Proof of the proposition before Theorem VI.8

Proposition 1. Let X be a Banach space and $T \in \mathcal{L}(X)$. Then

(a) If λ is in the residual spectrum of T, then λ is in the point spectrum of T'.

(b) If λ is in the point spectrum of T, then λ is in either the point or the residual spectrum of T'.

Proof. Let $S := \lambda I - T$. Then $S' = \lambda I - T'$.

We use the bracket notation $\langle f, x\rangle$ for $f \in X'$, $x \in X$. Observe that

$$\langle S'f, x\rangle = \langle f, Sx\rangle \quad \text{for all } f \in X',\ x \in X.$$

Hence

$$\operatorname{Ker} S' = \{f \in X' : \langle f, Sx\rangle = 0\ \forall x \in X\} =: (\operatorname{Ran} S)^{\perp}.$$

The latter is not a statement, it is a definition of $(\operatorname{Ran} S)^{\perp}$, which, by itself has no meaning in a Banach space. In particular,

$$\operatorname{Ker} S' = \{0\} \quad \Longleftrightarrow \quad \overline{\operatorname{Ran} S} = X.$$

Now, consider (a). Suppose λ is in the residual spectrum of T. Then S is injective and $\overline{\operatorname{Ran} S} \neq X$ by definition. By Hahn–Banach (Corollary 3 in III.3), there exists $f \in X'$, $f \neq 0$, such that

$$\langle f, y\rangle = 0 \quad \text{for all } y \in \operatorname{Ran} S.$$

Equivalently,

$$\langle f, Sx\rangle = 0 \quad \text{for all } x \in X,$$

so

$$\langle S'f, x\rangle = 0 \quad \text{for all } x \in X,$$

and therefore $S'f = 0$. Thus $\operatorname{Ker} S' \neq \{0\}$, so S' is not injective. Hence λ is in the point spectrum of T'. Note that if X was Hilbert, we would not need Hahn–Banach since we can just take any $f \perp \operatorname{Ran} S$ (with the brackets there replaced by the inner product parentheses).

(b) Suppose λ is in the point spectrum of T. Then there exists $x \neq 0$ such that $Sx = 0$.

We claim that S' is not surjective. Choose $g \in X'$ such that $\langle g, x\rangle \neq 0$ (by Hahn–Banach again). If S' were onto, there would exist $f \in X'$ such that $S'f = g$. Then

$$\langle g, x\rangle = \langle S'f, x\rangle = \langle f, Sx\rangle = \langle f, 0\rangle = 0,$$

a contradiction. Thus S' is not onto. In a Hilbert space, we would just choose $g = x$.

If S' is not injective, then λ is in the point spectrum of T'. If S' is injective, then $\operatorname{Ker} S' = \{0\}$, hence $\overline{\operatorname{Ran} S} = X$, but since S' is not onto, λ is not in the resolvent set of T'. Therefore λ belongs to the residual spectrum of T'.

□