Can There Be a General Theory of Fourier Integral Operators?

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Conference on Inverse Problems

in Honor of Gunther Uhlmann

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How I started working with Gunther

On L^2-estimates for singular Radon transforms

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What is (should be) a 'theory of FIOs' ?

Subject: oscillatory integral operators
$\hookrightarrow$ **phase functions and amplitudes**

- **A symbol calculus**
- **Composition of operators and parametrices**
- **Estimates:** L^2 **Sobolev ...**
- **Examples and applications**

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- **Examples and applications**

Standard Fourier Integral Operator Theory

• **Fourier integral (Lagrangian) distributions and symbol calculus**

• **FIOs: ops whose Schwartz kernels are Lagrangian distributions**

• **Composition: transverse intersection calculus (Hörmander) and clean intersection calculus (Duistermaat-Guillemin; Weinstein)**

• **Paired Lagrangian distributions and operators:**

$$\subseteq 2^{\{\textbf{Guillemin, Melrose, Mendoza, Uhlmann}\}}$$

$\hookrightarrow$ **Parametrices for real- and complex-principal type operators, some ops. with involutive multiple characteristics**

$\hookrightarrow$ **Conical refraction: FIOs with conical singularities**

Compositions Outside Transverse/Clean Intersection

- **Inverse problems $\Longrightarrow$ Focus on normal operators A^*A**
- **A degenerate $\Longrightarrow$ A^*A not covered by transverse/clean calculus**
- **Typically, A^*A propagates singularities: $WF(A^*A) \not\subset \Delta$**

$\hookrightarrow$ **A^*A has a non-ΨDO component**

$\hookrightarrow$ **Imaging artifacts**

Problem. **Describe A^*A: microlocal location and strength of artifacts, embed in an operator class to allow possible removal**

- **Emphasis on generic geometries**

$\hookrightarrow$ **Express conditions in language of C^∞ singularity theory**

Crash Course on FIOs

Fourier integral distributions:

Manifold X^n**,** T^*X**,** $\Lambda \subset T^*X \setminus 0$ **smooth, conic Lagrangian, order** $m \in \mathbb{R}$

$I^m(X;\Lambda) = I^m(\Lambda) = m^{\textbf{th}}$ **order Fourier integral distributions** $\subset \mathcal{D}'(X)$

Local representations:

$$u(x) = \int_{\mathbb{R}^N} e^{i\phi(x,\theta)} a(x,\theta)\, d\theta, \quad a \in S^{m-\frac{N}{2}+\frac{n}{4}}$$

with $\left\{ d_{x,\theta}\left(\frac{\partial \phi}{\partial \theta_j}\right) \right\}_{j=1}^{N}$ **linearly indep. on** $\left\{ d_\theta \phi(x,\theta) = 0 \right\}$

Fourier integral operators:

$X \times Y$, $C \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$ **a canonical relation**

$$I^m(C) = I^m(X, Y; C) = \{A : \mathcal{D}(Y) \longrightarrow \mathcal{E}(Y) \,|\, K_A \in I^m(X \times Y; C')\}$$

- **Inherits symbol calculus from** $I^m(C')$
- $X = Y$, $C = \Delta_{T^*Y} \implies I^m(\Delta_{T^*Y}) = \Psi^m(Y)$
- **Compositions. Transverse/clean intersection calculus: if**

$$(C_1 \times C_2) \cap (T^*X \times \Delta_{T^*Y} \times T^*Z) \text{ cleanly with excess } e \in \mathbb{Z}_+$$

then $C_1 \circ C_2 \subset T^*X \times T^*Z$ **is a smooth canonical relation and**

$$A \in I^{m_1}(X, Y; C_1),\ B \in I^{m_2}(Y, Z; C_2) \implies AB \in I^{m_1+m_2+\frac{e}{2}}(X, Z; C_1 \circ C_2)$$

Nondegenerate FIOs

Suppose $dim\, X = n_X \geq dim\, Y = n_Y, \quad C \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$

$$\begin{array}{ccccc} & & C & \hookrightarrow & T^*X \times T^*Y \\ & {}^{\pi_L}\swarrow & & \searrow^{\pi_R} & \\ T^*X & & & & T^*Y \end{array}$$

Projections: $\pi_L : C \longrightarrow T^*X,\ \pi_R : C \longrightarrow T^*Y$

Note: $dim\, T^*Y = 2n_Y \leq dim\, C = n_X + n_Y \leq dim\, T^*X = 2n_X$

Def. Say that C **is a nondegenerate canonical relation if**

(*) π_R **a submersion** $\iff$ π_L **an immersion**

C **nondegenerate** $\implies$ $C^t \circ C$ **covered by clean intersection calculus, with excess** $e = n_X - n_Y$

If strengthen (*) to

(**) π_L is an injective immersion,

then $C^t \circ C \subset \Delta_{T^*Y}$ **and**

$$A \in I^{m_1 - \frac{e}{4}}(C),\, B \in I^{m_2 - \frac{e}{4}}(C) \implies A^* B \in I^{m_1 + m_2}(\Delta_{T^*Y}) = \Psi^{m_1+m_2}(Y)$$

• **Integral geometry:** For a generalized Radon transform $R : \mathcal{D}(Y) \longrightarrow \mathcal{E}(X)$, (**) is the **Bolker condition** of Guillemin,

$$R^*R \in \Psi(Y) \implies \text{parametrices and local injectivity}$$

• **Seismology:** For the linearized scattering map F, under various acquisition geometries, (**) is the **traveltime injectivity condition** (Beylkin, Rakesh, ten Kroode-Smit-Verdel, Nolan-Symes),

$$F^*F \in \Psi(Y) \implies \text{singularities of sound speed}$$

are determined by singularities of pressure measurements

Q: What happens if Bolker/T.I.C. are violated?

A: **Artifacts**

Problem. (1) Describe **structure** and **strength** of the artifacts

(2) Remove (if possible)

Q. A general theory of FIOs?

In general, if $C \subset T^*X \times T^*Y$, $A \in I^{m_1}(C)$, $B \in I^{m_2}(C)$, then

$$WF(K_{A^*B}) \subseteq C^t \circ C \subset T^*Y \times T^*Y$$

is some kind of Lagrangian variety, containing points in Δ_{T^*Y}, but other points as well.

A general theory of FIOs would have to:

(1) describe such Lagrangian varieties,

(2) associate classes of Fourier integral-like distributions,

(3) describe the composition of operators whose Schwartz kernels are such, and

(4) give L^2 Sobolev estimates for these.

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A. For arbitrary C, fairly hopeless, but can begin to see some structure by looking at FIOs arising in applications with least degenerate geometries (given dimensional restrictions).

Restricted X-ray Transforms

Full X-ray transf. In $\mathbb{R}^n$: $\mathcal{G} = (2n-2)$-dim Grassmannian of lines.
More generally, on (M^n, g): $\mathcal{G} = S^*M/H_g$ local space of geodesics

$$Rf(\gamma) = \int_\gamma f\,ds$$

$R \in I^{-\frac{1}{2}-\frac{n-2}{4}}(C)$ **with** $C \subset T^*\mathcal{G} \times T^*M$ **nondeg.** $\implies R^*R \in \Psi^{-1}(M)$

Restricted X-ray transf. $K^n \subset \mathcal{G}$ **a line/geodesic complex**

$\hookrightarrow R_K f = Rf|_K$**,** $\quad R_K \in I^{-\frac{1}{2}}(C_K), \quad C_K \subset T^*K \times T^*M$

Gelfand's problem: For which K does $R_K f$ determine f?

G. - Uhlmann: K **well-curved** $\Longrightarrow \pi_R : C_K \longrightarrow T^*M$ **is a fold**

Gelfand cone condition $\Longrightarrow \pi_L : C_K \longrightarrow T^*\mathcal{G}$ **is a blow-down**

Form general class of canonical relations $C \subset T^*X \times T^*Y$ **with this blowdown-fold structure, cf. Guillemin; Melrose.**

$C^t \circ C$ **not covered by clean intersection calculus**

Theorem. **(i)** $C^t \circ C \subset \Delta_{T^*Y} \cup \widetilde{C}$**, with** $\widetilde{C}$ **the (smooth) flowout generated by the image in** T^*Y **of the fold points of** C**. Furthermore,** $\Delta \cap \widetilde{C}$ **cleanly in codimension 1.**

(ii) $A \in I^{m_1}(C)$, $B \in I^{m_2}(C) \quad \Longrightarrow \quad A^*B \in I^{m_1+m_2,0}(\Delta, \widetilde{C})$ **(paired Lagrangian class of Melrose-Uhlmann-Guillemin)**

A union of two cleanly intersecting canonical relations, such as $\Delta \cup \widetilde{C}$**, should be thought of as a Lagrangian variety.**

Inverse problem of exploration seismology

- **Earth** $= Y = \mathbb{R}^3_+ = \{y_3 > 0\}$**,** $c(y)$ $=$ **unknown sound speed**

$$\hookrightarrow \quad \Box_c = \frac{1}{c(y)^2}\partial_t^2 - \Delta_y \textbf{ on } Y \times \mathbb{R}$$

Problem: **Determine** $c(y)$ **from seismic experiments**

- **Fix source** $s \in \partial Y \sim \mathbb{R}^2$ **and solve**

$$\Box_c p(y,t) = \delta(y-s)\delta(t), \quad p \equiv 0 \textbf{ for } t < 0$$

- **Record pressure (solution) at receivers** $r \in \partial Y, \quad 0 < t < T$

Seismic data sets

- $\Sigma_{r,s} \subset \partial Y \times \partial Y$ **source-receiver manifold**
  $\hookrightarrow$ **data set** $X = \Sigma_{r,s} \times (0,T)$
- **Single source geometry:** $\Sigma_{r,s} = \left\{(r,s)|s = s^0\right\} \longrightarrow \mathit{dim}\, X = 3$
- **Full data geometry :** $\Sigma_{r,s} = \partial Y \times \partial Y \longrightarrow \mathit{dim}\, X = 5$
- **Marine geometry: A ship with an airgun trails a line of hydrophones, makes repeated passes along parallel lines.**

$$\Sigma_{r,s} = \{(r,s) \in \partial Y \times \partial Y \,|\, r_2 = s_2\} \quad \hookrightarrow \quad \mathit{dim}\, X = 4$$

Problem: For any of these data sets , determine $c(y)$ **from** $p|_X$

Linearized Problem

- **Assume** $c(y) = c_0(y) + (\delta c)(y)$, **background** c_0 **smooth and known**
- δc **small, singular, unknown** $\hookrightarrow \quad p \sim p_0 + \delta p$

  where $p_0 =$ **Green's function for** $\square_{c_0}$

Goal: (1) Determine δc **from** $\delta p|_X$**, or at least**

(2) Singularities of δc **from singularities of** $\delta p|_X$

High frequency linearized seismic inversion

Microlocal analysis

δp **induced by** δc **satisfies**

$$\Box_{c_0}(\delta p) = \frac{2}{(c_0)^3} \cdot \frac{\partial^2 p_0}{\partial t^2} \cdot \delta c, \quad \delta p \equiv 0, \quad t < 0,$$

Linearized scattering operator $F : \delta c \longrightarrow \delta p|_X$

- **For single source, no caustics for background** $c_0(y) \implies$

  $F \in I^1(C)$, C **a local canonical graph**, $F^*F \in \Psi^2(Y)$ **(Beylkin)**

- **Mild assumptions** $\implies F$ **is an FIO (Rakesh)**

Traveltime Injectivity Condition $\implies F \in I^m(C)$, C **nondeg.**
$\implies F^*F \in \Psi(Y)$ **(ten Kroode - Smit -Verdel; Nolan - Symes)**

- **TIC can be weakened to just:** π_L **an immersion, and then**

$$F^*F = \Psi\text{DO} + \text{ smoother FIOs (Stolk)}$$

But: TIC unrealistic - need to deal with caustics.

• Low velocity lens $\Longrightarrow F^*F$ doesn't satisfy expected estimates and can't be a ΨDO (Nolan–Symes)

• **Problem.** Study F for different data sets and for backgrounds with generic and nonremovable caustics (conjugate points, multipathing): folds, cusps, swallowtails, ...

(1) What is the structure of C?

(2) What can one say about F^*F? Where are the artifacts and how strong are they?

(3) Can F^*F be embedded in a calculus?

(4) Can the artifacts be removed?

Caustics of fold type

• **Single source data set in presence of (only) fold caustics for** c_0 $\Longrightarrow$ C **is a two-sided fold:** $\pi_L, \pi_R \in S_{1,0}$ **(Nolan)**

• **General class of such** $C's$ **studied by Melrose-Taylor; noted that**

$$C^t \circ C \nsubseteq \Delta_{T^*Y}$$

• **In fact,** $C^t \circ C \subseteq \Delta_{T^*Y} \cup \widetilde{C}$ **where** $\widetilde{C} \subset T^*Y \times T^*Y$ **is another two-sided fold, intersecting** Δ **cleanly at the fold points (Nolan).**

• **Thm. (Nolan; Felea) If** $C \subset T^*X \times T^*Y$ **is a two-sided fold,** $A \in I^{m_1}(C), \quad B \in I^{m_2}(C)$**, then**

$$A^*B \in I^{m_1+m_2,0}(\Delta_{T^*Y}, \widetilde{C}).$$

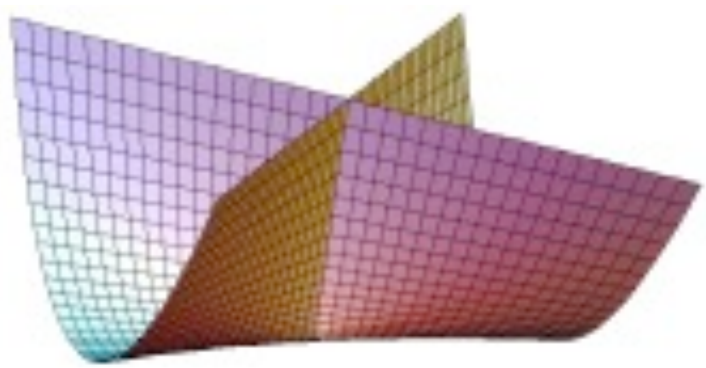
• **For 3D linearized single source seismic problem, the presence of fold caustics thus results in strong, nonremovable artifacts:**

$$F^*F \in I^{2,0}(\Delta, \widetilde{C}) \hookrightarrow I^2(\Delta \setminus \widetilde{C}) + I^2(\widetilde{C} \setminus \Delta)$$

Caustics of fold type - Marine data set

(Felea-G.) Now use 4-dim. marine data set X, and suitable interpretation of fold caustics. Then:

- **For $C \subset T^*X \times T^*Y$, $\pi_R : C \longrightarrow T^*Y$ is a submersion with folds and $\pi_L : C \longrightarrow T^*X$ is a cross-cap (or Whitney/Cayley umbrella)**



• **Define a general class of folded cross-cap canonical relations** $C^{2n+1} \subset T^*X^{n+1} \times T^*Y^n$

• **For these** $C^t \circ C \subseteq \Delta_{T^*Y} \cup \widetilde{C}$ **where** $\widetilde{C} \subset T^*Y \times T^*Y$ **is another two-sided fold, intersecting** Δ **cleanly at the fold points.**

• **If** $A \in I^{m_1}(C)$, $B \in I^{m_2}(C)$, **then** $A^*B \in I^{m_1+m_2-\frac{1}{2},\frac{1}{2}}(\Delta_{T^*Y}, \widetilde{C})$

• **N.B. Need to establish, work with a weak normal form for** C**.**

• **For the seismology problem,**

$$F^*F \in I^{\frac{3}{2},\frac{1}{2}}(\Delta, \widetilde{C}) \hookrightarrow I^2(\Delta \setminus \widetilde{C}) + I^{\frac{3}{2}}(\widetilde{C} \setminus \Delta)$$

• **The artifact is formally 1/2 order smoother, but actually removing it seems to be very challenging!**

Problem. **Develop an effective functional calculus for** $I^{p,l}(\Delta, \widetilde{C})$**.**

Estimates. **Model operators on $\mathbb{R}^2 \leftrightarrow$ translations of cubic (t, t^3)**

$$\phi(x, y; \xi; \eta) = (x_1 - y_1)\eta + (x_2 - y_2 - (x_1 - y_1)^3)\xi$$

is a multiphase parametrizing $(\widetilde{C}_0, \Delta)$ in the sense of Mendoza.

$T \in I^{p,l}(\Delta, \widetilde{C}_0) = I^{p+l,-l}(\widetilde{C}_0, \Delta)$ **can be written**

$$Tf(x) = \int e^{[(x_1-y_1)\eta + (x_2-y_2-(x_1-y_1)^3)\xi]} a(x, y; \xi; \eta) f(y)\, d\eta\, d\xi\, dy$$

where the amplitude is product-type, $a(x, y; \xi; \eta) \in S^{p+\frac{1}{2}, l-\frac{1}{2}}$**,**

$$\left|\partial_x^\gamma \partial_\eta^\beta \partial_\xi^\alpha a(x; \xi; \eta)\right| \lesssim \Big(1 + |\xi| + |\eta|\Big)^{p+\frac{1}{2}-|\alpha|} \Big(1 + |\eta|\Big)^{l-\frac{1}{2}-|\beta|}$$

Thm. (Felea-G.-Pramanik) If $T \in I^{p,l}(\Delta, \widetilde{C})$, $\widetilde{C} = \widetilde{C}_0,\ \widetilde{C}_{ss}$ or $\widetilde{C}_{mar}$, then $T : H^s \longrightarrow H^{s-r}$ for

$$\begin{aligned} r &= p + \frac{1}{6}, & l < -\frac{1}{2} \\ &= p + 1/6 + \epsilon, & l = -\frac{1}{2}, \quad \forall \epsilon > 0 \\ &= p + (l+1)/3, & -\frac{1}{2} < l < \frac{1}{2} \\ &= p + l, & l \geq \frac{1}{2}. \end{aligned}$$

Idea of proof: Combine parabolic cutoff with Phong-Stein-Cuccagna decomposition. Pick $\frac{1}{3} \leq \delta \leq \frac{1}{2}$. Localize to $|\xi| \sim 2^j$, $|\eta| \sim 2^k$:

$$T = T_0 + \sum_{j=0}^{\infty} \sum_{k=\delta j}^{j} T_{jk} + T_\infty$$

where $T_0 \in I^{m_\delta}(C)$, $T_\infty \in I^{p+l}(\Delta)$ and T_{jk} can be shown to satisfy almost orthogonality. Optimize over δ.

Caustics of cusp type

(G.-Felea) Single source geometry, but now assume that rays from source form a cusp caustic in Y. The $F \in I^1(C)$ with C having the following structure.

Def. If X and Y are manifolds of dimension $n \geq 3$, then a canonical relation $C \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$ is a flat two-sided cusp if

(i) both $\pi_L : C \longrightarrow T^*X$ and $\pi_R : C \longrightarrow T^*Y$ have at most cusp singularities;

(ii) the left- and right-cusp points are equal:

$$\Sigma_{1,1}(\pi_L) = \Sigma_{1,1}(\pi_R) := \Sigma_{1,1}; \text{ and}$$

(iii) $\pi_L(\Sigma_{1,1}) \subset T^*X$ and $\pi_R(\Sigma_{1,1}) \subset T^*Y$ are coisotropic (involutive) nonradial submanifolds.

Model operators. **Translations of cubic** (t, t^2, t^4) **in** $\mathbb{R}^3 \hookrightarrow$

$A \in I^m(C_{mod})$ **can be written**

$$Af(x) = \int_{\mathbb{R}^2} e^{i\phi_{mod}(x,y,\theta)} a(x,y,\theta) f(y) d\theta, \quad a \in S^0$$

$$\phi_{mod}(x,y,\theta) = \left(x_2 - y_2 - (x_1 - y_1)^2)\right) \theta_2 + \left(x_3 - y_3 - (x_1 - y_1)^4\right) \theta_3$$

For $A \in I^m(C_{mod})$**,**

$$K_{A^*A}(x,y) = \int_{\mathbb{R}^3} e^{i\widetilde{\phi}} a \, d\theta_2 \, d\theta_3 \, d\tau \text{ **with**}$$

$$\widetilde{\phi} = (x_2 - y_2 + \frac{\tau}{\theta_3}(x_1 - y_1))\theta_2 + (x_3 - y_3 + \frac{1}{2}(x_1 - y_1)(\frac{\tau}{\theta_3})^3 + \frac{1}{2}\frac{\tau}{\theta_3}(x_1 - y_1)^3)\theta_3$$

- $\widetilde{\phi}$ **is degenerate:**

- $Crit(\widetilde{\phi}) = \{d_{\theta_2,\theta_3,\tau}\widetilde{\phi} = 0\}$ **has normal crossings:**

$$\left\{d_\tau\widetilde{\phi} = 0\right\} = \left\{x_1 - y_1 = 0\right\} \cup \left\{\frac{\theta_2}{\theta_3} + \frac{3}{2}\frac{\tau^2}{\theta_3^2} + \frac{1}{2}(x_1 - y_1)^2 = 0\right\}$$

- **First surface** $\longrightarrow \Delta$, **but parametrized via a cusp map**

- **Second surface** $\longrightarrow \widetilde{C} =$ **an open umbrella**

  $=$ **simplest kind of singular Lagrangian**

Open umbrellas

- **Closed umbrella (Whitney-Cayley, crosscap)** $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$
  $f(x, y) = (x^2, y, xy) = (u, v, w)$ **with image** $\{w^2 = uv^2\}$

- **Immersion away from origin, rank**$(df(0)) = 1$
  Embedding off of $\{y = 0\}$**, where 2-1**

- **Lift to Lagrangian map** $g : \mathbb{R}^2 \longrightarrow (\mathbb{R}^4, \omega)$**,** $\omega = d\xi_1 \wedge dx_1 + d\xi_2 \wedge dx_2$

$$g(x, y) = (x^2, y; xy, \tfrac{2}{3}x^3) = (x_1, x_2; \xi_1, \xi_2)$$

- **Image is a smooth Lagrangian (** $g^*\omega = 0$**) away from the non-removable isolated singularity at origin (Givental)**

- **General** $\Lambda^n \subset (M^{2n}, \omega)$**, umbrella tip** Σ_1 **is codim 2**

Can put a general flat two-sided cusp into a weak normal form close to the model above:

Prop. **For any flat two-sided cusp** $C \subset T^*X \times T^*Y$ **, there exist canonical transformations on left and right so that** C **is microlocally parametrized by a phase function**

$$\phi(x, y, \theta) \; = \; (x_3 - y_3)\theta_3 + (x_1 - y_1)^4 S_3 + (S_2 - y_2 + (x_1 - y_1)^2 S_4)\theta_2,$$

$$\partial_{x_2} S_2|_{\Sigma_{1,1}} \neq 0, \qquad S_3 \neq 0.$$

Thm. **If** $C \subset T^*X \times T^*Y$ **is a flat two-sided cusp, then**

$$C^t \circ C \subset \Delta_{T^*Y} \cup \widetilde{C}$$

with $\widetilde{C}$ **an open umbrella. If** $A \in I^{m_1}(C)$, $B \in I^{m_2}(C)$, **then** A^*B **has an oscillatory representation with a phase function having normal crossings.**

Some problems

1. Describe classes of canonical relations C by demanding that π_L and π_R be Morin singularities of orders $l, r \in \mathbb{N}$, resp., plus appropriate additional conditions, such that $C^t \circ C \subset \Delta \cup \widetilde{C}$, where $\widetilde{C}$ is a union of higher order open umbrellas.

2. Associate classes of 'Fourier integral operators' to the Lagrangian varieties $\Delta \cup \widetilde{C}$, including a symbol calculus.

3. Prove estimates and establish some semblance of a functional calculus for these operators.

4. Apply these results to inverse problems!

Thank you, and

Happy Birthday, Gunther !