Lesson 6: Differential Equations: Solutions, Growth Decay.

- Goals: + solve basic separable equations
  + use Newton's law of cooling

Ex 1: \( \frac{dy}{dt} = ky \). Find \( y \)

- get all \( y \)'s to one side only using multiplication/division.
- get everything to the other side.

\[ \frac{dy}{y} = k \, dt \]

\[ \int \frac{1}{y} \, dy = \int k \, dt \]

\[ \ln |y| = kt + C \]

\[ y = e^{kt} + C \]

\[ y = C \left( e^{kt} \right) \rightarrow \text{general solution} \]

If \( y(0) = 2 \) and \( y'(0) = 2 \), then we can find \( C \) and \( k \)

\[ 2 = C \cdot e^{k \cdot 0} \implies C = 2 \]

\[ 2 = 2 \cdot k \cdot e^{k \cdot 0} \implies k = \frac{1}{2} \]

So \( y = 2e^{t/2} \rightarrow \text{particular solution} \)

Ex 2: \( \frac{dy}{dt} = t^2 e^{t^3} - y = t^2 e^{t^3} e^{-y} \)

\[ \int e^{y} \, dy = \int t^2 e^{t^3} \, dt \]

\[ e^{y} = \frac{1}{3} e^{t^3} + C \]

\[ y = \ln \left| \frac{1}{3} e^{t^3} + C \right| \rightarrow \text{general solution} \]
Ex 3: A bacterial culture grows at the rate proportional to its population. If \( t=0 \), population is 2000 and \( t=2 \), population is 5000. Find the population after 3 hours, \( t \) in hours.

\[
\frac{dP}{dt} = kP
\]

\[
\int \frac{dP}{P} = \int k \, dt
\]

\[
\ln |P| = kt + C
\]

\[
P = e^{kt} \cdot C
\]

\( k \) is positive if increasing

\( t=0 \), \( P=2000 = e^{k \cdot 0} \cdot C \) so \( C = 2000 \)

\( t=2 \), \( P=5000 = 2000 \cdot e^{2k} \)

\[
\frac{5}{2} = e^{2k}
\]

\[
\ln \left( \frac{5}{2} \right) = 2k
\]

\( k = 0.458 \)

So \( P = 2000 \cdot e^{0.458 \cdot t} \)

at \( t=3 \), \( P = 2000 \cdot e^{0.458 \cdot 3} \approx 7906 \)

Ex 4: A pizza comes out a 400°F oven into a 65°F room. If it is 200°F after 10 minutes, how much longer until it is 100°F?

\[
\frac{dy}{dt} = k(y - T_{\text{surrounding}})
\]

\[
y = C e^{kt} + T_{\text{surrounding}}
\]

Newton's Law of Cooling.

\( T \): temp, not time
\[ C = 400 - 65 = 335 \]
\[ y = 335 \, e^{kt} + 65 \]
\[ 200 = 335 \, e^{10k} + 65 \]

\[ k = \frac{1}{10} \ln \left( \frac{135}{235} \right) = -0.091 \]

Now if \[ y = 100 = 335 \, e^{-0.091t} + 65 \]
\[ t \approx 24.82 \]

but 10 mins has past so \( t \approx 14.82 \) more minutes.

**Ex 5:** If at \( t=1 \), there are 100 g of a radioactive element and at \( t=2 \), there are 75 g. How many grams were there initially?

\[ W = Ce^{kt} \quad (k \text{ is negative if decreasing}) \]

\[ 100 = Ce^k \]
\[ 75 = Ce^{2k} \]

Divide them: \[ \frac{3}{4} = e^k \]
\[ k = \ln \left( \frac{3}{4} \right) \]

\[ 100 = C \cdot e^{\ln(\frac{3}{4})} \]
\[ C = \frac{400}{3} \]

So \[ W = \frac{400}{3} e^{\ln(\frac{3}{4})} \cdot t \]

when \( t=0 \), \[ W = \frac{400}{3} \]

**Ex 6:** A radioactive with a half life of 5 years. If \( W = 7 \) lbs at \( t=0 \) what is the amount of the element after 11.7 years.

\[ W = Ce^{kt} \]

7 = \[ C e^{0.0} = C \]
3.5 = \[ 7 e^{k \cdot 5} \]
\[ k = \frac{1}{5} \ln(\frac{1}{2}) \]
\[ k = -0.087 \]

\[ W = 7 e^{-0.087 \cdot t} \]

at 11.7 then \[ W = 7 e^{-0.087 \cdot (11.7)} = 2.53 \text{ lbs} \]