs of volume preserving diffeomorphisms of $\mathbb{R}^2$.

$$\begin{array}{ccccc}
\widetilde{\mathrm{B}\Gamma_2^{\mathrm{vol}}} & \longrightarrow & \mathrm{B}\Gamma_2^{\mathrm{vol}} & \xrightarrow{e+v} & K(\mathbb{R},2) \\
& \beta \searrow & \downarrow \theta & & \\
& & \mathrm{BSL}_2(\mathbb{R}), & &
\end{array}$$

Stable results

Let Γ_2^{vol} be the Haefliger groupoid of germs of volume preserving diffeomorphisms of $\mathbb{R}^2$.

$$\begin{array}{ccccc} \widetilde{\mathrm{B}\Gamma_2^{\mathrm{vol}}} & \longrightarrow & \mathrm{B}\Gamma_2^{\mathrm{vol}} & \xrightarrow{e+v} & K(\mathbb{R},2) \\ & \beta \searrow & \downarrow \theta & & \\ & & \mathrm{BSL}_2(\mathbb{R}), & & \end{array}$$

Let γ be the tautological 2-plane bundle over $\mathrm{BSL}_2(\mathbb{R})$.

Definition

$$\mathrm{MT}\theta := \text{ Thom spectrum of the bundle } \theta^*(-\gamma)$$

$$\mathrm{MT}\beta := \text{ Thom spectrum of the bundle } \beta^*(-\gamma)$$

Theorem (N)

There is a diagram

$$\begin{array}{ccc} \mathrm{B}\widetilde{\mathrm{Ham}}^{\delta}(\Sigma, \partial) & \longrightarrow & \Omega^{\infty}_{\bullet}\mathrm{MT}\beta \\ \downarrow & & \downarrow \\ \mathrm{BSymp}^{\delta}(\Sigma, \partial) & \longrightarrow & \Omega^{\infty}_{\bullet}\mathrm{MT}\theta, \end{array}$$

whose horizontal maps are homology isomorphisms in the stable range.

Theorem (N)

There is a diagram

$$\begin{array}{ccc} \mathrm{B}\widetilde{\mathrm{Ham}}^{\delta}(\Sigma, \partial) & \longrightarrow & \Omega^{\infty}_{\bullet}\mathrm{MT}\beta \\ \downarrow & & \downarrow \\ \mathrm{BSymp}^{\delta}(\Sigma, \partial) & \longrightarrow & \Omega^{\infty}_{\bullet}\mathrm{MT}\theta, \end{array}$$

whose horizontal maps are homology isomorphisms in the stable range.

Corollary

$H_2(\mathrm{Symp}^{\delta}(\Sigma, \partial); \mathbb{Q}) \xrightarrow{\cong} H_4(B\Gamma_2^{vol}; \mathbb{Q})$

Theorem (N)

There is a diagram

$$\begin{array}{ccc} \mathrm{B}\widetilde{\mathrm{Ham}}^{\delta}(\Sigma, \partial) & \longrightarrow & \Omega^{\infty}_{\bullet}\mathrm{MT}\beta \\ \downarrow & & \downarrow \\ \mathrm{BSymp}^{\delta}(\Sigma, \partial) & \longrightarrow & \Omega^{\infty}_{\bullet}\mathrm{MT}\theta, \end{array}$$

whose horizontal maps are homology isomorphisms in the stable range.

Corollary

The geometric meaning of this theorem is that , up to torsion, any codimension 2 foliation $\mathcal{F}$ with transverse volume form on a 4-manifold M is cobordant to a symplectic surface bundle if and only if $\langle p_1(M) - p_1(\nu\mathcal{F}), [M]\rangle = 0$.

There is an action of $\mathrm{B}\mathbb{R}^\delta$ on $\Omega^\infty_\bullet \mathrm{MT}\beta$.

Theorem (N)

For a closed surface Σ, there is a homotopy commutative diagram

$$\begin{array}{ccc} \mathrm{B}\widetilde{\mathrm{Ham}}^\delta(\Sigma, \text{rel } D^2) & \longrightarrow & \Omega^\infty \mathrm{MT}\beta \\ \downarrow & & \downarrow \\ \mathrm{B}\widetilde{\mathrm{Ham}}^\delta(\Sigma) & \longrightarrow & \mathrm{B}\mathbb{R}^\delta \backslash\!\backslash \Omega^\infty \mathrm{MT}\beta, \end{array}$$

where the horizontal maps induce stable homology isomorphisms.

Extended hamiltonians do not have homological stability with respect to the last boundary component.

MMM-classes

Theorem (N)

The map

$$\mathbb{R}[\kappa_1, \kappa_2, \dots] \to H^*(\mathrm{B}\widetilde{\mathrm{Ham}}^{\delta}(\Sigma, \text{rel } D^2)),$$

is zero.

Theorem (N)

In the stable range, the fiber of the map

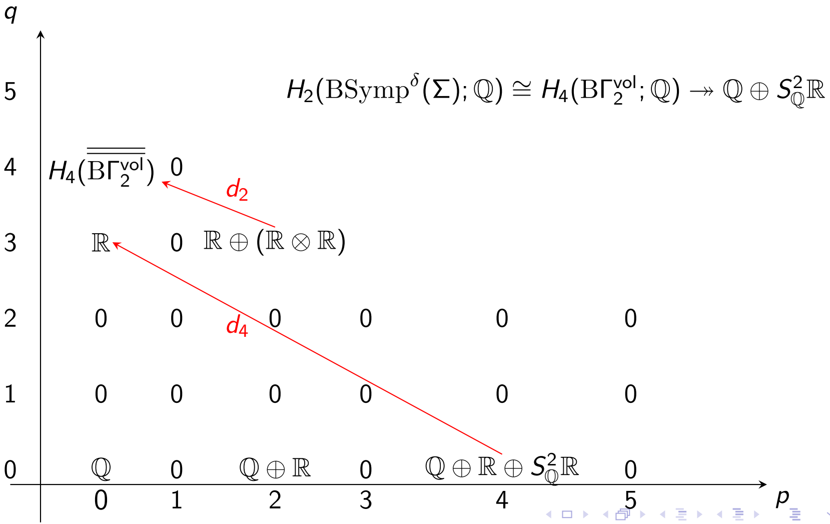
$$\mathrm{B}\widetilde{\mathrm{Ham}}^{\delta}(\Sigma) \xrightarrow{\frac{\kappa_1}{4-4g(\Sigma)}} K(\mathbb{R}, 2).$$

is homology isomorphic to $\mathrm{B}\widetilde{\mathrm{Ham}}^{\delta}(\Sigma, \text{rel } D^2)$.

KM-classes

KM-classes

$$\theta \times v : \mathrm{B}\Gamma_2^{\mathrm{vol}} \to \mathrm{BSL}_2(\mathbb{R}) \times K(\mathbb{R}, 2),$$



Thank you!