MA-54600 Homework 4 Spring 2026

1. Let $A : X \to Y$ be bounded with X, Y Banach spases. Prove that the following statements are equivalent:

(a) There exists $C > 0$ so that

$$\|x\|_X \leq C\|Ax\|_Y, \quad \forall x \in X;$$

(b) A is injective and $\operatorname{Ran} A$ is closed in Y;

(c) A has a bounded left inverse $B : \operatorname{Ran} A \to X$.

2. Let $A : f(x) \mapsto \phi(x)f(x)$ be the operator in $L^2(\mathbf{R}^n)$ of multiplication with the smooth real-valued bounded function ϕ.

(a) Determine the spectrum $\sigma(A)$ of A in terms of ϕ.

(b) Let λ be a non-degenerate value of ϕ, i.e., $\{x|\ \phi(x) = \lambda\}$ is non-empty, and $\phi = \lambda \Rightarrow \nabla\phi \neq 0$. Is λ in $\sigma(A)$? If so, is it an eigenvalue, or is it in the residual spectrum, or is it a spectral value so that $\operatorname{Ran}(A - \lambda)$ is dense but not closed?

(c) Let $\{x|\ \phi(x) = \lambda\}$ have a non-empty interior (it is not non-degenerate then). How do you answer the question in (b) then?

(d) Let $\phi(x) = e^{-|x|^2}$. Is $\lambda = 0$ in the spectrum? How do you answer the question in (b) then?

3. p.216, #10.

4. Let X, Y, Z be Banach spaces, let $A : X \to Y$ be linear and bounded, and let $K : X \to Z$ be a compact linear operator. Let

$$\|f\|_X \leq C\left(\|Af\|_Y + \|Kf\|_Z\right), \quad \forall f \in X.$$

Assume that A is injective. Prove that

$$\|f\|_X \leq C'\|Af\|_Y, \quad \forall f \in X.$$

with a possibly different constant C'.

5. Let A be a bounded operator in a Hilbert space $\mathcal{H}$. Let λ be an isolated eigenvalue, i.e., it is an eigenvalue which is not a limit point of $\sigma(A)$. Let $C \subset \mathbb{C}$ be a positively oriented contour encircling λ, so that the rest of $\sigma(A)$ is outside it.

(a) Show that

$$\Pi := \frac{1}{2\pi i}\oint_C (z - A)^{-1}\, dz$$

is a projection.

(b) Show that if x is an associated eigenvector, then $\Pi x = x$.

(c) Is Π the projection onto the eigenspace corresponding to λ?