HW #1

(1) Factoring polynomials (with real coefficients) leads to irreducible factors that are either linear $ax + b$ or quadratic $ax^2 + bx + c$ with $b^2 - 4ac < 0$. The Fundamental Theorem of Algebra (see also §2.2 in the book) says that over the filed of the complex numbers, we can do the factorization with linear factors only. In this exercise, you are asked to do it for some polynomials. Example: $z^2 + 1 = (z - i)(z + i)$.

Factor the following polynomials:

(a) $z^2 + z + 1$

(b) $z^4 - 1$

(c) $z^4 + 1$ (corrected) (you would need to find $\sqrt{i}$ here (there are two such roots!). It is OK to use polar representations which we have not covered yet but you can also use undetermined coefficients.

(d) $z^4 + 5z^2 + 4$

(e) $z^2 + (i - 1)z + 2 - 2i$

(2) Prove directly that $|zw| = |z|.|w|$. The book does it in an elegant way, you are supposed to do it with a brute force. Derive from here that $|z/w| = |z|/|w|$ when $w \neq 0$.

(3) The vector space $\mathbb{C}^n$ is usually equipped with the scalar product $(v, w) = v_1\bar{w}_1 + \ldots v_n\bar{w}_n$. This defines a norm: $|v| = (v, v)^{1/2}$. Prove the Cauchy-Schwartz inequality:

$$|(v, w)| \leq |v|.|w|, \quad \forall v, w \in \mathbb{C}^n.$$

(4) Assume that the series $\sum_{k=1}^{\infty} a_k$ converges. Prove that the power series $\sum_{k=1}^{\infty} a_k z^k$ has a radius of convergence $R \geq 1$. Is it possible for R to be strictly greater than one?

(5) Prove the "Parallelogram law," `https://en.wikipedia.org/wiki/Parallelogram_law` using complex numbers. In the second figure there, you can assume that D is the origin, and C and A represent complex numbers. You do not have to assume that C is real, i.e., that DC is along the real axis.

Note: The proof you will provide is the proof in inner product spaces there in disguise. Complex numbers become a more interesting tool in planar geometry when we involve rotations realized as multiplications but I am avoiding this at the moment.