Some integral geometry problems on Finsler and Riemannian surfaces

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Joint work with Nurlan Dairbekov

X-ray transform

I recall that the Radon transform makes from a function on $\mathbb{R}^2$ a function on straight lines:

$$\mathcal{R}f(\ell) = \int_\ell f\, d,$$

and the inverse problem is the problem of reconstructing f from $\mathcal{R}f$.
More generally, the geodesic X-ray transform on a Riemannian manifold makes from a function on the manifold a function on the set of geodesics running between boundary points. Clearly, this also makes sense for other families of curves, for example, magnetics geodesics. A beautiful Mukhometov's theorem of 1975 solves this problem for an arbitrary regular family of curves on subdomains of the Euclidean plane.

Mukhometov's theorem

I will not formulate Mukhometov's theorem in exact form, but instead I reformulate it in a way convenient for my further talk.

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Let M be a bounded simply connected set in $\mathbb{R}^2$ with smooth boundary ∂M. Consider a family of curves Γ joining boundary points in M which satisfies the following conditions:

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3. *All curves in Γ are parametrized by arclength with respect to the Euclidean metric.*

If $f \in C^\infty(M)$ has zero integrals over the curves in Γ,

$$\int_\gamma f(\gamma)\, ds = 0, \quad \gamma \in \Gamma,$$

then f is itself zero.

This theorem solves the scalar integral geometry problem for a regular family of curves in subdomains of $\mathbb{R}^2$ with flat metric and these subdomains are convex with respect to this family. A similar theorem was later proved by Anikonov for one-forms instead of functions, a vector integral geometry problem or Doppler transform.

No theorem of such generality is known in higher dimensions. Almost all results concern geodesic, or magnetic geodesic, case, except for results of Frigyik-Stefanov-Uhlmann and Holman-Stefanov where they consider the scalar and vector integral geometry problems for a real analytic regular family of curves.

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If these conditions are satisfied Γ is called a regular family of curves.
The conjecture is that the both scalar and vector integral geometry problems can be solved.

X-ray transform over Γ

Since the integration of scalar function depends on parametrization we make the assumption that all curves in Γ are parametrized by arclength with respect to any Finsler metric F.

Theorem (A.-Dairbekov)

Let M be a two-dimensional manifold with boundary. Consider a regular family of curves Γ on M and assume that all curves of Γ parametrized by arclength with respect to any Finsler metric. Then a sum of function $f \in C^\infty(M)$ and smooth one-form β on M integrates to zero over the curves in Γ,

$$\int_\gamma f(\gamma) + \beta_\gamma(\dot\gamma)\, ds = 0, \quad \gamma \in \Gamma,$$

if and only if $f = 0$ and $\beta = dh$ for some $h \in C^\infty(M)$ such that $h|_{\partial M} = 0$.

X-ray transform over Γ

When we consider the purely vector problem we do not need any parametrization. So, in this case the conjecture is true.

Theorem (A.-Dairbekov)

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Thermostats

If the curves of regular family are parametrized by arclength of some Finsler metric then Γ defines a flow on unit sphere bundle SM.
Indeed, any $\lambda \in C^\infty(SM)$ defines the flow ϕ on SM by the following Newton's equation:

$$\frac{D\dot\gamma}{dt} = \lambda(\gamma, \dot\gamma)\dot\gamma^\perp \tag{1}$$

to be called the flow of the thermostat (M, F, λ).
Given a general family of curves Γ with curves parametrized by arclength of Finsler metric, we define

$$\lambda(x, \xi, t) = \left\langle \frac{D\dot\gamma_{x,\xi}(t)}{dt}, \dot\gamma^\perp_{x,\xi}(t) \right\rangle_{\dot\gamma_{x,\xi}(t)}.$$

Since there is at most one curve $\gamma \in \Gamma$, up to a shift of the parameter, passing through x in the direction ξ for every point $x \in M$ and every direction ξ, function λ does not depend on t. Then Γ becomes a family of curves satisfying (1).

Elimination of convexity

The elimination of the convexity condition is possible due to the same technique that was firstly introduced by Sharafutdinov. This is the second crucial step in our argument.

Lemma

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$$\int_\gamma f(\gamma) + \beta_\gamma(\dot\gamma)\, ds = 0, \quad \gamma \in \Gamma,$$

then $f(x) + \beta_x(\xi) = 0$ for all $(x, \xi) \in S(\partial M)$.

The lemma implies that $f|_{\partial M} = 0$.
We will use the following consequence of Sharafutdinov's result, which shows that a certain correction can be added to $f(x) + \beta_x(\xi)$ to make it vanish on ∂M.

Elimination of convexity

Introduce the following notation $\mathbf{w} = f + \beta$.

Lemma

Let g be a Riemannian metric on M. For every a smooth 1-form ω, there is $\varphi \in C_0^\infty(M)$ such that

$$d_x\varphi(\nu) = \beta_x(\nu)$$

for all $x \in \partial M$ and every vector $\nu \in T_xM$ orthogonal to ∂M with respect to g.

Let $\nu(x)$ be an inward unit normal vector field to ∂M, i.e. $\nu \in T_xM$, $x \in \partial M$ such that $g_{\nu(x)}(\nu(x), \xi) = 0$ for all $\xi \in T_x\partial M$.
We construct Riemannian metric in Lemma above as follows: restrict fundamental tensor g_{ij} of Finsler metric F to any smooth vector field $\tilde{\nu}(x)$ on M such that $\tilde{\nu}|_{\partial M} = \nu$.
Write

$$\tilde{\mathbf{w}}(x, v) := f(x) + \beta_x(\xi) - d_x\varphi(v).$$

Then $\tilde{\mathbf{w}}(x, \nu) = 0$ for $x \in \partial M$ since $f|_{\partial M} = 0$.
We may henceforth assume that the field $\mathbf{w}$ itself vanishes on the boundary $\mathbf{w}|_{\partial M} = 0$.

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Now, we extend $\mathbf{w}$ from M to all of U by zero, denoting it again by $\mathbf{w}$ which is continuous on the whole U and contains in $H^1(SU)$.
Given $(x, v) \in SM$, let $\gamma_{x,v}$ be the complete curve of Γ in U issuing from (x, v), $\dot{\gamma}_{x,v}(0) = \xi$. Therefore, for any (x, v) there is a number $l_{x,v}$ such that $\gamma_{x,v}(l_{x,v}) \notin M$. We define a function $u : SM \to \mathbb{R}$ to be

$$u(x, v) = \int_{l_{x,v}}^{0} \mathbf{w}(\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))\, dt.$$

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Call a point $(x, v) \in SM$ *regular* if the curve $\gamma_{x,v}$ of Γ intersects ∂M transversally from either side and the open segment of $\gamma_{x,v}$ between the basepoint x and the point of intersection lies entirely in M^{int}. We denote by $RM \subset SM$ the set of all regular points. It is clear that RM is open in SM and has full measure in SM.

Short proof

Lemma (A.-Dairbekov)

The function u has the following properties:

1. $u|_{S(U\setminus M)} = 0$.
2. $u \in H^1(SU) \cap C(SU) \cap C^\infty(RM)$.
3. *u is* C^1 *smooth along the lifts of curves of* Γ *to* SM *and satifies*

$$\mathbf{F}u(x,v) = \mathbf{w}(x,v) \text{ on } SM.$$

Using smoothening techniques we show that the following integral identity holds for u

$$\int_{SM} (\mathbf{F}Vu)^2\, d\mu - \int_{SM} \mathbb{K}(Vu)^2\, d\mu = 0, \quad V := (v^\perp)^i \frac{\partial}{\partial v^i}$$

This implies that $Vu \equiv 0$ on RM. Then u independent of v almost everywhere. Since $u \in C(SM)$, then this holds everywhere. But in this case $\mathbf{F}u = du_x(v) = \mathbf{w}(x,v)$.

Magnetic Flows

On a compact oriented Riemannian manifold (M, g) consider closed 2-form Ω and **magnetic flow** ϕ_t on TM described by Newton's law of motion

$$\nabla_{\dot\gamma}\dot\gamma = Y(\dot\gamma),$$

where ∇ is the Levy-Civita connection of g and $Y : TM \to TM$ is the **Lorentz force** associated with Ω, i.e., the bundle map uniquely determined by

$$\Omega_x(\xi, \eta) = \langle Y_x(\xi), \eta\rangle$$

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In inverse problems magnetic flows were considered by Dairbekov-Paternain-Stefanov-Uhlmann.

Simple magnetic systems

We call (M, g, Ω) a **simple** magnetic system if

- For any two points $x, y \in \partial M$ there is unique magnetic geodesic connecting x, y and depending smoothly on x, y.
- The boundary ∂M is **strictly magnetic convex**, that is $\Lambda(x, \xi) > \langle Y(\xi), \nu(x) \rangle$, $(x, \xi) \in TM$, where Λ is the second fundamental form on ∂M and ν is the unit inward normal.

In this case, M is diffeomorphic to the unit ball of $\mathbb{R}^n$ and therefore Ω is exact i.e., is of the form

$$\Omega = d\omega$$

where is 1-form on M — **magnetic potential**. We call (M, g, ω) a simple magnetic system on M. The notion of simplicity arises naturally in the context of the boundary rigidity problem.

Attenuated magnetic X-ray transform

Let $h \in C^\infty(M)$ and α be a smooth 1-form on M. Consider an attenuation coefficient $\mathfrak{a}$ as a combination of h and α, i.e. $\mathfrak{a}(x,\xi) = h(x) + \alpha_x(\xi)$ for $(x,\xi) \in SM$. Let $\psi : SM \to \mathbb{R}$ be a smooth function on SM. Define the **attenuated magnetic X-ray transform** of ψ by

$$I^{\mathfrak{a}}\psi(x,\xi) := \int_0^{\tau(x,\xi)} \psi(\phi_t(x,\xi)) \exp\left[\int_0^t \mathfrak{a}(\phi_s(x,\xi))\, ds\right] dt, \quad (x,\xi) \in \partial_+ SM$$

where $\partial_+ SM$ denotes the set inward vectors and $\tau(x,\xi)$ is the time when the magnetic geodesic $\gamma_{x,\xi}(t)$ such that $x = \gamma_{x,\xi}(0)$, $\xi = \dot{\gamma}_{x,\xi}(0)$ exits is finite for each $(x,\xi) \in \partial_+ SM$.

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For $k = 0, 1, 2, \ldots$ denote by $C^\infty(S^k M)$ the space of symmetric covariant tensor fields on M of rank k, when $k = 0$, we abbreviate this to $C^\infty(M)$.

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For any $m \geq 0$ we are interested in $I^{\mathfrak{a}}$ applied to the functions on SM of the following type

$$\psi(x,\xi) = \sum_{k=0}^{m} f^k_{i_1 \cdots i_k}(x) \xi^{i_1} \cdots \xi^{i_k}, \tag{2}$$

where $f^k \in C^\infty(S^k M)$ for every $0 \leq k \leq m$.

It is easy to see that $I^{\mathfrak{a}}$ has nontrivial kernel since

$$\int_0^{\tau(x,\xi)} (\mathbf{G}_\mu + \sigma \circ \mathfrak{a}) \left(\sum_{k=0}^{m-1} h^k_{i_1 \cdots i_k}(\gamma_{x,\xi}(t)) \dot{\gamma}_{x,\xi}(t)^{i_1} \cdots \dot{\gamma}_{x,\xi}(t)^{i_k} \right) \cdot \\ \cdot \exp \left[\int_0^t \mathfrak{a}(\gamma_{x,\xi}(s), \dot{\gamma}_{x,\xi}(s))\, ds \right] dt = 0$$

for $h^r \in C^\infty(S^r M)$, $0 \le r \le m-1$, such that $h^r|_{\partial M} = 0$. Here and futher σ denotes the symmetrization.

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Theorem (A.)

Let (M, g, ω) be a simple 2-dimensional magnetic system. Consider h and α to be a smooth complex function and 1-form (resp.) on M, and denote $\mathfrak{a} = h + \alpha$. If ψ is a smooth function on SM of type (2) such that $I^{\mathfrak{a}}\psi \equiv 0$, then

$$(\mathbf{G}_\mu + \sigma \circ \mathfrak{a}) \left(\sum_{k=0}^{m-1} h^k_{i_1 \cdots i_k}(x) \xi^{i_1} \cdots \xi^{i_k} \right) = \sum_{k=0}^{m} f^k_{i_1 \cdots i_k}(x) \xi^{i_1} \cdots \xi^{i_k}$$

for some $h^r \in C^\infty(S^r M)$ such that $h^r|_{\partial M} = 0$, $0 \le r \le m-1$.

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for some $h^r \in C^\infty(S^r M)$ such that $h^r|_{\partial M} = 0$, $0 \le r \le m-1$.

Proof follows the same scheme as in the papers of Paternain-Salo-Uhlmann.

Holomorphic functions

Since M is assumed to be oriented, there is a circle action on the fibers of SM with infinitesimal generator V called the vertical vector field.
The space $L^2(SM)$ decomposes orthogonally as a direct sum

$$L^2(SM) = \bigoplus_{k \in \mathbb{Z}} H_k$$

where H_k is the eigenspace of $-iV$ corresponding to the eigenvalue k. Any function $u \in C^\infty(SM)$ has a Fourier series expansion

$$u = \sum_{k=-\infty}^{\infty} u_k,$$

where $u_k \in \Omega_k := C^\infty(SM) \cap H_k$.
A function u on SM is called **holomorphic** if $u_k = 0$ for all $k < 0$. Similarly, we say that a function u on SM is called **antiholomorphic** if $u_k = 0$ for all $k > 0$.

Holomorphic integrating factors

By a (anti)holomorphic **integrating factor** we mean a complex function $w \in C^\infty(SM)$ which is (anti)holomorphic and such that

$$\mathbf{G}_\mu w = -\mathfrak{a} \text{ in } SM.$$

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Theorem (Ainsworth)

Let (M, g, ω) be a simple magnetized Riemannian surface and let $\mathfrak{a} \in C^\infty(SM)$ be the sum of a function on M and a 1-form on M. Then there exist a holomorphic $w \in C^\infty(SM)$ and antiholomorphic $\tilde{w} \in C^\infty(SM)$ such that $\mathbf{G}_\mu w = \mathbf{G}_\mu \tilde{w} = -\mathfrak{a}$.

We say that $f \in C^\infty(SM)$ has **degree** m if $f_k = 0$ for $|k| \geq m+1$. The identification between real-valued symmetric m-tensor fields and smooth real valued functions on SM with degree m was shown by Paternain-Salo-Uhlmann. This reduces Theorem A for the proof of the following

Lemma

Let (M, g, ω) be a simple 2-dimensional magnetic system, and assume that $u \in C^\infty(SM)$ satisfies $\mathbf{G}_\mu u + \mathfrak{a} u = -\psi$ in SM with $u|_{\partial(SM)} = 0$.

(a) *If $m \geq 0$ and if $\psi \in C^\infty(SM)$ is such that $\psi_k = 0$ for $k \leq -m-1$, then $u_k = 0$ for $k \leq -m$.*

(b) *If $m \geq 0$ and if $\psi \in C^\infty(SM)$ is such that $\psi_k = 0$ for $k \geq m+1$, then $u_k = 0$ for $k \geq m$.*

We will only prove item (a) of the lemma, the proof of item (b) is completely analogous. Suppose that u is a smooth solution of $\mathbf{G}_\mu u + \mathfrak{a} u = -\psi$ in SM where $\psi_k = 0$ for $k \leq -m-1$ and $u|_{\partial(SM)} = 0$. We choose a nonvanishing function $v \in \Omega_m$ and define the 1-form

$$A := -v^{-1} \mathbf{G}_\mu v.$$

Then vu solves the problem

$$(\mathbf{G}_\mu + \mathfrak{a} + A)(vu) = -v\psi \text{ in } SM, \quad vu|_{\partial(SM)} = 0.$$

Note that $v\psi$ is a holomorphic function. There exists a holomorphic $w \in C^\infty(SM)$ with $\mathbf{G}_\mu w = \mathfrak{a} + A$. The function $e^w vu$ then satisfies

$$\mathbf{G}_\mu(e^w vu) = -e^w v\psi \text{ in } SM, \quad e^w vu|_{\partial(SM)} = 0.$$

The right hand side $e^w v\psi$ is holomorphic, then $e^w vu$ is also holomorphic and $(e^w vu)_0 = 0$. Looking at Fourier coefficients shows that $(vu)_k = 0$ for $k \leq 0$, and therefore $u_k = 0$ for $k \leq -m$ as required.

Let f and β be a symmetric m-tensor and $m-1$-tensor field on M and suppose that $I^{\mathfrak{a}}(f+\beta)=0$. We write

$$u(x,\xi) := \int_0^{\tau(x,\xi)} \left(\sum_{k=0}^{m} f^k_{i_1\cdots i_k}(\gamma_{x,\xi}(t))\dot{\gamma}_{x,\xi}(t)^{i_1}\cdots\dot{\gamma}_{x,\xi}(t)^{i_k} \right) \cdot \\ \cdot \exp\left[\int_0^t \mathfrak{a}(\gamma_{x,\xi}(s),\dot{\gamma}_{x,\xi}(s))\,ds \right] dt = 0, \quad (x,\xi)\in SM.$$

Then $u|_{\partial(SM)}=0$, and also $u\in C^\infty(SM)$. Now

$$\sum_{k=0}^{m} f^k_{i_1\cdots i_k}(x)\xi^{i_1}\cdots\xi^{i_k}$$

has degree m, and u satisfies $\mathbf{G}_\mu u + \mathfrak{a}u = -(f+\beta)$ in SM with $u|_{\partial(SM)}=0$. Then u has degree $m-1$. We let $p:=-u$. Now decompose p into its Fourier components, and components in Ω_i and Ω_{-i} associate a symmetric i-tensor, denoted by h^i. This proves the theorem.