Inverse Scattering in Classical Mechanics

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I. Forward Problem

- Multidimensional relativistic Newton equation in a static external electromagnetic field [Einstein, 1907]

(1)
$$\dot{p} = -\nabla V(x) + \frac{1}{c} B(x)\dot{x},$$
$$p = \frac{\dot{x}}{\sqrt{1 - \frac{|\dot{x}|^2}{c^2}}},\ x(t) \in \mathbb{R}^n,\ n \geq 2.$$

- Smoothness and short-range assumptions for the external field

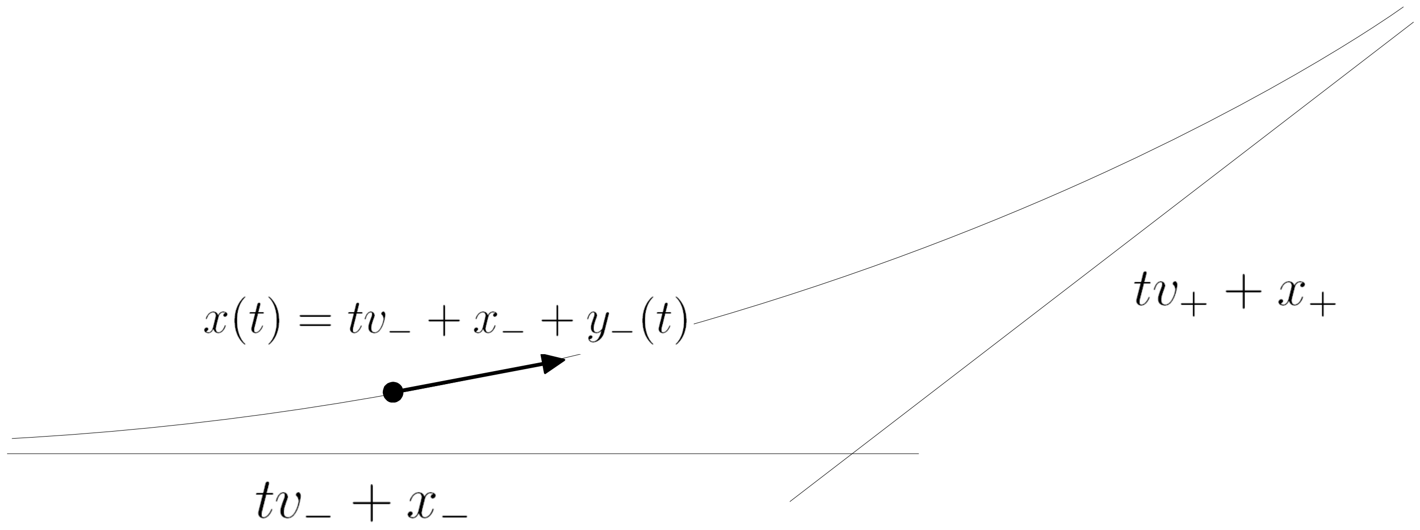
(2)
$$V \in C^2(\mathbb{R}^n, \mathbb{R}),\ B(x) = (B_{i,k}) \in C^1(\mathbb{R}^n, A_n(\mathbb{R})),$$
$$\frac{\partial B_{i,k}}{\partial x_l}(x) + \frac{\partial B_{l,i}}{\partial x_l}(x) + \frac{\partial B_{k,l}}{\partial x_i}(x) = 0,$$
$$|\partial_x^{j_1} V(x)| \leq \beta_{|j_1|}(1 + |x|)^{-(\alpha + |j_1|)},$$
$$|\partial_x^{j_2} B_{i,k}(x)| \leq \beta_{|j_2|+1}(1 + |x|)^{-\alpha - |j_2| - 1},$$

for $|j_1| \leq 2$, $|j_2| \leq 1$, $i, k, l = 1 \ldots n$ and for some $\alpha > 1$, where $j = (j^1, \ldots, j^n) \in (\mathbb{N} \cup \{0\})^n$, $|j| = \sum_{i=1}^n j^i$ and where $\beta_{|j|}$ are positive constants).

- Integral of motion, the energy of the classical relativistic particle

(3)
$$E = c^2\sqrt{1 + \frac{|p(t)|^2}{c^2}} + V(x(t))$$

• Existence of scattering states and asymptotic completeness [Yajima, 1982] :



• Scattering map and scattering data for equation (1) :

$$S(v_-, x_-) := (v_+, x_+) =: (v_- + a_{sc}(v_-, x_-), x_- + b_{sc}(v_-, x_-))$$

Remark : it is enough to know S on $\mathcal{D}(S) \cap \mathcal{M}$ where $\mathcal{M} := \{(v, x) \in B_c \times \mathbb{R}^n \mid v \cdot x = 0\}$.

• Direct problem : Given (V, B), find S.

Inverse problem : Given S, find (V, B).

II. Inverse scattering at high energies

• X-ray transform : $$Pf(\theta,x)=\int_{-\infty}^{+\infty} f(t\theta+x)dt,\ \ (\theta,x)\in T\mathbb{S}^{n-1}.$$

for $f\in C(\mathbb{R}^n,\mathbb{R}^m)$, $f(x)=O(|x|^{-1-\varepsilon})$ when $|x|\to+\infty$, $\varepsilon>0$,

and where $T\mathbb{S}^{n-1}:=\{(\theta',x')\in\mathbb{S}^{n-1}\times\mathbb{R}^n \mid \theta'\cdot x'=0\}$.

First study and inversion of P in $\mathbb{R}^2$: Radon (1917).
Application to X-ray Tomography : Cormack (1963).

II.1 Asymptotic of the scattering data

Theorem 1 [J1]. *Let* $(\theta,x)\in T\mathbb{S}^{n-1}$ *and* $0<r\le 1$, $r<\frac{c}{\sqrt{2}}$. *Under conditions* (2) *we have*

$$\lim_{\substack{s\to c\\ s<c}}\frac{s}{\sqrt{1-\frac{s^2}{c^2}}}a_{sc}(s\theta,x)=-P(\nabla V)(\theta,x)+\int_{-\infty}^{+\infty}B(x+\tau\theta)\theta d\tau,\qquad \textit{and}$$

$$\left|\frac{s}{\sqrt{1-\frac{s^2}{c^2}}}a_{sc}(s\theta,x)+P(\nabla V)(\theta,x)-\int_{-\infty}^{+\infty}B(x+\tau\theta)\theta d\tau\right|\le\frac{n^3 2^{2\alpha+7}c(1+\frac{1}{c})^2\tilde\beta^2(\frac{c}{\sqrt{2}}+1-r)^2}{\alpha(\alpha-1)(\frac{s_1}{\sqrt{2}}-r)^4(1+\frac{|x|}{\sqrt{2}})^{2\alpha-1}\sqrt{1+\frac{s^2}{4(c^2-s^2)}}}$$

for $s_1(c,n,\beta_1,\beta_2,\alpha,|x|,r)<s<c$, $(\tilde\beta=\max(\beta_1,\beta_2))$ *; In addition*

$$\lim_{\substack{s\to c\\ s<c}} \frac{s^2}{\sqrt{1-\frac{s^2}{c^2}}} b_{sc}(s\theta,x) = PV(\theta,x)\theta + \int_{-\infty}^{0}\int_{-\infty}^{\tau}(-\nabla V)(\sigma\theta+x)d\sigma d\tau$$

$$-\int_{0}^{+\infty}\int_{\tau}^{+\infty}(-\nabla V)(\sigma\theta+x)d\sigma d\tau + \int_{-\infty}^{0}\int_{-\infty}^{\tau} B(\sigma\theta+x)\theta d\sigma d\tau \quad -\int_{0}^{+\infty}\int_{\tau}^{+\infty} B(\sigma\theta+x)\theta d\sigma d\tau$$

Proposition 1 [J1]. *Under conditions (2) we have*

$$P(\nabla V)(\theta,x) = -\frac{1}{2}\left(\omega_1(V,B,\theta,x)+\omega_1(V,B,-\theta,x)\right).$$

for $(\theta,x)\in T\mathbb{S}^{n-1}$ *; in addition*

$$P(B_{i,k})(\theta,x) = \frac{\theta_k}{2}\left(\omega_1(V,B,\theta,x)_i - \omega_1(V,B,-\theta,x)_i\right)$$
$$-\frac{\theta_i}{2}\left(\omega_1(V,B,\theta,x)_k - \omega_1(V,B,-\theta,x)_k\right)$$

for $i,k=1\ldots n$ *and for every* $(\theta,x)\in T\mathbb{S}^{n-1}$, $\theta=(\theta_1,\ldots,\theta_n)$ *such that* $\theta_j=0$ *for* $j\neq i$ *and* $j\neq k$.

II.2 Idea of the proof

Theorem 1 was obtained by developing the method of R. Novikov (1999).
Equation (1) is rewritten in an integral equation and we have

$$(y_-, \dot{y}_-) = A_{v_-,x_-}(y_-, \dot{y}_-), \qquad \text{where} \quad A_{v_-,x_-} = (A^1_{v_-,x_-}, A^2_{v_-,x_-})$$

$$\begin{cases} A^1_{v_-,x_-}(f,h)(t) = \displaystyle\int_{-\infty}^{t} A^2_{v_-,x_-}(f,h)(\sigma)d\sigma, \\ A^2_{v_-,x_-}(f,h)(t) = g\left(g^{-1}(v_-) + \displaystyle\int_{-\infty}^{t} F(x_- + \sigma v_- + f(\sigma), v_- + h(\sigma))d\sigma\right) - v_-, \end{cases}$$

and where $g(z) := \frac{z}{\sqrt{1+\frac{z^2}{c^2}}}$ for $z \in \mathbb{R}^n$, $F(x,v) = -\nabla V(x) + \frac{1}{c}B(x)v$ for $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$.

We consider the operator A_{v_-,x_-} on the complete metric space

$$M_r := \{(f,h) \in C(\mathbb{R}, \mathbb{R}^n)^2 \mid \|(f,h)\|_\infty := \max\left(\sup_{t\in\mathbb{R}} |h(t)|, \sup_{t\in\mathbb{R}} |f(t) - th(t)|\right) \le r\},\ 0 < r < 1.$$

Hence we study a small angle scattering regime.

- Quantum analogs : Born, Faddeev (1956), Henkin-Novikov (1988), Enss-Weder (1995),
  H. Ito (1995), etc...

III Inverse scattering at fixed energy

III.1 Statement of the problem

For $E > c^2$, $\mathcal{D}(S_E) := \{(v_-, x_-) \in \mathcal{D}(S) \mid |v_-| = c\sqrt{1 - (\frac{c^2}{E})^2}\}$, $\quad S_E := S_{|\mathcal{D}(S_E)}$

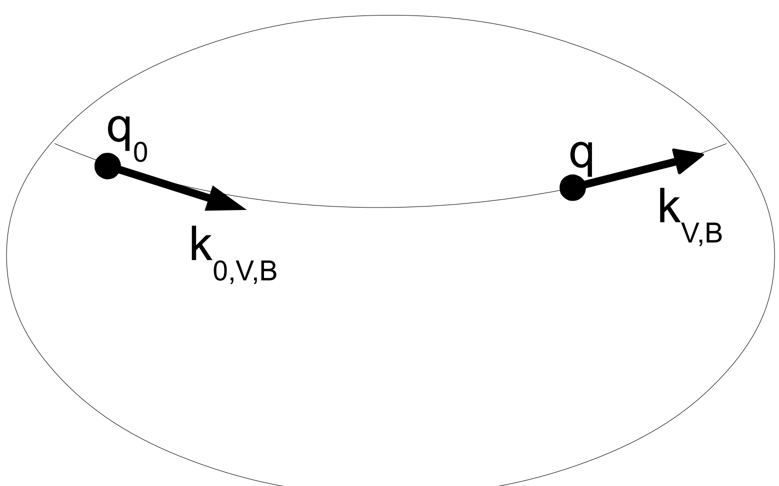
Given S_E at fixed energy $E > c^2$, find (V, B).

Remarks :
- if $(V_1, B_1) \not\equiv (V_2, B_2)$ then there exists an energy E such that $S_{V_1,B_1,E} \neq S_{V_2,B_2,E}$.
- if $E < c^2 + \sup_{x \in \mathbb{R}^n} V(x)$ then S_E does not determine uniquely (V, B).

III.2 An inverse boundary value problem

D strictly convex (in the strong sense) and bounded open subset of $\mathbb{R}^n$, $n \geq 2$, with a C^2 boundary

At $E > E(\|V\|_{C^2}, \|B\|_{C^1}, D)$



$$k_{0,V,B}(E, q_0, q) \in C^1((\bar{D} \times \bar{D}) \backslash \bar{G}, \mathbb{R}^n)$$

$$|k_{0,V,B}(E, q_0, q)| = c\sqrt{1 - \left(\frac{c^2}{E - V(q_0)}\right)^2}$$

Statement of the problem :

Given $k_{V,B}(E, q_0, q)$ (resp. $k_{0,V,B}(E, q_0, q)$), $(q_0, q) \in \partial D \times \partial D$, $q_0 \neq q$, find $\quad (V, B)$.

$$k_{0,V,-B}(E,q_0,q) = -k_{V,B}(E,q,q_0),\ \ s_{V,B}(E,q_0,q) = s_{V,-B}(E,q,q_0),\ \ \text{for } (q_0,q) \in \bar{D}^2,\ q_0 \neq q.$$

$$|k_{0,V,B}(E,q_0,q)| = c\sqrt{1 - \left(\frac{c^2}{E - V(q_0)}\right)^2}$$

Theorem 2 [J2] **.** *At fixed energy* $E > E(\|V\|_{C^2}, \|B\|_{C^1}, D)$*, the boundary data* $k_{V,B}(E,q_0,q)$ *(resp.* $k_{0,V,B}(E,q_0,q)$*),* $(q_0,q) \in \partial D \times \partial D$*,* $q_0 \neq q$*, uniquely determine* (V,B)*.*

Theorem 2 was obtained by developing the approach of Gerver-Nadirashvili (1983) and results of Muhometov-Romanov (1978), Beylkin (1979) and Bernstein-Gerver (1980).

• Boundary rigidity problem with magnetic field: Dairbekov-Paternain-Stefanov-Uhlmann (2007).

• Quantum analogs for the inverse boundary value problem : Novikov (1988), Nachman-Sylvester-Uhlmann (1988), Nakamura-Sun-Uhlmann (1995).

III.3 Idea of the proof

Time-independent Hamiltonian $H(P,x) = c^2\sqrt{1+\frac{\left|P-\frac{A(x)}{c}\right|^2}{c^2}} + V(x),\ P\in\mathbb{R}^n,\ x\in D,$

where A is a C^1 magnetic potential for B in $\bar{D}$.

$$P := \frac{\dot{x}}{\sqrt{1-\frac{|\dot{x}|^2}{c^2}}} + \frac{1}{c}A(x)$$

Reduced action S_0 **at fixed energy** E **:**

$$S_0(q_0,q) = \int_0^{s(E,q_0,q)} P(t,E,q_0,q)\cdot\dot{x}(t,E,q_0,q)dt.$$

Properties of the reduced action : $S_0 \in C(\bar{D}\times\bar{D},\mathbb{R}),\ S_0\in C^2((\bar{D}\times\bar{D})\backslash\bar{G},\mathbb{R}),$

$$\frac{\partial S_0}{\partial\zeta^i}(\zeta,x) = -\bar{k}_0^i(E,\zeta,x) - \frac{1}{c}A^i(\zeta), \qquad \frac{\partial S_0}{\partial x^i}(\zeta,x) = \bar{k}^i(E,\zeta,x) + \frac{1}{c}A^i(x),$$

$$\left|\frac{\partial^2 S_0}{\partial x^i\partial\zeta^i}(\zeta,x)\right| \le \frac{M}{|\zeta-x|},\quad \text{for } \zeta=(\zeta^1,\ldots,\zeta^n),\ x=(x^1,\ldots,x^n)\in\bar{D},\ \zeta\neq x, \text{ and } i,j=1\ldots n.$$

Remark : $B_{i,j}(x) = -c\left(\frac{\partial\bar{k}^j}{\partial x^i} - \frac{\partial\bar{k}^i}{\partial x^j}\right)(E,\zeta,x)$ for $(\zeta,x)\in\bar{D}^2,\ \zeta\neq x.$

$$\bar{k} = \frac{k}{\sqrt{1-\frac{k^2}{c^2}}}$$

Differential forms on $(\partial D \times \bar{D})\backslash\bar{G}$: $\beta_\mu(\zeta,x) = \sum_{j=1}^{n} \bar{k}_\mu^j(E,\zeta,x)dx^j,\ \mu = 1,2,$

$$\Phi_0(\zeta,x) = -(-1)^{\frac{n(n+1)}{2}}(\beta_2-\beta_1)(\zeta,x)\wedge d_\zeta(S_{0,1}-S_{0,2})(\zeta,x)\wedge \sum_{p+q=n-2}\left(dd_\zeta S_{0,1}\right)^p(\zeta,x)\wedge\left(dd_\zeta S_{0,2}\right)^q(\zeta,x),$$

$$\Phi_1(\zeta,x) = -(-1)^{\frac{n(n-1)}{2}}\Big(\beta_1(\zeta,x)\wedge(dd_\zeta S_{0,1})^{n-1}(\zeta,x) + \beta_2(\zeta,x)\wedge(dd_\zeta S_{0,2})^{n-1}(\zeta,x)$$

$$-\beta_1(\zeta,x)\wedge(dd_\zeta S_{0,2})^{n-1}(\zeta,x) - \beta_2(\zeta,x)\wedge(dd_\zeta S_{0,1})^{n-1}(\zeta,x)\Big),$$

We have $\int_{\partial D\times\partial D}\mathrm{incl}^*(\Phi_0) = \int_{\partial D\times\bar{D}}\Phi_1,$ where $\mathrm{incl} : (\partial D\times\partial D)\backslash\partial G \to (\partial D\times\bar{D})\backslash\bar{G}.$

Uniqueness and stability results

$$\frac{1}{(n-1)!}\Phi_1(\zeta,x) = r_1(x)^n\omega_1(\zeta,x)+r_2(x)^n\omega_2(\zeta,x)-(\bar{k}_1\cdot\bar{k}_2)(E,\zeta,x)\left(r_1(x)^{n-2}\omega_2(\zeta,x)+r_2(x)^{n-2}\omega_1(\zeta,x)\right),$$

$$r_\mu = c\sqrt{(\frac{E-V_\mu}{c^2})^2-1},\qquad \nu(\zeta,x) = -(\frac{k}{|k|})(E,\zeta,x),\ \ (\zeta,x)\in\partial D\times D.$$

$$\int_D (r_1-r_2)(r_1^{n-1}-r_2^{n-1})dx \le \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}(n-1)!}\int_{\partial D\times\partial D}\mathrm{incl}^*(\Phi_0).$$

III.4 Uniqueness results

Theorem 3. *Let $R > 0$ and $\lambda > 0$. There exists $E(\lambda, R) > 0$ such that for any $E > E(\lambda, R)$ and for any (V_i, B_i), $i = 1, 2$, satisfying condition (2) with* $\max(\beta_0, \beta_1, \beta_2) \leq \lambda$ *we have*

$$\left.\begin{array}{c}(V_1, B_1) \equiv (V_2, B_2) \ \text{on}\ \mathbb{R}^n \backslash B(0, R) \\ S_E^1 = S_E^2\end{array}\right\} \Rightarrow (V_1, B_1) \equiv (V_2, B_2),$$

where S_E^i is the scattering map at fixed energy E for (V_i, B_i), $i = 1, 2$.

Theorem 4. *Let $R > 0$ and $\lambda > 0$. There exists $E(\lambda, R) > 0$ such that for any $E > E(\lambda, R)$ and for any $(V_i, 0)$, $i = 1, 2$, satisfying condition (2) with* $\max(\beta_0, \beta_1, \beta_2) \leq \lambda$ *we have*

$$\left.\begin{array}{c}V_1 \ \text{and}\ V_2 \ \text{are spherical symmetric on}\ \mathbb{R}^n \backslash B(0, R) \\ S_E^1 = S_E^2\end{array}\right\} \Rightarrow V_1 \equiv V_2,$$

where S_E^i is the scattering map at fixed energy E for $(V_i, 0)$, $i = 1, 2$.

Remark : The geometry may not be simple.

III.5 Idea of the proof of Theorem 4 (for the nonrelativistic case)

$$E = \frac{\dot{r}^2}{2} + \frac{q^2}{2r^2} + V(r), \ r^2\dot{\theta} = q, \ \theta_q = \int_{-\infty}^{+\infty} \frac{dt}{r_q(t)^2}.$$

Lemma. *Let $E > 0$. There exists $q_{E,\beta,\alpha}$ (also denoted q_E) such that $r_{\min,q} := \min_{t\in\mathbb{R}} r_q(t)$ is C^1 strictly increasing on $[q_E, +\infty)$. In addition*

$$E = \frac{q^2}{2r_{\min,q}^2} + V(r_{\min,q}), \qquad \frac{dr_{\min,q}}{dq} = \frac{q r_{\min,q}}{q^2 - r_{\min,q}^3 V'(r_{\min,q})}, \ q \in [q_E, +\infty)$$

$$r_{\min,q} = \frac{q}{\sqrt{2E}} + O(q^{1-\alpha}), \ q \to +\infty$$

We introduce
$$\chi(\sigma) = \frac{1}{r_{\min,\sigma^{-\frac{1}{2}}}}, \ \chi : [0, q_E^{-2}) \to [0, r_{\min,q_E}),$$

$$H(\sigma) := \int_0^\sigma \frac{\theta_{u^{-\frac{1}{2}}} \, du}{2\sqrt{u}\sqrt{\sigma - u}} = \pi \int_0^{\chi(\sigma)} \frac{ds}{\sqrt{2(E - V(s^{-1}))}},$$

We have
$$\frac{\sqrt{2E}\chi(\sigma)}{\sqrt{\sigma}} = 1 + 0(\sigma^{\frac{\alpha}{2}}), \qquad \ln\left(\frac{\sqrt{2E}\chi(\sigma)}{\sqrt{\sigma}}\right) \to 0, \ \sigma \to 0^+,$$

$$\frac{1}{\pi\sqrt{\sigma}} \frac{dH}{d\sigma}(\sigma) = \frac{d}{d\sigma} \ln(\chi(\sigma)) \quad \text{for } \sigma \in [0, q_E^{-2}).$$

$$\chi(\sigma) = (2E)^{-\frac{1}{2}}\sigma^{\frac{1}{2}}e^{\int_0^\sigma \left(\frac{1}{\pi\sqrt{s}}\frac{dH}{ds}(s)-\frac{1}{2s}\right)ds} \quad \text{for } \sigma \in [0, q_E^{-2}).$$

- When $V(r)$ is assumed to be positive and monotonically decreasing, see Firsov (1953).

- For $B \equiv 0$, R. Novikov (1999) studied the nonrelativistic inverse scattering problem at fixed energy and gave relations between this problem and the nonrelativistic inverse boundary value problem.

- Quantum analogs for the inverse scattering problem at fixed energy : Henkin-Novikov (1987), Novikov (1988), Eskin-Ralston (1995), Isozaki (1997).

- Open question

  - Can we prove a uniqueness result for the inverse scattering at fixed energy under the only Condition (2) ?

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