

## MA-54600 Homework 2 Spring 2026

1. Let  $X$  be a normed linear space, and let  $0 \neq l \in X^*$ . Show that  $\text{Ker}(l)$  is closed and nowhere dense.
2. Let  $\mathcal{H} = L^2[-1, 1]$ , and

$$M_1 := \left\{ f \in \mathcal{H} \mid \int_{-1}^1 f(t) dt = 0 \right\}$$
$$M_2 := \left\{ f \in \mathcal{H} \mid f(t) = f(-t), \forall t \right\}$$

- (a) Show that  $M_1$  is closed and describe  $M_1^\perp$ .
  - (b) Show that  $M_2$  is closed and describe  $M_2^\perp$ .
  - (c) Find the distance from  $f = t^2$  to  $M_1$ .
3. Page 64, #6: Let  $M$  be a linear subspace of a Hilbert space  $\mathcal{H}$ . Prove that  $M^\perp$  is a closed linear subspace and that  $\bar{M} = (M^\perp)^\perp$ .
  4. Page 64, #10:
  5. Page 65, #16:
  6. Page 86, #8:
    - (a) Prove that there is a nonzero bounded functional on  $L^\infty(\mathbb{R})$  which vanishes on  $C(\mathbb{R})$ .
    - (b) Prove that there is a bounded functional  $\lambda$  on  $L^\infty(\mathbb{R})$  such that  $\lambda(f) = f(0)$  for each  $f \in C(\mathbb{R})$ .

7. Do the second part of Example 3 on p. 73, i.e., prove that  $\ell_1^* = \ell_\infty$ , and that this is an isometric isomorphism (which allows us to use the equality sign in the first place).

**Hint:** To summarize the approach in the book, if you have to find  $X^*$  with  $X$  being one of those sequence spaces, you argue like this. Take  $a = \{a_k\}$  and write  $a = \sum a_k e^k$ . Then we must have  $\Lambda a = \sum \lambda_k a_k$  at least when the sum is finite, where  $\lambda_k = \Lambda e^k$ , just by linearity. Now, the subtle part: in the case  $X = c_0$ , the little  $f$  space is dense in  $X$ . Then that series converges on every  $a \in X$  because  $\Lambda$  is bounded. Here,  $\{e^k\}$  is an example of a Schauder basis, which does not exist in all Banach spaces but in this case, it does. Thus we can identify (meaning, we construct an one-to-one map  $\lambda \mapsto \Lambda$ ) such a functional  $\Lambda$  with the sequence  $\lambda = \{\lambda_k\}$ . It remains to find out where that sequence belongs.

In the first part,  $X = c_0$ , and they show that every  $\{\lambda_k\} \in \ell_1$  defines an element in  $c_0^*$  in the first sentence. This is the easy part. The more involved part is to show that every functional  $\Lambda \in X^* = c_0^*$  is given by some  $\lambda = \{\lambda_k\} \in \ell_1$ . On finite sequences, where convergence is not a problem, we just identify  $\lambda$  with  $\{\lambda_k\}$  as above. So we know what  $\{\lambda_k\}$  is, and we have to show that it is in  $\ell_1$  now. To do that, we use continuity, and apply  $\lambda$  to  $f^\ell$ , where  $f^\ell$ ,  $\ell \rightarrow \infty$ , is a truncated family of the “signs” of  $\lambda_k$  so that  $\lambda(f^\ell) = \sum_{i=1}^\ell |\lambda_k|$ . Since  $\lambda$  is bounded, that sum is uniformly bounded, hence  $\{\lambda_k\} \in \ell_\infty$ . Note that  $f^\ell$  is chosen carefully to get exactly that sum. They prove next that the map  $\ell_\infty \ni \lambda \mapsto \Lambda \in c_0^*$

is an isometric isomorphism by showing that  $\|\lambda\|_\infty = \|\Lambda\|_{c_0^*}$ . The identity  $c_0^* = \ell_\infty$  actually means that these two are isometrically isomorphic.

Just to note that choosing  $\ell$  in  $f^\ell$  is not a great notation since one can confuse the number  $\ell$  with one of those spaces  $\ell_\infty$ , etc. A better notation perhaps would be  $f^N$ .

A final note: All this fails if you want to characterize  $\ell_\infty^*$ ! Then  $f$  is not dense in it (you cannot approximate, for example,  $(1, 1, 1, 1, \dots)$  with finite sequences in the sup norm), and there are linear functionals on  $\ell_\infty$  not of the form  $\Lambda a = \sum \lambda_k a_k$ . In particular,  $\{e^k\}$  is not a Schauder basis (and such basis does not exist). One can show that  $\ell_\infty^*$  is strictly larger than  $\ell_1$  (but it contains the latter), i.e.,  $\ell_\infty^* \supsetneq \ell_1$ . This is similar to what happens with  $(L^\infty)^* \supsetneq L^1$ . The proofs of both facts use the Hahn-Banach theorem typically.

8. Prove that all norms on a finitely dimensional linear normed space are equivalent. You can just work with  $\mathbb{C}^n$ ,  $n$  fixed.