HW #3

(1) Prove that the equation $z^n = w$, with $\mathbb{C} \ni w \neq 0$ fixed, and n a positive integer, has exactly n distinctive solutions. This shows in particular, that $z^{1/n}$ has no unique meaning for any fixed $z \neq 0$.

(2) Find all complex numbers that can be called $1^{1/3}$ in the context of the previous exercise.

(3) One can define $a^z = e^{z \log a}$ for $a > 0$ fixed. Show that a^z is analytic in $\mathbb{C}$. When $a = 1$, we have $1^z = 1$ for every z. Then $1^{1/3} = 1$, and this is a unique value. Does that contradict the previous two exercises?

(4) Define a branch of the square root function on $\Omega = \mathbb{C} \setminus \bar{\mathbb{R}}_+$, called $\tilde{\sqrt{z}}$, as an inverse of the square function so that $\tilde{\sqrt{-1}} = i$.
(a) Prove that this is a correctly defined analytic function.
(b) Find its range.
(c) Compute $\tilde{\sqrt{i}}$, and $\tilde{\sqrt{-i}}$.
(d) Compare those two values with $\sqrt{i}$, and $\sqrt{-i}$ with $\sqrt{z}$ defined as in the book (a cut along $\bar{\mathbb{R}}_-$).
Note that neither (c) or (d) gives us the "right values" of $\sqrt{i}$, and $\sqrt{-i}$. In each case, there are two possible ones. Uniqueness comes from fixing a branch of a square root function.

(5) We computed i^i as $f(i)$ with $f(z) = e^{i \log z}$, where log is the branch defined in the book (and nothing says that this is the only way!). Now, interpret $w = \log z$ as any w for which $z = e^w$. Show that this equation has infinitely many solutions for w when $z \neq 0$ is fixed, and find all of them. They give us all possible values of $\log z$ (when the latter is considered as the inverse of the exponential function). Then find all values of i^i interpreted as $f(i)$ in this sense.

(6) Suggest a way to define 1^i and compute it. Is that value unique?

(7) The printed version of the book: §1.5/#5.

(8) The printed version of the book: §1.5/#21.

(9) The printed version of the book: §1.5/#7. This problem requires more efforts than the rest.