MA-504, Spring 2025, Practice Problems for Test 1

Problem 1. Prove that if $a_n \to 0$, and b_n is bounded, then $a_n b_n \to 0$.

Problem 2.
(a) Show that $1/2$ is a limit point of the sequence $a_n = \cos(n\pi/3 + 1/n)$. You can use the fact that cos is a continuous function. Construct a subsequence a_{n_k} so that $a_{n_k} \to 1/2$.
(b) Find all limit points of a_n.

Problem 3. Construct a nested sequence $\{F_n\}$ (i.e., $F_{n+1} \subset F_n$, $\forall n$) of closed sets with an empty intersection $\cap_{i=1}^{\infty} F_n$.

Problem 4. Find the limit of $a_n = \sqrt{n+1} - \sqrt{n}$, and prove your answer using the definition.

Problem 5. Is every one of the following sets open/closed (both or neither could be true) in the metric space X with the usual distance?
(a) $(0,1)$ in $X = (0,\infty)$?
(b) $(0,1]$ in $X = (0,\infty)$?
(c) $\{x \in \mathbf{R}^2;\ |x| < 1\} \setminus \{0\}$ in $X = \mathbf{R}^2$?
(d) The plane x_1x_2 with the line $x_2 = 0$ removed, in $X = \mathbf{R}^2$?
(e) The curve $\{(x, \sin x) |\ x \in \mathbf{R}\}$ in the plane?
(f) The line segment $\{x_2 = 0,\ 0 < x_1 < 1\}$ in the plane?
(g) The spiral $\{e^{-t}(\cos t, \sin t) |\ t \in \mathbf{R}\}$ in the plane?

Problem 6. Let $A \subset \mathbf{R}^n$, $B = \mathbf{R}^n \setminus A$, both assumed non-empty. Find $\inf\{\mathrm{dist}(x,y) |\ x \in A,\ y \in B\}$. Is it a minimum? Is that true in any metric space?

Problem 7. Prove that

$$\big||x| - |y|\big| \le |x - y|$$

in $\mathbf{R}^n$.