On support theorems for the X-Ray transform with incomplete data

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Introduction
Weighted X-ray Transform

- $X \subset \mathbb{R}^n$ is open
- $Y \subset \mathbb{G}_n$ is an immersed real-analytic $n$-dimensional submanifold of the set of lines—*line complex*
- $Z = \{ (x, l) \in X \times Y \mid x \in l \}$—the incidence relation
- $\mu(x, l) \in C^\infty(Z)$ is a weight function
- $l(a, \xi) = \{ x = a + \xi t \}$ is a line parameterization
- $R_\mu f(l) = R_\mu f(a, \xi) = \int_{l(a, \xi)} f(x) \mu(x, l(a, \xi)) \, dt$
Boman-Quinto support theorems [BQ]
Introduction
Boman-Quintosupport theorems [BQ]

Type I: Given a non-planar real analytic surface $W \subset \mathbb{R}^3$. $Y$ is the set of all lines $l$, tangent to $W$, such that $W$ has nonzero directional curvature along $l$ at point of tangency.

Type II: Given a nonsingular real analytic curve $\gamma \in \mathbb{R}^3$. $Y$ is the set of lines intersecting this curve non-tangentially.

Type III: Given a closed simple nonsingular real analytic curve of directions $\theta \subset S^2$. $Y$ is the set of lines with directions on $\theta$. 
Theorem 1. Let $Y$ be an open connected subset of type I complex defined by $W$. Assume that $Y$ is an embedded submanifold of the set of all lines. In case there is a plane $\mathcal{P}$ tangent to $W$ at non-discrete set of points, assume that no line in $Y$ is contained in $\mathcal{P}$.

Let $X$ be an open set in $\mathbb{R}^3$ disjoint from $W$ and let $\mu(x, l)$ be real analytic function on $Z$ that is never zero. Let $f \in \mathcal{E}'(X)$. If $R_\mu f|_Y = 0$ and some line in $Y$ is disjoint from $\text{supp } f$, then every line in $Y$ is disjoint from $\text{supp } f$. 
**Theorem 2.** Let $Y$ be an open connected subset of type II complex defined by $\gamma$. Assume that $Y$ is an embedded submanifold of the set of all lines. If $\gamma$ is a plane curve, assume that no line in $Y$ is contained in a plane containing $\gamma$. Let $X$ be an open set in $\mathbb{R}^3$ disjoint from $\gamma$ and let $\mu(x, l)$ be real analytic function on $Z$ that is never zero. Let $f \in \mathcal{E}'(X)$. If $R_\mu f|_Y = 0$ and some line in $Y$ is disjoint from $\text{supp } f$, then every line in $Y$ is disjoint from $\text{supp } f$. 
**Theorem 3.** Let $Y$ be an open connected subset of type II complex defined by $\theta$. Assume that $\theta$ is not a great circle of $\mathbb{S}^2$.

Let $X$ be an open set in $\mathbb{R}^3$ disjoint from $\gamma$ and let $\mu(x, l)$ be real analytic function on $Z$ that is never zero. Let $f \in \mathcal{E}'(X)$. If $R_\mu f|_Y = 0$ and some line in $Y$ is disjoint from $\text{supp } f$, then every line in $Y$ is disjoint from $\text{supp } f$. 
Theorem of Hörmander

Theorem 4. Let $X$ be an open subset of $\mathbb{R}^n$, $f \in \mathcal{D}'(x)$, and $x_0$ a boundary point of the support of $f$, and assume that there is a $C^2$ function $F$ such that $F(x_0) = 0$, $dF(x_0) \neq 0$, and $F(x) \leq 0$ on supp $f$. Then $(x_0, \pm dF(x_0)) \in WF_A(f)$. 
Double fibration

- \( N^*(Z) \subset T^*X \setminus 0 \times T^*Y \setminus 0 \)
- \( p_X : Z \to X \) has surjective differential (\( Y \) is a regular line complex)
Admissible complexes

- cone $C_x = \bigcup_{Y} p_Y(p_X^{-1}(x)) \subset X$
- for non-critical $x$ $C_x$ is two-dimensional
- $l \in Y$ is non-critical, if not all of its points are critical.
- complex of lines is admissible, if $\forall$ non-critical $x \in l C_x$ has the same tangent plane along $l$
**Proposition 5** (cf. [GU]). Let $Y$ be a regular real analytic admissible line complex. Let $l_0 \in Y$ and assume $f \in \mathcal{E}'(X)$ and $R_{\mu}f(l) = 0$ for all $l \in Y$ in a neighborhood of $l_0$. Let $x \in l_0 \cap X$ and let $\xi \in T^*_x(X)$ be conormal to $l_0$, but not conormal to the tangent plane to $C_x$ along $l_0$. Then $(x, \xi) \notin WF_A(f)$. 
Proof of the proposition

- Let $\Lambda_0 \subset \Lambda$ be a set of $(x, \xi, l, \eta)$ such that $\xi$ is not conormal to $C_x$ along $l$.
- $R_\mu$ as a Fourier integral operator with Lagrangian manifold $\Lambda$
- $\Lambda_0$ is a local canonical graph
- $R_\mu$ is analytic elliptic, when microlocally restricted to $\Lambda_0$
- $R_\mu f = 0$ near $l_0 \Rightarrow (x, \xi) \notin WFA(f)$ for $(x, \xi, l_0, \eta) \in \Lambda_0$
**Characteristic paths**

Let \( x_0 \in \mathbb{RP}^n \). *Characteristic path* with pivot point \( x_0 \) is the smooth path in \( p_Y(p_X^{-1}(x_0)) \).

**Proposition 6.** Let the hypotheses of theorem Type I (Type II, Type III) hold. Let \( f \in \mathcal{E}'(X) \) and assume \( R_\mu f = 0 \) on \( Y \). Let \( l(s) : [a, b] \to Y \) be a characteristic path and assume \( l(a) \) does not meet \( \text{supp} \, f \) and the pivot point of the path is disjoint from \( \text{supp} \, f \). Then

\[
l(s) \cap \text{supp} \, f = \emptyset \text{ for } a \leq s \leq b
\]
Proof of proposition 6

- Reduce to the case of pivot point at infinity
- Construct a “wedge neighbourhood” of \( l(s) \) in \( X \):
  - \( D(s, \tau), D(s, 0) = l(s), (\tau = (\tau_1, \tau_2), \|\tau\| \leq \varepsilon) \)
  - \( D(a, \tau) \cap \text{supp } f = \emptyset \)
  - no conormal \( \bar{\xi} \) to \( \partial D(\bar{s}) \) at \( \bar{x} \) is conormal to \( C_{\bar{x}} \) along \( l(\bar{s}) \ni \bar{x} \)
- Let \( \bar{s} = \sup \{ s_1 \in [a, b] \mid D(s) \cap \text{supp } f = \emptyset \text{ for } a \leq s \leq s_1 \} \)
- \( D(\bar{s}) \) meets \( \text{supp } f \) at some point \( \bar{x} \in \partial D(\bar{s}), \bar{\xi} \perp \partial D(\bar{s}) \)
- Proposition 11 implies that \( (\bar{x}, \bar{\xi}) \notin WF_A(f) \)
- Hörmander’s theorem implies that \( f = 0 \) near \( \bar{x} \)
- The only possibility is \( \bar{s} = b \). So, \( l(b) \cap \text{supp } f = \emptyset \)
Boman-Quinto support theorems—revised
Proposition—revised
Completeness condition

- Let $Y$ be a $n$-dimensional complex of lines
- $x(t) = \xi(u)t + \beta(u)$ be a local parameterization
- $y_0 \in Y$, $\omega \in \mathbb{R}^{n^*}$, $\omega \neq 0$, $\omega \perp y_0$

**Definition 7** (cf. [Pa]). Line $y_0$ satisfies a *weak completeness condition* for $\omega$ at $x_0 = x(t_0) \in y_0 = y(u_0)$, if a germ of the map $\Pi_\omega : Y \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}^n$, $\Pi_\omega : (u, t) \mapsto (\langle \omega, \dot{x} \rangle, x(t))$ is a diffeomorphism at $(u_0, t_0)$.
\(\omega\)-critical points

- \(Y\) is an \(n\)-dimensional line complex, \(y \in Y, \omega \perp y\)

**Definition 8.** Point \(x(t) \in y\) is \(\omega\)-critical, if the weak completeness condition is not held at \(h\)

- Line \(y\) is \(\omega\)-critical, if all its point are \(\omega\)-critical

- The set of conormals \(\omega \perp y\) for which \(y\) is \(\omega\)-critical is called the set of critical conormals, and is denoted by \(\Omega_y\)
**ω-critical lines**

Lemma 9. Let \( y_0 = y(u_0) \in Y, \omega \perp y_0 \). A point \( x = x(u_0, t_0) \) is \( \omega \)-critical \iff \( P_\omega(t_0) = 0 \), where polynomial

\[
P_\omega(t) = \left\langle \omega, \sum_{k=1}^{n} \frac{\partial \xi}{\partial u_k} P_k(t) \right\rangle.
\]

**Proof.**

\[
P_\omega(t) = \det \begin{vmatrix}
\left\langle \omega, \frac{\partial \xi}{\partial u^1} \right\rangle & \left\langle \omega, \frac{\partial \xi}{\partial u^2} \right\rangle & \ldots & \left\langle \omega, \frac{\partial \xi}{\partial u^n} \right\rangle & 0 \\
\frac{\partial \xi^1}{\partial u^1} t + \frac{\partial \beta^1}{\partial u^1} & \frac{\partial \xi^1}{\partial u^2} t + \frac{\partial \beta^1}{\partial u^2} & \ldots & \frac{\partial \xi^1}{\partial u^n} t + \frac{\partial \beta^1}{\partial u^n} & \xi^1 \\
\frac{\partial \xi^2}{\partial u^1} t + \frac{\partial \beta^2}{\partial u^1} & \frac{\partial \xi^2}{\partial u^2} t + \frac{\partial \beta^2}{\partial u^2} & \ldots & \frac{\partial \xi^2}{\partial u^n} t + \frac{\partial \beta^2}{\partial u^n} & \xi^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{\partial \xi^n}{\partial u^1} t + \frac{\partial \beta^n}{\partial u^1} & \frac{\partial \xi^n}{\partial u^2} t + \frac{\partial \beta^n}{\partial u^2} & \ldots & \frac{\partial \xi^n}{\partial u^n} t + \frac{\partial \beta^n}{\partial u^n} & \xi^n
\end{vmatrix}
\]
Theorem 10. Let $Y$ be an $n$-dimensional line complex in $\mathbb{R}^n$. The following properties are equivalent:

1. $Y$ is admissible

2. For all non-critical line $y \in K$, for all $\omega \in \mathbb{R}^{n^*}$, $\omega \perp y$, either $y$ is $\omega$-critical, or all its $\omega$-critical points are critical.

3. For all non-critical $y \in Y$ $\dim \Omega_y = n - 2$. 

Admissible complexes and critical normals
Proof of the theorem 10

- **Tangent plane to** $C_{x_0}$ **is spanned on vectors** $\xi$ **and**
  \[
  \sum_{k=1}^{n} \left( \frac{\partial \xi}{\partial u^k} t + \frac{\partial \beta}{\partial u^k} \right) P_k(t_0) = \\
  \left( \sum_{k=1}^{n} \frac{\partial \xi}{\partial u^k} P_k(t) \right) (t - t_0) - P_0(t) \xi,
  \]

- **So,** $\Omega_y = \left\{ \omega \in \mathbb{R}^{n*} \mid \langle \omega, \xi \rangle = 0, P_\omega(t) \equiv 0 \right\} = \\
  \bigcap_{t \in \mathbb{R}} \left\{ \omega \in \mathbb{R}^{n*} \mid \omega \perp \xi, \omega \perp \sum_{k=1}^{n} \frac{\partial \xi}{\partial u^k} P_k(t) \right\} = \\
  \bigcap_{t \in \mathbb{R} \setminus \text{Crit}_y} \left\{ \omega \in \mathbb{R}^{n*} \mid \omega \perp TC_{x(t)} \right\} = \\
  \bigcap_{t \in \mathbb{R} \setminus \text{Crit}_y} (TC_{x(t)})^\perp = \left( \bigcup_{t \in \mathbb{R} \setminus \text{Crit}_y} TC_{x(t)} \right)^\perp
Proposition 11. Let $Y$ be a real analytic $n$-dimensional line complex in $\mathbb{R}^n$. Let $l_0 \in Y$ and assume $f \in \mathcal{E}'(X)$ and $R_{\mu}f(l) = 0$ for all $l \in Y$ in a neighborhood of $l_0$. Let $x_0 \in l_0 \cap X$ and let $\xi \in T^*_{x_0}(X)$ be conormal to $l_0$, and such that $x_0$ is not $\xi$-critical point for $l_0$. Then $(x_0, \xi) \notin WF_A(f)$. 
Proof of the revised proposition

- Let $\Lambda_0 \subset \Lambda$ be a set of $(x, \xi, l, \eta)$ such that $x$ is not $\xi$-critical for $l$.
- $R_\mu$ as a Fourier integral operator with Lagrangian manifold $\Lambda$
- $\Lambda_0$ is a local canonical graph
- $R_\mu$ is analytic elliptic, when microlocally restricted to $\Lambda_0$
- $R_\mu f = 0$ near $l_0 \Rightarrow (x, \xi) \notin WF_A(f)$ for $(x, \xi, l_0, \eta) \in \Lambda_0$
A common way to prove a support theorem

Principle 12. Let $Y$ be an $n$-dimensional complex of lines in $\mathbb{R}^n$. Let $f \in \mathcal{E}'(X)$ and assume $R_{\mu}f = 0$ on $Y$. Let $l(s) : [a, b] \to Y$ be a path and assume $l(a)$ does not meet $\text{supp } f$. Suppose that there exists a “wedge neighbourhood” of $l(s)$ in $X$, such that

1. $D(s, \tau), D(s, 0) = l(s)$
2. $D(a, \tau) \cap \text{supp } f = \emptyset$
3. for no conormal $\bar{\xi}$ to $\partial D(\bar{s})$ at $\bar{x}$, line $l(\bar{s}) \ni \bar{x}$ is $\bar{\xi}$-critical

Then

$$l(s) \cap \text{supp } f = \emptyset \text{ for } a \leq s \leq b$$
Admissible complexes—characteristic paths

- \( \Omega_y = \left( \bigcup_{t \in \mathbb{R} \setminus \text{Crit}_y} TC_{x(t)} \right)^\perp \)
- \( \dim \Omega_y = n - 2 \)
- for each critical point \( x_j \in y \), \( \dim T_yY \cap T_yG_{x_j} = 1 + r_j \), where \( r_j \) is the multiplicity of \( x_j \), \( \sum r_j = n - 2 \)
Non-admissible complex

\[ x = \begin{pmatrix} u_3 t + u_1 \\ u_1 t + u_2 \\ t \end{pmatrix} \]

\[ P_\omega(t) = \omega_1 - t\omega_2 \]

- you can choose \( \varepsilon(s) \) such that it will be internal point of \( D(s, \tau) \).
Generalizations

- hyperbolic complexes of lines
- complex $(\mathbb{C})$ critical points
By a complex of lines we understand a submanifold $K \subset Y$, $\dim K \geq n - 1$

**Definition 13.** Let $y \in K \subset Y$ and $\dim K = n - 1 + r$. A point $x \in y$ is a *critical* for the complex $K$ if

$$R(x) = \dim (T_yK \cap T_yY_x) > r.$$ A number $k(x) = R(x) - r$ is called the *multiplicity* of $x$. 

**Complexes of lines and critical points**
Admissible complexes—revised [De]
Complexes of curves

- Let a smooth manifold $X$, $\dim X = n$ be given.
- Let $Y$ be a family of curves on $X$:
  - $\forall L \subset T_x X$, $\dim L = 1$ there is exactly one curve $y \in Y$, such that $x \in y$ and $T_x y = L$.
  - then $\dim Y = 2n - 2$.
- Assume that $\pi_Y : N^* Z \rightarrow T^* Y \setminus 0$ is bijective immersion.
- $\Sigma = \text{Im} \pi_Y \subset T^* Y \setminus 0$ is the characteristic surface.
- Covector $\xi \in T^*_y Y$ is called characteristic, if $(y, \xi) \in \Sigma$.
- Let $Y_x = \{ y \in Y \mid y \ni x \} \subset Y$.
- $\Sigma = \bigcup_{x \in y} N^*_y Y_x$. 

Introduction

Boman-Quinto support theorems [BQ]

Boman-Quinto support theorems—revised

Bibliography
Complexes of curves and critical points

- By a complex of curves we understand a submanifold $K \subset Y$, $\dim K \geq n - 1$

**Definition 14.** Let $y \in K \subset Y$ and $\dim K = n - 1 + r$. A point $x \in y$ is a critical for the complex $K$ if $R(x) = \dim(T_yK \cap T_yY_x) > r$. A number $k(x) = R(x) - r$ is called the multiplicity of $x$
Critical points and characteristic covectors

Lemma 15 (cf. [Gu]). There is a critical point $x \in y \in K$ of multiplicity $k(x) = k$ if and only if there is subspace $L_x \subset N^*yK$ with $\dim L_x = k$ that consists of characteristic covectors.

Proof.

- $\dim K = n - 1 + r$, $\dim (T_yK \cap T_yY_x) = r + k$
- $(T_yK \cap T_yY_x)^\perp = (T_yK)^\perp \cup (T_yY_x)^\perp$
- $2n - 2 - r - k = (n - 1 - r) + (n - 1) - \dim (N^*yK \cap N^*Y_x)$
Definition 16. A complex of curves $K$ is hyperbolic, if $\forall y \in K$ there exist critical points $x_1, \ldots, x_s \in y$ such that

$$N_y^* K = L_{x_1} \oplus \cdots \oplus L_{x_s},$$

where $L_{x_j}$ is the characteristic subspace corresponding to $x_j$.

Definition 17. A complex of curves $K$ is characteristic,

$$N^* K \subset \Sigma.$$

Lemma 18. $K$ is characteristic $\iff \forall y \in K$ there exists critical point $x \in y$ of multiplicity $\text{codim } K$

Remark. The notion of hyperbolic complex of curves differs from the notion of admissible complex of curves.

For $\mathbb{C}$-complexes of lines notions coincide.
**Regular non-splitting critical points**

**Definition 19.** A critical point $x \in y$ is *non-singular*, if there is a neighborhood $W \subset K$ of a curve $y$ such that $\forall \xi \in L_x \quad T_\xi(N^*W \cap \Sigma) = T_\xi(N^*W) \cap T_\xi(\Sigma)$

**Definition 20.** Let $K$ be a hyperbolic complex, $y_0 \in K$. We say that $y_0$ has *non-splitting* critical points if there is a neighborhood $W \subset K$ of a curve $y_0$ such that for $y \in W$ there are $s$ non-singular critical points $x_1, \ldots, x_s \in y$ smoothly dependent in $y$ with constant multiplicities $k_1, \ldots, k_s$ ($\sum k_i = \text{codim } K$) for which $N^*_y(W) = L_{x_1} \oplus \cdots \oplus L_{x_s}$
Theorem 21. Suppose that $K$ is a hyperbolic complex, $y_0 \in K$ is a curve with non-splitting critical points. Then there exists $s$ characteristic complexes $W_j$, such that $W = \cap W_j$ and $\forall y \in W$ the critical point $x_j \in y$ will be critical for exactly one complex $W_j$ with the same multiplicity.

Conversely, suppose that $W = \cap W_j$, where the $W_j$ are hyperbolic complexes, and $\dim W \geq n - 1$. Then $W$ is hyperbolic, and any $y \in W$ will have as critical points all the critical points of all the $W_j$ with corresponding multiplicity.
Proof of the theorem 21

- \( N^*W \cap \Sigma = \cap V_j \), where \( V_j \) is the bundle of characteristic covectors corresponding to the critical point \( x_j = x_j(y) \)
  - \( \dim V_j = 2n - 2 + k_j - k \)
  - \( V_j \) is isotropic submanifold in \( T^*Y \setminus 0 \)

- \( \Sigma \) is an involutary submanifold of \( T^*Y \setminus 0 \), \( \text{codim } \Sigma = n - 2 \)
  - Ideal \( J \) of functions vanishing on \( \Sigma \) corresponds to the Lie algebra \( \mathcal{V} \) of vector fields tangent to \( \Sigma \):
    - \( J \ni f \mapsto \text{sgrad } f \in \mathcal{V} \) (\( \text{sgrad } f \vee \omega = -df \))

- Act on \( V_j \) by the Hamiltonian flow corresponding to \( \mathcal{V} \)
- We obtain a Lagrange manifold \( \overline{W}_j = N^*W_j \)
Local structure of characteristic complexes

**Theorem 22.** Let $K$ be a characteristic complex, codim $K = k$. Then in a neighborhood of non-singular curve $K$ consists

1. for $k > 1$ of curves intersecting given submanifold $M \subset X$, codim $M = k + 1$
2. for $k = 1$ of either curves intersecting a given submanifold $M \subset X$, codim $M = 2$, or curves tangent to a given submanifold $M \subset X$, codim $M = 1$

**Proof.** Compute a rank of $\varphi : y \mapsto x(y)$, where $x(y)$ is a critical point of $y$. $M = \text{Im} \varphi$. $\square$
Theorem 23 (cf. [Ma]). Let $K$ be a hyperbolic complex, codim $K = k$, and $y_0 \in K$ be a curve with non-splitting critical points. Then the critical points $x_j(y)$ circumscribe $s$ manifolds $M_j \subset X$. Moreover, if $k_j > 1$ then codim $M_j = k_j + 1$ and curves in $W$ intersects $M_j$ transversally; if $k_j = 1$, then either codim $M_j = 2$ and curves in $W$ intersect $M_j$ transversally, or codim $Y = 1$ and curves in $W$ are tangent to $M_j$. 
**Theorem 24** (cf. [Ma]). Conversely, let $s$ submanifolds $M_j \subset X$ be given and let $k_j = \max \{ 1, \text{codim } M_j - 1 \}$; if $\sum k_j = k$ and the set of curves intersecting the submanifolds of codimension $k_j + 1$ and tangent to submanifolds of codimension 1 forms a submanifold in $Y$ of codimension $k$, then this is a hyperbolic complex with critical points of multiplicities $k_j$ lying on the $M_j$. 


