MA-54600 Homework 3 Spring 2022

1. Let $Af(x) = a(x)f(x)$ be the operator of multiplication by the function $a(x)$ on $L^p(\Omega) \ni f$, where $\Omega \subset \mathbf{R}^n$ is open, and $p \geq 1$, including $p = \infty$. Prove that A is bounded on $L^p(\Omega)$ if and only if $a \in L^\infty(\Omega)$, and $\|A\|_{\mathcal{L}(L^p(\Omega))} = \|a\|_{L^\infty(\Omega)}$.

2. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator. Prove that

(a) $\operatorname{Ker} T^* = (\operatorname{Ran} T)^\perp$,

(b) $\overline{\operatorname{Ran} T^*} = (\operatorname{Ker} T)^\perp$.

Is the closure above really needed for (b) to be true? Why not closure in (a)?

3. p.216, #5.

4. Let T_t be as above but now the question is different. Is $t \to T_t$ continuous in the uniform topology? In the strong one? In the weak operator one?

5. p.216, #7.

6. p.216, #8.

7. p.216, #9. Read it, do not write anything. This is a good fact to memorize. The easiest proof is through the spectral theorem. Assume, for example, that A is a Hermitian matrix. Then you can choose a basis in which it is diagonal, with real entries (eigenvalues), and then the statement is trivial. The spectral theorem (to be covered later) replaces that argument in the infinitely dimensional case.

8. p.219, #41.

9. Let $A : L^2(Y) \to L^2(X)$, with X, Y domains in $\mathbf{R}^n$ with possibly different dimensions, be the operator

$$Af(x) = \int_Y K(x,y)f(y)dy, \quad x \in X$$

where $K \in L^2(X \times Y)$ is called the integral kernel of A. Show that

(a) A is bounded, and

(b) the Banach adjoint A' and the Hilbert adjoint A^* are of the same type and find their integral kernels. You may want to use the BLT theorem to work on a dense subset of functions first.

Note: Such operators are called Hilbert-Schmidt (see the book as well). The condition $K \in L^2(X \times Y)$ is sufficient but not necessary for A to be bounded. Also, notice some lack of consistency in the notation compared to that in the book. There, $A : X \to Y$ (and X, Y are the Banach spaces, not the domains as here), but I switched X and Y. The notation I used is more consistent with the modern one, where one usually takes $A : Y \to X$, or $L^2(Y) \to L^2(X)$, etc., see, e.g., the book by Friedlander and Joshi.

10. Let A be a bounded operator on a Hilbert space $\mathcal{H}$.

(a) Prove that there is a unique decomposition

$$A = A_1 + iA_2,$$

where A_1 and A_2 are self-adjoint. Sometimes this is written as $A = A_1 + \tilde{A}_2$, where A_1 is self-adjoint, and $\tilde{A}_2$ is skew adjoint, i.e., $\tilde{A}_2^* = -\tilde{A}_2$.

(b) Is it true that $A^*A = A_1^*A_1 + A_2^*A_2$?