A note on the Laurent series argument

What is a Laurent series? Here is a good reference:
`https://mtaylor.web.unc.edu/wp-content/uploads/sites/16915/2018/04/complex.pdf`, p. 130. For the purpose of this note, we consider

$$R_\lambda = \frac{1}{\lambda}\left\{I + \sum_{n=1}^{\infty} \lambda^{-n} T^n\right\} \tag{1}$$

as a Laurent series about $\lambda = \infty$, i.e., as a power series of $z = 1/\lambda$ about $z = 0$. A basic fact about power series is that each one converges inside a disk, diverges outside it, and the boundary behavior can be complicated (but we are not concerned about that for the moment). The radius of that disk is called the radius of convergence. Another basic fact is that if a function is holomorphic in the annulus $\mathcal{A} = \{z \in \mathbb{C} | \ r_0 < |z| < r_1\}$, where $0 \le r_0 < r_1 \le \infty$, then its Laurent series must converge in it. Note that this is a kind of a converse statement: our starting point is not the series, it is the function now. Our case corresponds to $r_1 = \infty$. Even simpler: with $z = 1/\lambda$ we are really working with a Taylor series in $|z| < C$.

Back to our case: we want to show that $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$. Clearly, the Laurent series for R_λ converges for $|\lambda| > \|T\|$ (a Neumann series) but a priori, it converges in $|\lambda| > A$ with some $A \le \|T\|$, where A^{-1} is the radius of convergence w.r.t. z.

Claim 1. $A^{-1} = r(T)$.

(a) Since the Laurent series (1) converges absolutely for $|\lambda| > A$, it is a resolvent there, indeed; therefore we have $A \ge r(T)$ just by the definition of $r(T)$.

(b) When $|\lambda| > r(T)$, the resolvent exists. Thus it is analytic there, and by the properties of Laurent/Taylor series mention above, the series (1) converges.

Claim 2. $A^{-1} = \lim_{n\to\infty} \|T^n\|^{1/n}$.

There is a theorem by Hadamard, which gives a formula for the radius of convergence of a power series about, say 0: $\limsup \|T^n\|^{1/n}$ in this case. Applied to our case, for $z = 1/\lambda$, we get $A^{-1} = \lim \|T^n\|^{1/n}$ (once we prove that this limsup is actually a limit).