Some integral geometry problems on Finsler and Riemannian surfaces

Yernat M. Assylbekov

Institute of Mathematics Informatics and Mechanics
Kazakhstan

Joint work with Nurlan Dairbekov
X-ray transform

I recall that the Radon transform makes from a function on $\mathbb{R}^2$ a function on straight lines:

$$\mathcal{R}f(\ell) = \int_{\ell} f \, d\ell,$$

and the inverse problem is the problem of reconstructing $f$ from $\mathcal{R}f$. More generally, the geodesic X-ray transform on a Riemannian manifold makes from a function on the manifold a function on the set of geodesics running between boundary points. Clearly, this also makes sense for other families of curves, for example, magnetics geodesics. A beautiful Mukhometov's theorem of 1975 solves this problem for an arbitrary regular family of curves on subdomains of the Euclidean plane.
Mukhometov’s theorem

I will not formulate Mukhometov’s theorem in exact form, but instead I reformulate it in a way convenient for my further talk.

**Theorem**

Let $M$ be a bounded simply connected set in $\mathbb{R}^2$ with smooth boundary $\partial M$. Consider a family of curves $\Gamma$ joining boundary points in $M$ which satisfies the following conditions:

1. For every interior point $x \in M$ and every direction $\xi$, there is exactly one curve of our family passing through $x$ in the direction $\xi$ (considering the curves obtained by shift of a parameter to be the same curve).

2. Any two points $(x, y) \in \partial M \times \partial M$ are joint by exactly one curve of $\Gamma$, which depends smoothly on $x$ and $y$.

3. All curves in $\Gamma$ are parametrized by arclength with respect to the Euclidean metric.

If $f \in C^\infty(M)$ has zero integrals over the curves in $\Gamma$,

$$\int_\gamma f(\gamma) \, ds = 0,$$

then $f$ is itself zero.
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If $f \in C^\infty(M)$ has zero integrals over the curves in $\Gamma$,

$$\int_\gamma f(\gamma) \, ds = 0, \quad \gamma \in \Gamma,$$

then $f$ is itself zero.
This theorem solves the scalar integral geometry problem for a regular family of curves in subdomains of $\mathbb{R}^2$ with flat metric and these subdomains are convex with respect to this family. A similar theorem was later proved by Anikonov for one-forms instead of functions, a vector integral geometry problem or Doppler transform.

No theorem of such generality is known in higher dimensions. Almost all results concern geodesic, or magnetic geodesic, case, except for results of Frigyik-Stefanov-Uhlmann and Holman-Stefanov where they consider the scalar and vector integral geometry problems for a real analytic regular family of curves.
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At first glance, Mukhometov's theorem has no underlying geometric structure for the family of curves in question. Our aim is to reveal this structure and, surely, to generalize it to curved surfaces rather than subdomains of the Euclidean plane. We consider any two-dimensional manifold with boundary, and we wish to eliminate the convexity condition.
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At first glance, Mukhometov’s theorem has no underlying geometric structure for the family of curves in question. Our aim is to reveal this structure and, surely, to generalize it to curved surfaces rather than subdomains of the Euclidean plane. We consider any two-dimensional manifold with boundary, and we wish to eliminate the convexity condition.
Regular family of curves

On any two-dimensional manifold $M$ with boundary consider a family of curves $\Gamma$ joining boundary points in $M$ which satisfies the following conditions:

1. For every interior point $x \in M$ and every direction $\xi$, there is exactly one curve of our family passing through $x$ in the direction $\xi$ (considering the curves obtained by shift of a parameter to be the same curve).

2. Any two points $(x, y) \in \partial M \times \partial M$ are joined by at most one curve of $\Gamma$, which depends smoothly on $x$ and $y$.

If these conditions are satisfied $\Gamma$ is called a regular family of curves. The conjecture is that both scalar and vector integral geometry problems can be solved.
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If these conditions are satisfied $\Gamma$ is called a regular family of curves. The conjecture is that the both scalar and vector integral geometry problems can be solved.
X-ray transform over $\Gamma$

Since the integration of scalar function depends on parametrization we make the assumption that all curves in $\Gamma$ are parametrized by arclength with respect to any Finsler metric $F$.

**Theorem (A.-Dairbekov)**

Let $M$ be a two-dimensional manifold with boundary. Consider a regular family of curves $\Gamma$ on $M$ and assume that all curves of $\Gamma$ parametrized by arclength with respect to any Finsler metric. Then a sum of function $f \in C^\infty(M)$ and smooth one-form $\beta$ on $M$ integrates to zero over the curves in $\Gamma$, $$\int_\gamma f(\gamma) + \beta_\gamma(\dot{\gamma}) \, ds = 0, \quad \gamma \in \Gamma,$$

if and only if $f = 0$ and $\beta = dh$ for some $h \in C^\infty(M)$ such that $h|_{\partial M} = 0$. 
When we consider the purely vector problem we do not need any parametrization. So, in this case the conjecture is true.

**Theorem (A.-Dairbekov)**

Let $M$ be a two-dimensional manifold with boundary and consider a regular family of curves $\Gamma$ on $M$. Then a smooth one-form $\beta$ on $M$ integrates to zero over the curves in $\Gamma$,

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if and only if $\beta = dh$ for some $h \in C^\infty(M)$ such that $h|_{\partial M} = 0$. 

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Integral geometry problems on Finsler and Riemannian surfaces
Thermostats

If the curves of regular family are parametrized by arclength of some Finsler metric then $\Gamma$ defines a flow on unit sphere bundle $SM$.
Indeed, any $\lambda \in C^\infty(SM)$ defines the flow $\phi$ on $SM$ by the following Newton’s equation:

$$\frac{D\dot{\gamma}}{dt} = \lambda(\gamma, \dot{\gamma}) \dot{\gamma}^\perp$$  \hspace{1cm} (1)

to be called the flow of the thermostat $(M, F, \lambda)$.

Given a general family of curves $\Gamma$ with curves parametrized by arclength of Finsler metric, we define

$$\lambda(x, \xi, t) = \left\langle \frac{D\dot{\gamma}_{x, \xi}(t)}{dt}, \dot{\gamma}_{x, \xi}(t) \right\rangle_{\dot{\gamma}_{x, \xi}(t)}.$$

Since there is at most one curve $\gamma \in \Gamma$, up to a shift of the parameter, passing through $x$ in the direction $\xi$ for every point $x \in M$ and every direction $\xi$, function $\lambda$ does not depend on $t$. Then $\Gamma$ becomes a family of curves satisfying (1).
Elimination of convexity

The elimination of the convexity condition is possible due to the same technique that was firstly introduced by Sharafutdinov. This is the second crucial step in our argument.

Lemma

If a sum of function \( f \in C^\infty(M) \) and smooth one-form \( \beta \) on \( M \) integrates to zero over the curves in \( \Gamma \),

\[
\int_\gamma f(\gamma) + \beta_\gamma(\dot{\gamma}) \, ds = 0, \quad \gamma \in \Gamma,
\]

then \( f(x) + \beta_x(\xi) = 0 \) for all \((x, \xi) \in S(\partial M)\).

The lemma implies that \( f|_{\partial M} = 0 \).

We will use the following consequence of Sharafutdinov’s result, which shows that a certain correction can be added to \( f(x) + \beta_x(\xi) \) to make it vanish on \( \partial M \).
Elimination of convexity

Introduce the following notation \( \mathbf{w} = f + \beta \).

**Lemma**

Let \( g \) be a Riemannian metric on \( M \). For every a smooth 1-form \( \omega \), there is \( \varphi \in C_0^\infty(M) \) such that

\[
d_x \varphi(\nu) = \beta_x(\nu)
\]

for all \( x \in \partial M \) and every vector \( \nu \in T_x M \) orthogonal to \( \partial M \) with respect to \( g \).

Let \( \nu(x) \) be an inward unit normal vector field to \( \partial M \), i.e. \( \nu \in T_x M, \ x \in \partial M \) such that \( g_{\nu(x)}(\nu(x), \xi) = 0 \) for all \( \xi \in T_x \partial M \).

We construct Riemannian metric in Lemma above as follows: restrict fundamental tensor \( g_{ij} \) of Finsler metric \( F \) to any smooth vector field \( \tilde{\nu}(x) \) on \( M \) such that \( \tilde{\nu}|_{\partial M} = \nu \).

Write

\[
\tilde{\mathbf{w}}(x, \nu) := f(x) + \beta_x(\xi) - d_x \varphi(\nu).
\]

Then \( \tilde{\mathbf{w}}(x, \nu) = 0 \) for \( x \in \partial M \) since \( f|_{\partial M} = 0 \).

We may henceforth assume that the field \( \mathbf{w} \) itself vanishes on the boundary \( \mathbf{w}|_{\partial M} = 0 \).
Elimination of convexity

Losing no generality, we assume that \((M, F)\) is a smooth subset of a compact smooth Finsler surface \(U\) without boundary and extend \(F, \Gamma M\) to \(U\).
Elimination of convexity

Losing no generality, we assume that $(M, F)$ is a smooth subset of a compact smooth Finsler surface $U$ without boundary and extend $F, \Gamma_M$ to $U$. Now, we extend $w$ from $M$ to all of $U$ by zero, denoting it again by $w$ which is continuous on the whole $U$ and contains in $H^1(SU)$. 

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Given $(x, v) \in SM$, let $\gamma_{x,v}$ be the complete curve of $\Gamma$ in $U$ issuing from $(x, v)$, $\dot{\gamma}_{x,v}(0) = \xi$. Therefore, for any $(x, v)$ there is a number $l_{x,v}$ such that $\gamma_{x,v}(l_{x,v}) \notin M$. We define a function $u : SM \rightarrow \mathbb{R}$ to be

$$u(x, v) = \int_{l_{x,v}}^{0} w(\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)) \, dt.$$
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Note that the value of \(u(x, v)\) is independent of the choice of \(l_{x,v}\).
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Note that the value of \(u(x, v)\) is independent of the choice of \(l_{x,v}\).

Call a point \((x, v) \in SM\) regular if the curve \(\gamma_{x,v}\) of \(\Gamma\) intersects \(\partial M\) transversally from either side and the open segment of \(\gamma_{x,v}\) between the basepoint \(x\) and the point of intersection lies entirely in \(M^{\text{int}}\). We denote by \(RM \subset SM\) the set of all regular points. It is clear that \(RM\) is open in \(SM\) and has full measure in \(SM\).
Lemma (A.-Dairbekov)

The function $u$ has the following properties:

1. $u|_{S(U \setminus M)} = 0$.
2. $u \in H^1(SU) \cap C(SU) \cap C^\infty(RM)$.
3. $u$ is $C^1$ smooth along the lifts of curves of $\Gamma$ to $SM$ and satisfies

$$F u(x, v) = w(x, v) \text{ on } SM.$$ 

Using smoothening techniques we show that the following integral identity holds for $u$

$$\int_{SM} (FVu)^2 d\mu - \int_{SM} K(Vu)^2 d\mu = 0, \quad V := (v^\perp)^i \frac{\partial}{\partial v^i}$$

This implies that $Vu \equiv 0$ on $RM$. Then $u$ independent of $v$ almost everywhere. Since $u \in C(SM)$, then this holds everywhere. But in this case $F u = du_x(v) = w(x, v)$. 

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Integral geometry problems on Finsler and Riemannian surfaces
Magnetic Flows

On a compact oriented Riemannian manifold \((M, g)\) consider closed 2-form \(\Omega\) and magnetic flow \(\phi_t\) on \(TM\) described by Newton’s law of motion

\[
\nabla \dot{\gamma} \dot{\gamma} = Y(\dot{\gamma}),
\]

where \(\nabla\) is the Levy-Civita connection of \(g\) and \(Y : TM \to TM\) is the Lorentz force associated with \(\Omega\), i.e., the bundle map uniquely determined by

\[
\Omega_x(\xi, \eta) = \langle Y_x(\xi), \eta \rangle
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for all \(x \in M\) and \(\xi, \eta \in T_xM\). Orbits of magnetic flow are referred to as magnetic geodesic.
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Magnetic flows were firstly considered by Anosov-Sinai and Arnold. It was shown in that they are related to dynamical systems, symplectic geometry, classical mechanics and mathematical physics.

In inverse problems magnetic flows were considered by Dairbekov-Paternain-Stefanov-Uhlmann.
Simple magnetic systems

We call \((M, g, \Omega)\) a **simple** magnetic system if

- For any two points \(x, y \in \partial M\) there is unique magnetic geodesic connecting \(x, y\) and depending smoothly on \(x, y\).

- The boundary \(\partial M\) is **strictly magnetic convex**, that is

  \[ \Lambda(x, \xi) > \langle Y(\xi), \nu(x) \rangle, \quad (x, \xi) \in TM, \]

  where \(\Lambda\) is the second fundamental form on \(\partial M\) and \(\nu\) is the unit inward normal.

In this case, \(M\) is diffeomorphic to the unit ball of \(\mathbb{R}^n\) and therefore \(\Omega\) is exact i.e., is of the form

\[ \Omega = d\omega \]

where is 1-form on \(M\) — **magnetic potential**. We call \((M, g, \omega)\) a simple magnetic system on \(M\). The notion of simplicity arises naturally in the context of the boundary rigidity problem.
Attenuated magnetic X-ray transform

Let $h \in C^\infty(M)$ and $\alpha$ be a smooth 1-form on $M$. Consider an attenuation coefficient $\alpha$ as a combination of $h$ and $\alpha$, i.e. $\alpha(x, \xi) = h(x) + \alpha_x(\xi)$ for $(x, \xi) \in SM$. Let $\psi : SM \to \mathbb{R}$ be a smooth function on $SM$. Define the attenuated magnetic X-ray transform of $\psi$ by

$$I^a\psi(x, \xi) := \int_0^{\tau(x, \xi)} \psi(\phi_t(x, \xi)) \exp \left[ \int_0^t \alpha(\phi_s(x, \xi)) \, ds \right] \, dt, \quad (x, \xi) \in \partial_+ SM$$

where $\partial_+ SM$ denotes the set inward vectors and $\tau(x, \xi)$ is the time when the magnetic geodesic $\gamma_{x, \xi}(t)$ such that $x = \gamma_{x, \xi}(0)$, $\xi = \dot{\gamma}_{x, \xi}(0)$ exits is finite for each $(x, \xi) \in \partial_+ SM$. 
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For $k = 0, 1, 2, \ldots$ denote by $C^\infty(S^k M)$ the space of symmetric covariant tensor fields on $M$ of rank $k$, when $k = 0$, we abbreviate this to $C^\infty(M)$. 

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For $k = 0, 1, 2, \ldots$ denote by $C^\infty(S^k M)$ the space of symmetric covariant tensor fields on $M$ of rank $k$, when $k = 0$, we abbreviate this to $C^\infty(M)$.

For any $m \geq 0$ we are interested in $I^a$ applied to the functions on $SM$ of the following type

$$\psi(x, \xi) = \sum_{k=0}^m f^k_{i_1 \ldots i_k}(x) \xi^{i_1} \cdots \xi^{i_k}, \quad (2)$$

where $f^k \in C^\infty(S^k M)$ for every $0 \leq k \leq m$. 

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Integral geometry problems on Finsler and Riemannian surfaces
It is easy to see that $I^a$ has nontrivial kernel since

$$
\int_0^{\tau(x,\xi)} (G_{\mu} + \sigma \circ a) \left( \sum_{k=0}^{m-1} h^k_{i_1 \cdots i_k} (\gamma_{x,\xi} (t)) \dot{\gamma}_{x,\xi} (t)^{i_1} \cdots \dot{\gamma}_{x,\xi} (t)^{i_k} \right) \cdot \\
\cdot \exp \left[ \int_0^t a(\gamma_{x,\xi} (s), \dot{\gamma}_{x,\xi} (s)) \, ds \right] \, dt = 0
$$

for $h^r \in C^\infty (S^r M)$, $0 \leq r \leq m - 1$, such that $h^r|_{\partial M} = 0$. Here and further $\sigma$ denotes the symmetrization.
It is easy to see that $l^a$ has nontrivial kernel since

$$\int_0^{\tau(x,\xi)} (G_\mu + \sigma \circ a) \left( \sum_{k=0}^{m-1} h_{i_1 \cdots i_k} (\gamma_{x}(t),\xi(t)) \dot{\gamma}_{x}(t)^{i_1} \cdots \dot{\gamma}_{x}(t)^{i_k} \right) \cdot \exp \left[ \int_0^t a(\gamma_{x}(s),\dot{\gamma}_{x}(s)) \, ds \right] \, dt = 0$$

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$$\cdot \exp \left[ \int_0^t a(\gamma_x, \xi(s), \dot{\gamma}_{x, \xi(s)}) \, ds \right] \, dt = 0$$

for $h^r \in C^\infty(S^r M)$, $0 \leq r \leq m - 1$, such that $h^r|_{\partial M} = 0$. Here and further $\sigma$ denotes the symmetrization.

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**Theorem (A.)**

Let $(M, g, \omega)$ be a simple 2-dimensional magnetic system. Consider $h$ and $\alpha$ to be a smooth complex function and 1-form (resp.) on $M$, and denote $a = h + \alpha$. If $\psi$ is a smooth function on $SM$ of type (2) such that $l^a \psi \equiv 0$, then

$$(G_\mu + \sigma \circ a) \left( \sum_{k=0}^{m-1} h_{i_1}^k \ldots i_k (x) \xi_{i_1}^1 \ldots \xi_{i_k}^k \right) = \sum_{k=0}^{m} f_{i_1}^k \ldots i_k (x) \xi_{i_1}^1 \ldots \xi_{i_k}^k$$

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Proof follows the same scheme as in the papers of Paternain-Salo-Uhlmann.
Holomorphic functions

Since $M$ is assumed to be oriented, there is a circle action on the fibers of $SM$ with infinitesimal generator $V$ called the vertical vector field. The space $L^2(SM)$ decomposes orthogonally as a direct sum

$$L^2(SM) = \bigoplus_{k \in \mathbb{Z}} H_k$$

where $H_k$ is the eigenspace of $-iV$ corresponding to the eigenvalue $k$. Any function $u \in C^\infty(SM)$ has a Fourier series expansion

$$u = \sum_{k=-\infty}^{\infty} u_k,$$

where $u_k \in \Omega_k := C^\infty(SM) \cap H_k$.

A function $u$ on $SM$ is called **holomorphic** if $u_k = 0$ for all $k < 0$. Similarly, we say that a function $u$ on $SM$ is called **antiholomorphic** if $u_k = 0$ for all $k > 0$. 

By a (anti)holomorphic \textbf{integrating factor} we mean a complex function $w \in C^\infty (SM)$ which is (anti)holomorphic and such that

$$G_\mu w = -a \text{ in } SM.$$
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**Theorem (Ainsworth)**

Let $(M, g, \omega)$ be a simple magnetized Riemannian surface and let $\alpha \in C^\infty (SM)$ be the sum of a function on $M$ and a 1-form on $M$. Then there exist a holomorphic $w \in C^\infty (SM)$ and antiholomorphic $\tilde{w} \in C^\infty (SM)$ such that $G_\mu w = G_\mu \tilde{w} = -\alpha$. 

Yermat M. Assylbekov

Integral geometry problems on Finsler and Riemannian surfaces
We say that $f \in C^\infty(SM)$ has degree $m$ if $f_k = 0$ for $|k| \geq m + 1$. The identification between real-valued symmetric $m$-tensor fields and smooth real valued functions on $SM$ with degree $m$ was shown by Paternain-Salo-Uhlmann. This reduces Theorem A for the proof of the following

**Lemma**

Let $(M, g, \omega)$ be a simple 2-dimensional magnetic system, and assume that $u \in C^\infty(SM)$ satisfies $G_{\mu} u + au = -\psi$ in $SM$ with $u|_{\partial(SM)} = 0$.

(a) If $m \geq 0$ and if $\psi \in C^\infty(SM)$ is such that $\psi_k = 0$ for $k \leq -m - 1$, then $u_k = 0$ for $k \leq -m$.

(b) If $m \geq 0$ and if $\psi \in C^\infty(SM)$ is such that $\psi_k = 0$ for $k \geq m + 1$, then $u_k = 0$ for $k \geq m$.

We will only prove item (a) of the lemma, the proof of item (b) is completely analogous. Suppose that $u$ is a smooth solution of $G_{\mu} u + au = -\psi$ in $SM$ where $\psi_k = 0$ for $k \leq -m - 1$ and $u|_{\partial(SM)} = 0$. We choose a nonvanishing function $v \in \Omega_m$ and define the 1-form

$$A := -v^{-1}G_{\mu} v.$$
Then $\nu u$ solves the problem

$$(G_\mu + a + A)(\nu u) = -\nu \psi \text{ in } SM, \quad \nu u|_{\partial (SM)} = 0.$$  

Note that $\nu \psi$ is a holomorphic function. There exists a holomorphic $w \in C^\infty (SM)$ with $G_\mu w = a + A$. The function $e^w \nu u$ then satisfies

$$(G_\mu (e^w \nu u) = -e^w \nu \psi \text{ in } SM, \quad e^w \nu u|_{\partial (SM)} = 0.$$  

The right hand side $e^w \nu \psi$ is holomorphic, then $e^w \nu u$ is also holomorphic and $(e^w \nu u)_0 = 0$. Looking at Fourier coefficients shows that $(\nu u)_k = 0$ for $k \leq 0$, and therefore $u_k = 0$ for $k \leq -m$ as required.
Let $f$ and $\beta$ be a symmetric $m$-tensor and $m-1$-tensor field on $M$ and suppose that $I^a(f + \beta) = 0$. We write

$$u(x, \xi) := \int_0^{\tau(x, \xi)} \left( \sum_{k=0}^m f_{i_1 \cdots i_k}^k (\gamma_{x, \xi}(t)) \dot{\gamma}_{x, \xi}(t)^{i_1} \cdots \dot{\gamma}_{x, \xi}(t)^{i_k} \right) \cdot \exp \left[ \int_0^t a(\gamma_{x, \xi}(s), \dot{\gamma}_{x, \xi}(s)) \, ds \right] \, dt = 0, \quad (x, \xi) \in SM.$$

Then $u|_{\partial(SM)} = 0$, and also $u \in C^\infty(SM)$. Now

$$\sum_{k=0}^m f_{i_1 \cdots i_k}^k (x) \xi^{i_1} \cdots \xi^{i_k}$$

has degree $m$, and $u$ satisfies $G_\mu u + au = -(f + \beta)$ in $SM$ with $u|_{\partial(SM)} = 0$. Then $u$ has degree $m - 1$. We let $p := -u$. Now decompose $p$ into its Fourier components, and components in $\Omega_i$ and $\Omega_{-i}$ associate a symmetric $i$-tensor, denoted by $h^i$. This proves the theorem.