MA-54600 Homework 5 Spring 2026

1. p. 245, #1. The integral there is called the Riesz-Dunford integral and defines a functional calculus but f needs to be analytic. The resolvent $(\lambda - A)^{-1}$ is an example where it applies; then $f(z) = (\lambda - z)^{-1}$.

2. p. 245, #2: read it only. It is related to a problem in the previous homework, where P was denoted by Π.

3. p. 246, read #3 and #5. They say that the Spectral Theorem can be generalized to normal operators.

4. p. 246, read #4. It says that functions of several operators are well defined *if the operators commute*. Think about the partial case of analytic functions of several variables: they have Taylor expansions in terms of $z_1^{k_1} \dots z_n^{k_n}$. Multiplication is a commutative operation on complex numbers, of course but if we want to replace z_j by A_j, the term $A_1^{k_1} \dots A_n^{k_n}$ depends on the other in which we apply the operators, in general. In particular, $z_1 z_2 = z_2 z_1$, but $A_1 A_2 \neq A_2 A_1$ in general.

5. p. 247, #12.

6. p. 247, #14.

7. Let

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

with $\lambda_1 \neq \lambda_2$, both real, in $\mathcal{H} = \mathbb{C}^2$.

(a) Find an if and only if condition on $a = (a_1, a_2) \in \mathbb{C}^2$ which makes it cyclic. (Strictly speaking, it should be $(a_1, a_2)^T$ but we omit the "T").

(b) Find the spectral measure $d\mu_a$ when a is cyclic.

(c) Write the spectral representation of A generated by this cyclic vector. You should obtain a single copy of $\mathbb{R}$ and a single measure on it. Compare it to Example 1 on p. 288, where the answer is given directly but the point here is to derive it starting from a fixed cyclic vector. Observe that the spectral measure is not unique because it depends on a but U depends on a as well.

(d) Repeat this constriction starting with $e_1 = (1, 0)$, which is not cyclic; and then using $e_2 = (0, 1)$ as well (still not cyclic). You will get two copies and two measures. This would be a different spectral representation but unitarily equivalent.

8. Let $A = id/dx$ on the space P_N of trigonometric polynomials of order $N \geq 1$ spanned by $\{e^{inx},\ n = 0, \pm 1, \cdots \pm N\}$. All those trig polynomials are 2π–periodic and we define the L^2 norm on P_N by $\|f\|^2 = \int_0^{2\pi} |f|^2\, dx$ (note that this is a norm on P_N even though we integrate over $[0, 2\pi]$ only). As a well-defined operator on a finite dimensional space, A is bounded.

(a) Give a simple formula for $e^{itA} f$ with $t \in \mathbb{R}$ fixed, which does not involve a Fourier expansion.

(b) Derive such a formula for $\sin(tA)/A$ applied to f, i.e., the function now is $f(\lambda) = \sin(t\lambda)/\lambda$ (note that the singularity at $\lambda = 0$ is removable).

(c) How does the Spectral Theorem apply to A? Which operator plays the role of U in order to reduce it to a multiplication by the spectral parameter?

As a remark, this question makes sense on the space of the 2π–periodic functions or on $L^2(\mathbb{R})$ or even for $-\Delta$ on $L^2(\mathbb{R}^n)$ but then A becomes unbounded. The answers are not much affected by that though.