MA-54600 Homework 1 Spring 2026

1. p.32, #1 from the book.

2. p.32, #3 from the book.

3. p.34, #27 from the book.

4. Find the norms of the following functionals on (the real) $C([a,b])$ with the norm $\|f\| = \max_{[a,b]} |f|$:

(a)

$$l(f) = \int_a^b f(t)\,dt;$$

(b)

$$l_g(f) = \int_a^b f(t)g(t)\,dt, \quad g \in C([a,b]) \text{ fixed;}$$

(c)

$$L(f) = \sum_{i=1}^{n} \lambda_i f(t_i),$$

where $\lambda_1, \lambda_2, \dots$ are fixed numbers and $a \le t_1 < t_2 < \cdots < t_n \le b$.

5. True or false?

(a) One can define a scalar product $(\cdot,\cdot)$ on $C([0,1])$ with the norm as above, to get a Hilbert space so that this norm is generated by the scalar product, i.e., $\|f\|^2 = (f,f)$, $\forall f \in C([0,1])$.

(b) If a metric space is a linear (vector) space, then one can define a norm $\|\cdot\|$ so that the metric ρ is generated by the norm, i.e., $\rho(x,y) = \|x-y\|$.

6. Let $H^1[0,1]$ be the (Sobolev) Hilbert space with norm $\|f\|_{H^1}^2 = \int_0^1 \left(|f|^2 + |f'(t)|^2\right)dt$, more precisely, it is the completion of $C^\infty([0,1])$ under that norm with the inner product $\int_0^1 \left(\bar{f}_1 f_2 + \bar{f}_1' f_2'\right)dt$.

(a) Prove that $H^1[0,1]$ can be naturally identified with a subspace of $L^2([0,1])$. Hint: Cauchy sequences in the H^1 norm are also Cauchy in L^2.

(b) Prove that $f \mapsto f(0)$ is a continuous linear functional on $H^1[0,1]$ (this follows from the trace theorem in 1D but you have to prove this statement without using the trace theorem).

(c) Let $H_0^1[0,1]$ be the subspace of all f with $f(0) = f(1) = 0$. Prove that it is a closed subspace.

(d) Express (in an explicit way) any $f \in H^1[0,1]$ as a sum $f = f_1 + f_2$, where $f_1 \in H_0^1[0,1]$, and $f_2 \perp H_0^1[0,1]$.

7. An exercise about the norm of a matrix as a linear operator.

(a) Express the norm of an $m \times n$ matrix A as an operator $A : \mathbf{R}^n \to \mathbf{R}^m$ (the Euclidean space has its usual norm here) in terms of the eigenvalues of A^*A. Here, for $A = \{a_{ij}\}$, we set $A^* = \{\bar{a}_{ji}\}$.

(b) Derive from here, that if $A = A^*$ (then A is square), we have $\|A\| = \max|\lambda_j|$, where the maximum is taken over all eigenvalues of A (which must be real).

(c) What is the norm of $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and how does it compare to the eigenvalues of A?