Hybrid Inverse Problems and Internal Functionals

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High Contrast and High Resolution in Imaging

Some imaging techniques are **high-contrast low-resolution**. Typically based on elliptic models that do not propagate singularities. WF(data) not affected by (most of) WF(parameters). Such methods include Optical, Electrical Impedance Tomography, and Elastography.

Other techniques are (sometimes) **high-resolution low-contrast**. They are based on the Fourier transform, wave propagation, or integral geometry. WF(data) determines WF(parameters) and injectivity sometimes holds. Such methods include M.R.I, Ultrasound, X-ray CT.

Hybrid (coupled-physics) Inverse Problems result from the physical coupling of one modality in each category. They combine high-contrast with high-resolution. Mathematically, they take the form of *inverse problems for the high-contrast parameters from knowledge of* **internal functionals** obtained from the high-resolution modality.

Hybrid Inverse Problems (HIP) are typically Low Signal.

Examples of physical couplings

In Photo-acoustic Tomography, light propagates through a domain. Absorbed photons generate a thermal expansion and the emission of ultrasound. This is the photo-acoustic effect. Boundary ultrasound measurements are first inverted to provide high resolution photon absorption maps, which are internal functionals of high-contrast optical parameters.

In Transient Elastography, elastic waves are generated. The resulting displacement is imaged by high-resolution ultrasound. The elastic displacement is a functional of high-contrast elastic parameters.

Other hybrid inverse problems include Acousto-Optics, Thermo-acoustics, Magnetic Resonance Elastography, Magnetic Resonance Electrical Impedanc Tomography, Ultrasound Modulated Tomography, etc.

Examples of Hybrid Inverse Problems

- We consider PDE models of the form:

$$-\nabla \cdot \gamma(x) \nabla u + \sigma(x) u = 0 \text{ in } X, \qquad u = f \text{ on } \partial X$$

$$-\nabla \times \nabla \times E + n(x) k^2 E + i\sigma(x) E = 0 \text{ in } X, \qquad \nu \times E = g \text{ on } \partial X$$

- We consider **internal functionals** of the form:

$H(x) = \Gamma(x)\sigma(x)\, u(x)$		Photo-acoustics
$H(x) = u(x)$		Transient (MR) Elastography
$H(x) = \sigma(x)\, \lvert u \rvert^2(x)$	or $\sigma(x) \lvert E \rvert^2(x)$	Thermo-acoustics
$H(x) = \gamma(x)\, \nabla u(x) \cdot \nabla u(x)$		Ultrasound Modulation

- We have one or several **illuminations** $f = f_j$ (and thus $H = H_j$).

Inverse Problems with Internal Functionals

- Applications in hydrology: $\nabla \cdot \gamma \nabla u = 0$, u known. Richter, SIAP 81; Alessandrini, ANS Pisa 87; Kohn&Lowe, M2AN 88
- MREIT and CDII ($H = \gamma |\nabla u|$; medical imaging); Nachman, Tamasan et al. IP 07, IP 09, 11; Seo et al. SIAM Rev 11
- UMEIT ($H = \gamma |\nabla u|^2$; medical); Ammari et al. SIAP 08; Gebauer&Scherzer SIAP 08; Capdeboscq et al. SIIS 09; Kuchment&Kunyansky, IP 11; Kuchment&Steinhauer 12
- TE/MRE ($H = u$); J. McLaughlin et al. IP 04; IP 09; IP 10; G. Nakamura et al. JAA 08; SIAP 11
- QPAT/QTAT and related ($H = \Gamma |u|^\alpha$; medical imaging); Cox et al. IP 07, JOSA 09; Ammari et al. 11; Gao et al. 11; Triki IP 11; Patrolia 12
- Books: Ammari, Springer 08; O. Scherzer (Handbook) Springer 11.
- Exponentially increasing Bio-Engineering literature.

- QPAT, MRE, TE

B.-Jollivet-Jugnon, Inverse transport theory of Photoacoustics, I.P. **26**, *025011, 2010*
B.-Uhlmann, Inverse Diffusion Theory of Photoacoustics, I.P. **26***(8), 085010, 2010*
B.-Ren, Multi-source Quantitative PAT in diffusive regime, I.P. **27***(7), 075003, 2011*
B.-Uhlmann, Reconstruction of coefficients in scalar second-order elliptic equations from knowledge of their solutions, CPAM, 2012

- QTAT

B.-Ren-Uhlmann-Zhou, Quantitative Thermo-acoustics and related problems, I.P. **27***(5), 055007, 2011* *B.-Zhou, 2013*
B.-Ren, Non-uniqueness results for a hybrid inverse problem, Cont. Math, **559**, *2011*

- Ultrasound Modulation

B., Cauchy problem and Ultrasound Modulated EIT, 2012
B.-Bonnetier-Monard-Triki, Inverse diffusion from knowledge of power densities 2012
Monard-B., Inverse diffusion problem with redundant internal information 2012
Monard-B., Inverse anisotropic diffusion in 2D, 2012; *M.-B. Higher dimensions, 2013*
B.-Guo-Monard 2013 *B.-Imperiale 2013* *B.-Moskow 2013*

- Review paper

B., Hybrid inverse problems and internal measurements, Inside Out 2012

Ultrasound modulation problem

Consider the problem

$$
\begin{aligned}
-\nabla \cdot \gamma(x) \nabla u_1 &= 0 && \text{in } X, && u_1 = f_1 \text{ on } \partial X \\
-\nabla \cdot \gamma(x) \nabla u_2 &= 0 && \text{in } X, && u_2 = f_2 \text{ on } \partial X \\
H_1(x) &= \gamma(x) \nabla u_1(x) \cdot \nabla u_1(x) && \text{in } X \\
H_2(x) &= \gamma(x) \nabla u_2(x) \cdot \nabla u_2(x) && \text{in } X
\end{aligned}
$$

Ultrasound modulation problem

Consider the problem

$$
\begin{aligned}
-\nabla \cdot \gamma(x) \nabla u_1 &= 0 && \text{in } X, && u_1 = f_1 \text{ on } \partial X \\
-\nabla \cdot \gamma(x) \nabla u_2 &= 0 && \text{in } X, && u_2 = f_2 \text{ on } \partial X \\
\gamma(x) \nabla u_1(x) \cdot \nabla u_1(x) &= H_1(x) && \text{in } X \\
\gamma(x) \nabla u_2(x) \cdot \nabla u_2(x) &= H_2(x) && \text{in } X
\end{aligned}
$$

Ultrasound modulation problem

Consider the problem

$$
\begin{aligned}
-\nabla \cdot \gamma(x) \nabla u_1 &= 0 && \text{in } X, && u_1 = f_1 \text{ on } \partial X \\
-\nabla \cdot \gamma(x) \nabla u_2 &= 0 && \text{in } X, && u_2 = f_2 \text{ on } \partial X \\
\gamma(x) \nabla u_1(x) \cdot \nabla u_1(x) &= H_1(x) && \text{in } X \\
\gamma(x) \nabla u_2(x) \cdot \nabla u_2(x) &= H_2(x) && \text{in } X
\end{aligned}
$$

The left-hand side is a **polynomial** of γ, u_j and their derivatives. This forms a 4×3 **redundant system of nonlinear PDEs** in X.

Systems of coupled nonlinear equations

The above hybrid inverse problems may be recast as

$$\mathcal{F}(\gamma, \{u_j\}_{1\leq j\leq J}) = \mathcal{H}, \tag{1}$$

where γ is the collection of unknown parameters and u_j are PDE solutions. For instance, for the ultrasound modulation problem, we have

$$\mathcal{F}(\gamma, \{u_j\}_{1\leq j\leq J}) = \begin{pmatrix} -\nabla\cdot\gamma\nabla u_j \\ \gamma|\nabla u_j|^2 \end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix} 0 \\ H_j \end{pmatrix}, \qquad 2J - \text{ rows }.$$

so that (1) is a redundant $2J\times(J+m)$ system of nonlinear equations with m number of unknowns in γ so that $m=1$ if γ is scalar.

HIP theory therefore concerns uniqueness, stability estimates, reconstruction procedures for *typically redundant (over-determined) systems* of the form (1) with appropriate boundary conditions.

The $0-$Laplacian with $J=1$

$$-\nabla\cdot\gamma(x)\nabla u=0, \qquad \gamma(x)|\nabla u|^2(x)-H(x)=0 \qquad u=f \text{ on } \partial X.$$

The elimination of γ is possible and yields the 0-Laplacian

$$-\nabla\cdot\frac{H(x)}{|\nabla u|^2}\nabla u=0 \text{ in } X, \qquad u=f \text{ on } \partial X.$$

The above equation *with Cauchy data* may be transformed as

$$(I-2\widehat{\nabla u}\otimes\widehat{\nabla u}):\nabla^2 u+\nabla \ln H\cdot\nabla u=0 \text{ in } X, \quad u=f \text{ and } \frac{\partial u}{\partial \nu}=j \text{ on } \partial X.$$

Here $\widehat{\nabla u}=\frac{\nabla u}{|\nabla u|}$. Thus $g^{ij}=(I-2\widehat{\nabla u}\otimes\widehat{\nabla u})_{ij}$ is a definite matrix of signature $(1,n-1)$. We thus have a quasilinear strictly hyperbolic equation with $\widehat{\nabla u}(x)$ the "time" direction. Cauchy data generate stable solutions on "space-like" part of ∂X for the *metric* g.

Stability on domain of influence

Let u and $\tilde{u}$ be two solutions of the hyperbolic equation and $v = u - \tilde{u}$.

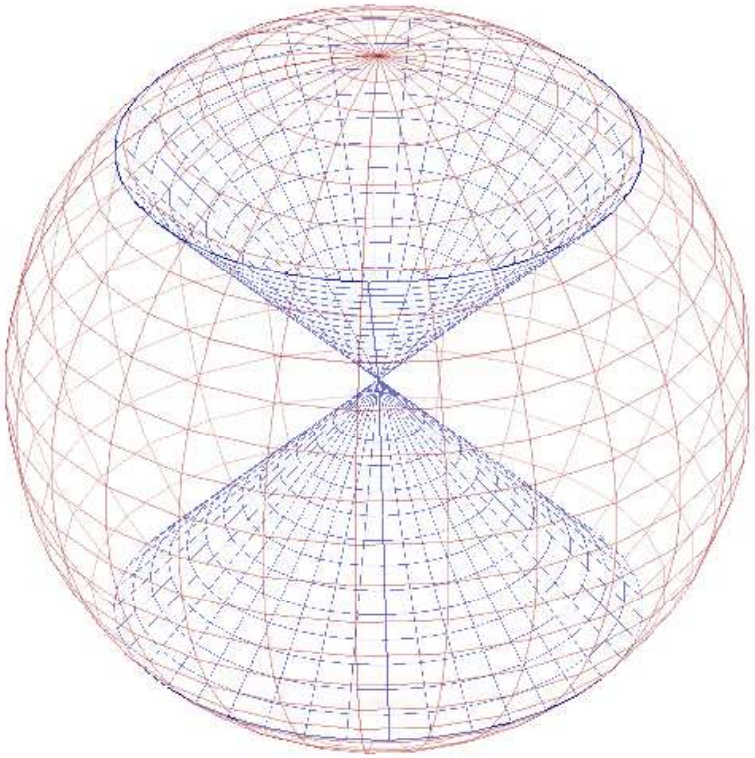
IF (appropriate) Lorentzian metric is uniformly strictly hyperbolic, then:

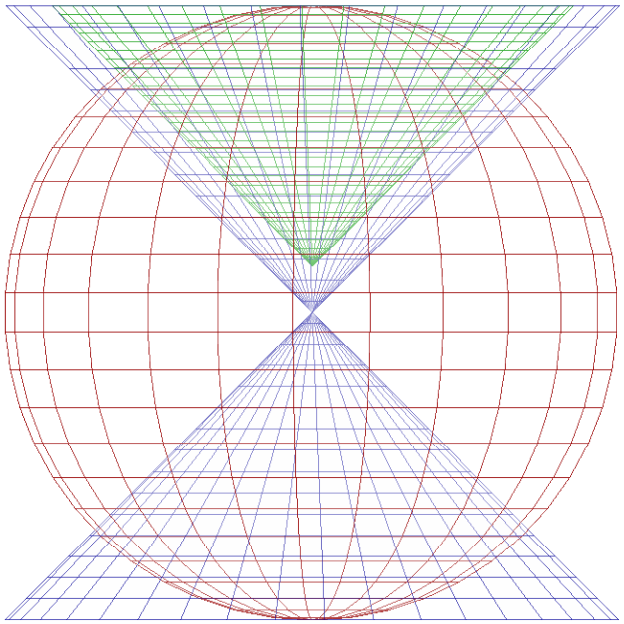
Theorem [B. 12]. Let $\Sigma_1 \subset \Sigma_g$ the space-like component of ∂X and $\mathcal{O}$ the **domain of influence** of Σ_1. For θ the distance of $\mathcal{O}$ to the boundary of the domain of influence of Σ_g, we have the local stability result:

$$\int_{\mathcal{O}} |v|^2 + |\nabla v|^2 + (\gamma - \tilde{\gamma})^2 \, dx \leq \frac{C}{\theta^2} \Big(\int_{\Sigma_1} |\delta f|^2 + |\delta j|^2 \, d\sigma + \int_{\mathcal{O}} |\nabla \delta H|^2 \, dx \Big),$$

where $\gamma = \frac{H}{|\nabla u|^2}$ and $\tilde{\gamma} = \frac{\tilde{H}}{|\nabla \tilde{u}|^2}$ are the reconstructed conductivities. We observe the loss of one derivative from δH to $\delta\gamma$ (sub-elliptic estimate).

Domain of Influence





Domain of influence (blue) for metric $g = I - 2e_z \otimes e_z$ on sphere (red). Null-like vectors (surface of cone) generate instabilities. Right: Sphere (red), domains of uniqueness (blue) and with controlled stability (green).

Linearization and Ellipticization

Consider as an example for $\mathbb{R} \ni \alpha > 0$ a redundant system of the form

$$-\nabla\cdot\gamma(x)\nabla u_j = 0, \quad \gamma^\alpha(x)|\nabla u_j|^2(x) - H_j(x) = 0, \quad u_j|_{\partial X} = f_j, \quad 1 \leq j \leq J.$$

With $1 \leq j \leq J = 1$ and $\alpha < 2$, the system is hyperbolic. How about the **redundant** system $2J \times (J+1)$ for $J \geq 2$? It may be **elliptic**.
Linearizing gives the system:

$$\begin{aligned} \nabla \cdot \delta\gamma \nabla u_j + \nabla \cdot \gamma \nabla \delta u_j &= 0 \qquad (2) \\ \alpha\gamma^{\alpha-1}\delta\gamma|\nabla u_j|^2 + 2\gamma^\alpha \nabla u_j \cdot \nabla \delta u_j &= \delta H_j. \qquad (3) \end{aligned}$$

Rewrite this system as $\mathcal{A}v = \mathcal{S}$ with $v = (\delta\gamma, \delta u_1, \ldots, \delta u_J)$. We recast the above system for v as

$$\mathcal{A}v := (\mathcal{P}_J + \mathcal{R}_J)v = \mathcal{S}$$

where $\mathcal{P}_J$ is the *principal part* and $\mathcal{R}_J$ is lower order.

Let us define $F_j = \nabla u_j$. The symbol of this $2J \times (J+1)$ system is:

$$\mathfrak{p}_J(x,\xi) = \begin{pmatrix} \alpha\gamma^{\alpha-1}|F_1|^2 & 2\gamma^\alpha F_1 \cdot i\xi & \dots & 0 \\ F_1 \cdot i\xi & -\gamma|\xi|^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha\gamma^{\alpha-1}|F_J|^2 & 0 & \dots & 2\gamma^\alpha F_J \cdot i\xi \\ F_J \cdot i\xi & 0 & \dots & -\gamma|\xi|^2 \end{pmatrix}.$$

The system is *elliptic* when $\mathfrak{p}_J(x,\xi)$ is maximal rank $(J+1)$ for all $\xi \in \mathbb{S}^{n-1}$.

Take the row $|F_1|^2$ and all rows with $\gamma|\xi|^2$. The determinant is proportional to $\alpha - 2(\hat{F}_1 \cdot \hat{\xi})^2$. When $\alpha > 2$, the system is elliptic with $J = 1$. Repeat for all rows with $|F_j|^2$. When $\alpha < 2$, the system becomes elliptic for a sufficiently rich family of $\hat{F}_j = \widehat{\nabla u_j}$ since $\alpha = 2(\hat{F}_j \cdot \hat{\xi})^2$ for all $1 \le j \le J$ cannot be satisfied by any $\hat{\xi} \in \mathbb{S}^{n-1}$.

(i) Redundant concatenation of hyperbolic systems becomes *elliptic*.

(ii) Possible **IF** we choose f_j s.t. the following qualitative statement on quadratic forms holds: $\alpha|\xi|^2 - 2(\hat{F}_j \cdot \xi)^2 = 0$, $1 \le j \le J$ implies $\xi = 0$.

Redundant elliptic systems

The system is elliptic in the sense of Douglis and Nirenberg: each row and column is given an index s_i and t_j and the principal term is the homogeneous differential operator of order $s_i + t_j$. For the above system, we choose $s_{2k+1} = 0$, $s_{2k} = 1$, $t_1 = 0$, $t_{k\geq 2} = 1$.

The system needs to be augmented with boundary conditions that satisfy the Lopatinskii condition. Dirichlet conditions on δu_j and no condition on $\delta\gamma$ satisfy the LC. Indeed, we need to show that $v(z) = (\delta\gamma(z), \ldots, \delta u_J(z)) \equiv 0$ is the only solution to

$$\delta u_j(0) = 0, \quad F_j \cdot N \partial_z \delta\gamma + \gamma \partial_z^2 \delta u_j = 0, \ |F_j|^2 \delta\gamma + 2\gamma F_j \cdot N \partial_z \delta u_j = 0, \ z > 0$$

vanishing as $z \to \infty$ for $N = \nu(x)$ at $x \in \partial X$ and z coordinate along $-N$. We observe that this is the case if $\alpha|F_j|^2 - 2(F_j \cdot N)^2 \neq 0$ for some j. This is the condition for joint ellipticity.

We can then use the theory of Agmon-Douglis-Nirenberg extended to redundant systems by Solonnikov (J. Sov. Math. 73) to obtain that the system $Av = S$ (including b.c.) admits a left-parametrix R so that $RA = I - T$, T compact.

Moreover, we have elliptic stability (regularity) estimates.

Elliptic stability estimates

Stability (regularity) estimates for redundant DN systems take the form

$$\sum_{j=1}^{J+1} \|v_j\|_{H^{l+t_j}(X)} \leq C \sum_{i=1}^{2J} \|S_i\|_{H^{l-s_i}(X)} + C_2 \sum_{t_j>0} \|v_j\|_{L^2(X)}.$$

For the example (2)-(3), we have for $v_1 = \delta\gamma$ that $s_i = 0$ when $S_i \neq 0$ and $t_1 = 0$. Thus,

$$\|\delta\gamma\|_{H^l(X)} + \sum_j \|\delta u_j\|_{H^{l+1}(X)} \leq C \sum_j \|\delta H_j\|_{H^l(X)} + C_2 \sum_j \|\delta u_j\|_{L^2(X)}.$$

This implies *no loss* of derivatives from δH to $\delta\gamma$.
However, we *do not have injectivity* of the system since $C_2 \neq 0$ a priori.
This shows that the system A can be inverted up to a finite dimensional kernel with RA Fredholm of index 0.

Invertibility and Local Uniqueness for Nonlinear I.P.

Let us recast the original nonlinear I.P. as

$$\mathcal{F}(v_0+v)=\mathcal{H}, \qquad \mathcal{H}_0:=\mathcal{F}(v_0),$$

with $A=\mathcal{F}'(v_0)$ the linearization about v_0. When A admits a bounded left inverse $(\mathcal{F}')^{-1}(v_0):=(I-T)^{-1}R$, then we can recast the I.P. as

$$v=\mathcal{G}(v):=(\mathcal{F}')^{-1}(v_0)(\mathcal{H}-\mathcal{H}_0)-(\mathcal{F}')^{-1}(v_0)\Big(\mathcal{F}(v_0+v)-\mathcal{F}(v_0)-\mathcal{F}'(v_0)v\Big).$$

Since $\mathcal{F}$ is polynomial and hence Lipschitz, we obtain that $\mathcal{G}(v)$ is a contraction when $\mathcal{H}-\mathcal{H}_0$ is sufficiently small. This provides a *local uniqueness result* for the nonlinear I.P.

To obtain such a result, we need a framework in which the linearized problem is *injective*. This is often a difficult problem.

Holmgren, Carleman, and Calderón

Assume that the system $\mathcal{A}$ is elliptic in the regular sense, i.e., $t_j = t$ and $s_i = 0$. (Else, $\mathcal{A}^t\mathcal{A}$ is not necessarily DN elliptic.) Consider, with $t = 2$, the two problems

$$\mathcal{A}v = S, \quad v|_{\partial X} = 0, \qquad \text{and} \qquad \mathcal{A}^t\mathcal{A}v = \mathcal{A}^t S, \quad v|_{\partial X} = \partial_\nu v|_{\partial X} = 0.$$

The second system is $(J+1)\times(J+1)$- determined even if the first one is $2J \times (J+1)$ redundant. It provides an *explicit* reconstruction procedure. Moreover, injectivity of the second one implies injectivity of the redundant (both in X and on ∂X) system:

$$\mathcal{A}v = 0, \quad v|_{\partial X} = \partial_\nu v|_{\partial X} = 0.$$

Injectivity for such a system can be proved by Holmgren's theorem when $\mathcal{A}$ has analytic coefficients and by Carleman estimates, as obtained for systems in Calderón's theorem, for a restricted class of operators $\mathcal{A}$.

Holmgren and local results

Holmgren's theorem can be used for $\mathcal{A}$ with analytic coefficients and $\mathcal{A}$ on a sufficiently small domain X.

When $\mathcal{A} = \mathcal{A}_A$ has analytic coefficients and $\mathcal{A}_A v = 0$, then an application of Hörmander's theorem shows that $WF_A(v) \subset WF_A(\det(\mathcal{A}_A^t \mathcal{A}_A)v)$ so that v is analytic. With vanishing Cauchy data, $v = 0$ and *injectivity* follows.
This provides *genericity* for hybrid inverse problems (invertibility of linear and nonlinear IP on open, dense, set).

When spatial domain X is small, write $\mathcal{A} = \mathcal{A}_0 + (\mathcal{A} - \mathcal{A}_0)$ with $\mathcal{A}_0$ the operator with coefficients frozen at $x = 0$. We then apply Holmgren's theorem to $\mathcal{A}_0$ and then to $\mathcal{A}$ by perturbation on a small domain.

Carleman estimates and Calderón's theorem

When $\mathcal{A}$ is not analytic and X is not small, proving **injectivity** is more difficult.

Recalling that $\mathcal{A} = \mathcal{P} + \mathcal{R}$ with $\mathcal{P}$ leading term, we seek *injectivity results* depending on leading term $\mathcal{P}$ and not $\mathcal{R}$. This essentially forces $\mathfrak{p}(\xi + \tau N)$ for $\xi \in \mathbb{S}^{n-1}$ and $N \in \mathbb{S}^{n-1}$ to be a diagonal (diagonalized) symbol with diagonal terms that are polynomials in τ with at most simple real roots and at most double complex roots. When these assumptions do not hold, then UCP depends on the structure of lower-order terms.

Applies to modified form of ultrasound modulation problem and systems of the form $\begin{pmatrix} P_1 & C \\ 0 & P_2 \end{pmatrix} u = 0$ with P_1 satisfying UCP, P_2 elliptic with *simple complex roots* (saving one to control C; all operators of order m here).

Summary of elliptization

• Non-elliptic determined systems can be augmented to redundant elliptic systems by acquiring more internal functionals.

• A-D-N theory (Solonnikov) provides left parametrix for system satisfying Lopatinskii conditions and optimal stability estimates but no injectivity.

• Injectivity of linear problem crucial to obtain local injectivity of nonlinear problem as dimension of kernel of linear problem is not stable (index is).

• Injectivity can be proved by augmenting the boundary conditions, for instance by solving $\mathcal{A}^t\mathcal{A}v = \mathcal{A}^t\mathcal{S}$ with Dirichlet B.C. (e.g., $v = \partial_\nu v = 0$).

• On small domain, Holmgren's theory applies. On larger domains, a UCP needs to be proved using Carleman/Calderón approach, which is more problem-dependent.

• Applied to reconstruction of optical parameters (γ, σ) (S. Moskow) and Maxwell systems (T. Zhou).

Constraints for ellipticity and beyond

• In some settings, ellipticity is *not true*, independent of J, although $\mathcal{A}$ is injective: the next-order term $\mathcal{P}_1$ after the principal part $\mathcal{P}$ is necessary to invert $\mathcal{P}+\mathcal{P}_1$ (with sub-elliptic estimates and the loss of one derivative; see Chenxi Guo's talk).

• In some settings, redundant functionals give more than *ellipticity*.

• The **ellipticity** of the linearized system requires certain constraints on the values of the functionals H_j (linear independence of gradients, quadratic forms, etc.). Such constraints depend on the choice of the boundary conditions $\{f_j\}$ that are obtained by explicit solutions such as, for instance, complex geometric optics (CGO) solutions.

Redundant Internal Functionals in UMEIT

$$-\nabla\cdot\gamma(x)\nabla u_j = 0 \ X, \quad u_j = f_j \ \partial X, \quad H_{ij}(x) = \gamma(x)^{2\alpha}\nabla u_i\cdot\nabla u_j(x), \ 1 \le i,j \le J.$$

Reconstructions in UMEIT ($\alpha = \frac{1}{2}$) are obtained by acquiring **redundant** *internal functionals* $H_{ij} = S_i \cdot S_j(x)$ with $S_i(x) = \gamma^\alpha \nabla u_i(x)$. Then

$$\nabla \cdot S_j = (\alpha - 1) F \cdot S_j, \quad dS_j^\flat = \alpha F^\flat \wedge S_j^\flat, \quad 1 \le j \le J, \qquad F = \nabla(\log \gamma).$$

Strategy: (i) *Eliminate* F and find closed-form equation for $S = (S_1|\dots|S_n)$

.

(ii) Solve for the redundant system of ODEs for S.

Works **IF** H is invertible in $\mathcal{M}(n;\mathbb{R})$, i.e., $\det(\nabla u_1, \dots, \nabla u_n) \neq 0$. This qualitative property on elliptic solutions holds for well-chosen $\{f_j\}$.

Elimination and system of ODEs in UMEIT

Lemma [B.-Bonnetier-Monard-Triki 12; Monard-B. 12]. Let $\Omega \subset X$. **IF** $\inf_{x\in\Omega} \det(S_1(x),\dots,S_n(x)) \geq c_0 > 0$, then with $D(x) = \sqrt{\det H(x)}$,

$$F(x) = \frac{c_F}{D} \sum_{i,j=1}^{n} \Big(\nabla(DH^{ij}) \cdot S_i(x)\Big) S_j(x), \; c_F = \frac{1}{1+(n-2)\alpha}, \; H^{-1} = (H^{ij}).$$

Moreover, we find $\nabla \otimes S_j = \sum_{i,k,l,m} H^{ik}(S_k \cdot \nabla S_j) \cdot S_l H^{lm} S_i \otimes S_m$ with

$$2(S_i \cdot \nabla S_j) \cdot S_k = S_i \cdot \nabla H_{jk} - S_j \cdot \nabla H_{ik} + S_k \cdot \nabla H_{ij} - 2F \cdot S_k H_{ij} + 2F \cdot S_j H_{ik}.$$

Theorem [idem; Capdeboscq et al. SIIS 09 in $n = 2$]. There exists an open set of illuminations f_j for $J = n$ in even dimension and $J = n+1$ in odd dimension such that for γ and γ' the conductivities corresponding to H and H', we have the following **global stability** result:

$$\|\gamma - \gamma'\|_{W^{1,\infty}(X)} \leq C\left(\varepsilon_0 + \|H - H'\|_{W^{1,\infty}(X)}\right)$$

$$\varepsilon_0 = |\gamma(x_0) - \gamma'(x_0)| + \textstyle\sum_{i=1}^{J} \|S_i(x_0) - S_i'(x_0)\|.$$

Anisotropic conductivities and internal data in $n = 2$

$$\nabla \cdot \gamma \nabla u_i = 0 \ X, \quad u_i = f_i \ \partial X, \qquad H_{ij} = \gamma \nabla u_i \cdot \nabla u_j, \ 1 \leq i, j \leq I.$$

Define $\gamma = A^2$ and $A = |A|\tilde{A}$ with $\det(\tilde{A}) = 1$. Then for appropriate boundary conditions f_i on ∂X and for $I = 4$, we have:

Theorem [Monard B. 12] The internal functionals $H = \{H_{ij}\}_{i,j=1}^4$ uniquely determine the tensor $\tilde{A}$ via explicit algebraic equations. Moreover, we have the (sub-elliptic) stability estimate

$$\|\tilde{A} - \tilde{A}'\|_{L^\infty(X)} \leq C \|H - H'\|_{W^{1,\infty}}.$$

Theorem [Monard B. 12] Let $\tilde{A}$ be known. Then $|A|$ is uniquely determined by $\{H_{ij}\}_{1 \leq i,j \leq 2} \in W^{1,\infty}$. Moreover, we have the (elliptic) estimate

$$\| |A| - |A'| \|_{W^{1,\infty}(X)} \leq C \|H - H'\|_{W^{1,\infty}}.$$

Reconstructions from solution measurements

Consider a *general scalar elliptic* equation

$$\nabla \cdot a\nabla u + b \cdot \nabla u + cu = 0 \quad \text{in } X, \qquad u = f \quad \text{on } \partial X$$

with a, b, c, $\nabla \cdot a$ of class $C^{0,\alpha}(\bar{X})$ for $\alpha > 0$, complex-valued, and $\alpha_0|\xi|^2 \leq \xi \cdot (\Re a)\xi \leq \alpha_0^{-1}|\xi|^2$. For τ a non-vanishing function on X, define

$$a_\tau = \tau a, \quad b_\tau = \tau b - a\nabla\tau, \quad c_\tau = \tau c$$

and the equivalence class $\mathfrak{c} := (a, b, c) \sim (a_\tau, b_\tau, c_\tau)$.

Let $I \in \mathbb{N}^*$ and $(f_i)_{1\leq i\leq I}$ be I boundary conditions. Define $\mathfrak{f} = (f_1, \ldots, f_I)$. The **measurement operator** $\mathfrak{M}_\mathfrak{f}$ is

$$\mathfrak{M}_\mathfrak{f} : \quad \mathfrak{c} \mapsto \mathfrak{M}_\mathfrak{f}(\mathfrak{c}) = (u_1, \ldots, u_I),$$

with $H_j(x) = u_j(x)$ **solution** of the above elliptic problem with $f = f_j$.

Unique reconstruction up to gauge transformation

$$\nabla \cdot a\nabla u_j + b \cdot \nabla u_j + cu_j = 0 \quad \text{in } X, \qquad u_j = f_j \quad \text{on } \partial X, \quad 1 \le j \le I.$$

We assume the above elliptic equation well posed for $\mathfrak{c} = (a, b, c)$.

Theorem [B. Uhlmann 2012]. Let $\mathfrak{c}$ and $\tilde{\mathfrak{c}}$ be two classes of coefficients with (a, b, c) and $\nabla \cdot a$ of class $C^{m,\alpha}(\bar{X})$ for $\alpha > 0$ and $m = 0$ or $m = 1$.

For I *sufficiently large* and an *open set of boundary conditions* $\mathfrak{f} = (f_j)_{1 \le j \le I}$, then $\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c})$ uniquely and stably determines $\mathfrak{c}$:

$$\begin{aligned} \|(a, b + \nabla \cdot a, c) - (\tilde{a}, \tilde{b} + \nabla \cdot \tilde{a}, \tilde{c})\|_{W^{m,\infty}(X)} &\le C\|\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c}) - \mathfrak{M}_{\mathfrak{f}}(\tilde{\mathfrak{c}})\|_{W^{m+2,\infty}(X)}, \\ \|b - \tilde{b}\|_{L^\infty(X)} &\le C\|\mathfrak{M}_{\mathfrak{f}}(\mathfrak{c}) - \mathfrak{M}_{\mathfrak{f}}(\tilde{\mathfrak{c}})\|_{W^{3,\infty}(X)}, \end{aligned}$$

for $m = 0, 1$ and for an appropriate $(\tilde{a}, \tilde{b}, \tilde{c})$ of $\tilde{\mathfrak{c}}$.

The *explicit reconstructions* and *proofs* are based on **linear independence** of gradients ∇u_j and Hessians $\nabla^{\otimes 2} u_j$ of solutions to elliptic model.

The IFs and the CGOs

Several HIPs require to verify qualitative properties of elliptic solutions:

- the absence of critical points in Photo-acoustics and Elastography
- the hyperbolicity of a given Lorentzian metric in UMOT
- the **linear independence of gradients** of elliptic solutions in UMOT
- the joint ellipticity of quadratic forms in UMEIT

(i) One way to solve this is to use **CGO** solutions whenever available: we verify the property on unperturbed CGOs (for constant-coefficient equation), then by continuity on perturbed CGOs, and then by continuity for **illuminations** f_j on ∂X close to CGO traces.

(ii) When CGO solutions are not available (anisotropic or complex valued leading coefficients), we construct local solutions (by freezing coefficients) that satisfy such conditions. We then use UCP and the Runge approximation to control such solutions from ∂X.

Vector fields and complex geometrical optics

• Take $\rho = (\rho_r + i\rho_i) \in \mathbb{C}^n$ with $\rho \cdot \rho = 0$. Then $\Delta e^{\rho\cdot x} = 0$. Let $u_1 = \Re e^{\rho\cdot x}$ and $u_2 = \Im e^{\rho\cdot x}$ so that $\nabla u_1 = e^{\rho_r\cdot x}\Big(\cos(\rho_i\cdot x\rho_r) - \sin(\rho_i\cdot x\rho_i)\Big)$ and $\nabla u_2 = e^{\rho_r\cdot x}\Big(\sin(\rho_i\cdot x\rho_r) + \cos(\rho_i\cdot x\rho_i)\Big)$. We thus find that

$$|\nabla u_1| > 0, \qquad |\nabla u_2| > 0, \qquad \nabla u_1 \cdot \nabla u_2 = 0.$$

• Let $u_\rho(x) = \gamma^{-\frac{1}{2}} e^{\rho\cdot x}\Big(1 + \psi_\rho(x)\Big)$ solution of $-\nabla\cdot\gamma\nabla u_\rho + \sigma u_\rho = 0$.

Theorem[B.-Uhlmann 10]. For q sufficiently smooth and $k \geq 0$, we have

$$|\rho|\|\psi_\rho\|_{H^{\frac{n}{2}+k+\varepsilon}(X)} + \|\psi_\rho\|_{H^{\frac{n}{2}+k+1+\varepsilon}(X)} \leq C\|q\|_{H^{\frac{n}{2}+k+\varepsilon}(X)}.$$

Thus the perturbed gradient directions $\theta_1 = \widehat{\nabla u_1}$ and $\theta_2 = \widehat{\nabla u_2}$ still satisfy $|\theta_1| > 0$, $|\theta_2| > 0$, and $|\theta_1 \cdot \theta_2| \ll 1$ locally so that (θ_1, θ_2) are linearly independent on the bounded domain X of interest.

First Conclusions

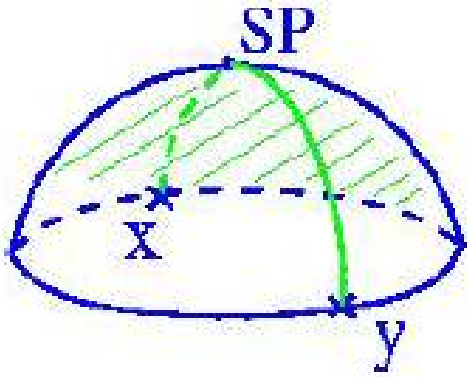
• Mathematically **hybrid imaging modalities** are stable inverse problems combining **high resolution** with **high contrast** (though they are Low Signal).

• They typically involve the analysis of redundant (overdetermined) systems of nonlinear partial differential equations, with optimal stability estimate are obtained for elliptic systems.

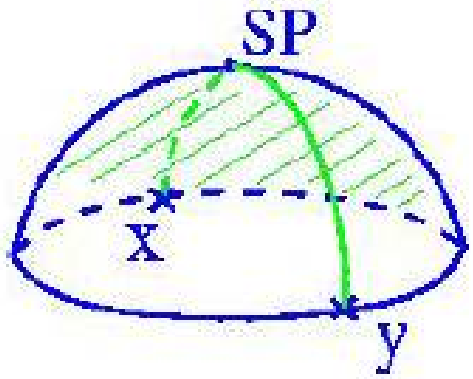
• Additional redundancy may provide algebraic/explicit reconstructions.

• **Tensors** and **Complex-valued** coefficients can be reconstructed to account for *anisotropy* and *dispersion* effects. The reconstruction of anisotropic coefficients may lead to non-elliptic, yet invertible, systems.

• Reconstructions require qualitative properties of PDE solutions (linear independence of gradients, Hessians). *Well-chosen* boundary conditions chosen by **CGO** constructions or local constructions with boundary control (by Runge approximation and UCP).



Can change metric near SP



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HAPPY BIRTHDAY GUNTHER !