

A note on double supremum

Claim: Let $f : A \times B \rightarrow \mathbb{R}$ with A, B non-empty sets. We claim that

$$\sup_{a \in A} \sup_{b \in B} f(a, b) = \sup_{b \in B} \sup_{a \in A} f(a, b) = \sup_{(a, b) \in A \times B} f(a, b).$$

Indeed, set

$$M_1 := \sup_{a \in A} \sup_{b \in B} f(a, b), \quad M_2 := \sup_{b \in B} \sup_{a \in A} f(a, b).$$

We prove first that $M_1 \leq M_2$. Fix $a \in A$. For every $b \in B$,

$$f(a, b) \leq \sup_{a' \in A} f(a', b).$$

Taking supremum over $b \in B$ gives

$$(1) \quad \sup_{b \in B} f(a, b) \leq \sup_{b \in B} \sup_{a' \in A} f(a', b) = M_2.$$

Since this holds for every $a \in A$, taking supremum over a yields

$$M_1 \leq M_2.$$

Since things are symmetric, we also have $M_2 \leq M_1$. Therefore, $M_1 = M_2$, and both equal

$$M := \sup_{(a, b) \in A \times B} f(a, b).$$

Why the latter? First, M_1 (and M_2) is less or equal than any upper bound of f . On the other hand, it is an upper bound by itself as it follows from (1). \square