Inverse boundary problems for magnetic Schrödinger operators with bounded magnetic potentials

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Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set with Lipschitz boundary. Consider the magnetic Schrödinger operator,

$$L_{A,q}(x, D) = \sum_{j=1}^{n} (D_j + A_j(x))^2 + q(x),$$

where $D_j = i^{-1}\partial_{x_j}$, $A = (A_1, \dots, A_n) \in L^\infty(\Omega, \mathbb{C}^n)$ is the magnetic potential, and $q \in L^\infty(\Omega, \mathbb{C})$ is the electric potential.

Let $u \in C_0^\infty(\Omega)$ and let us write

$$L_{A,q}u = -\Delta u + A \cdot Du + D \cdot (Au) + (A^2 + q)u.$$

Since $Au \in L^\infty(\Omega) \cap \mathcal{E}'(\Omega) \subset L^2(\mathbb{R}^n) \cap \mathcal{E}'(\Omega)$, we see that

$$L_{A,q} : C_0^\infty(\Omega) \to H^{-1}(\mathbb{R}^n) \cap \mathcal{E}'(\Omega)$$

is a bounded operator. Here

$$\mathcal{E}'(\Omega) := \{v \in \mathcal{D}'(\Omega) : \mathrm{supp}(v) \text{ is compact}\}.$$

Let $u \in H^1(\Omega)$ be a solution to

$$L_{A,q} u = 0 \quad \text{in} \quad \Omega.$$

The set of the Cauchy data is given by

$$C_{A,q} := \{(u|_{\partial\Omega}, (\partial_\nu u + i(A \cdot \nu)u)|_{\partial\Omega}) : u \in H^1(\Omega) \text{ and } L_{A,q} u = 0 \text{ in } \Omega\}.$$

Here $u|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$, and we set

$$\langle \partial_\nu u + i(A \cdot \nu)u, g\rangle_{\partial\Omega} := \int_\Omega (\nabla u \cdot \nabla G + iA \cdot (u \nabla G - G \nabla u) + (A^2 + q)uG)\, dx,$$

where $g \in H^{1/2}(\partial\Omega)$ and $G \in H^1(\Omega)$ is such that $G|_{\partial\Omega} = g$.

It follows that $(\partial_\nu u + i(A \cdot \nu)u)|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$.

Inverse boundary value problem : Determine A and q in Ω from the set of the Cauchy data $C_{A,q}$.

- Z. Sun (1993) : There is an obstruction to uniqueness.

Let $\psi \in W^{1,\infty}(\Omega)$. Then

$$e^{-i\psi} L_{A,q} e^{i\psi} = L_{A+\nabla\psi,q},$$

and

$$e^{-i\psi}(\partial_\nu + i(A\cdot\nu))e^{i\psi}u = (\partial_\nu + i(A+\nabla\psi)\cdot\nu)u \text{ on } \partial\Omega.$$

Thus, if $\psi|_{\partial\Omega} = 0$, then

$$C_{A,q} = C_{A+\nabla\psi,q}.$$

Hence, given $C_{A,q}$, we may only hope to recover the magnetic field dA in Ω, which is defined by

$$dA = \sum_{1\le j<k\le n} (\partial_{x_j} A_k - \partial_{x_k} A_j) dx_j \wedge dx_k.$$

Here $A = \sum_{j=1}^n A_j dx_j$.

Indeed, $C_{A,q}$ determines dA and q in Ω under some regularity assumptions on A and q.

- Z. Sun (1993) :
    - $A \in W^{2,\infty}$ with $\|dA\|_{L^\infty}$ is small, and $q \in L^\infty$.
- G. Nakamura, Z. Sun, G. Uhlmann (1995) :
    - $A \in C^\infty$ and $q \in C^\infty$;
    - $A \in C^2$ and $q \in L^\infty$.
- C. Tolmasky (1998) :
    - $A \in C^1$ and $q \in L^\infty$.
- A. Panchenko (2002) :
    - $A \in L^\infty$, with some additional assumptions, and in particular, a smallness condition, and $q \in L^\infty$.
- M. Salo (2004) :
    - A Dini continuous and $q \in L^\infty$.

Theorem (K., Uhlmann, 2012). Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set with Lipschitz boundary, and assume that $A_1, A_2 \in L^\infty(\Omega, \mathbb{C}^n)$ and $q_1, q_2 \in L^\infty(\Omega, \mathbb{C})$. If $C_{A_1,q_1} = C_{A_2,q_2}$, then $dA_1 = dA_2$ and $q_1 = q_2$ in Ω.

Sketch of proof

- Step 1. Construction of complex geometric optics solutions (CGO solutions) to the equation $L_{A,q}u = 0$ in Ω with $A \in L^\infty$ and $q \in L^\infty$.

The use of such solutions in inverse boundary value problems has a long and distinguished tradition going back to the fundamental works of A. Calderón (1980), J. Sylvester and G. Uhlmann (1987), ...

CGO solutions are solutions of the form,

$$u(x,\zeta;h) = e^{x\cdot\zeta/h}(a(x,\zeta;h) + r(x,\zeta;h)),$$

where $\zeta \in \mathbb{C}^n$, $\zeta\cdot\zeta = 0$, $|\zeta| \sim 1$, $0 < h \le h_0$, a is a smooth amplitude, and r is a remainder, which tends to zero as $h \to 0$.

Our approach to the construction of CGO solutions is based on the method of Carleman estimates.

To construct CGO solutions using this method, one wants the conjugated operator

$$e^{-\varphi/h} h^2 L_{A,q} e^{\varphi/h}, \quad \varphi(x) = x \cdot \mathrm{Re}\zeta,$$

to be solvable in Ω in the semiclassical sense.

The point of the Carleman estimate is exactly to provide us with the appropriate tool for deducing semiclassical solvability of the conjugated operator.

Our starting point : the Carleman estimate for $-h^2\Delta$ due to M. Salo and L. Tzou (2009), which is a generalization of the corresponding estimate of C. Kenig, J. Sjöstrand, and G. Uhlmann (2007).

Proposition. (Salo–Tzou, 2009). Let $\varphi(x) = \alpha \cdot x$, $\alpha \in \mathbb{R}^n$, $|\alpha| \sim 1$, and let $\varphi_\varepsilon = \varphi + \frac{h}{2\varepsilon}\varphi^2$ be a convexification of φ. Then for $0 < h \ll \varepsilon \ll 1$ and $s \in \mathbb{R}$, we have

$$\frac{h}{\sqrt{\varepsilon}}\|u\|_{H^{s+2}_{\mathrm{scl}}(\mathbb{R}^n)} \leq C\|e^{\varphi_\varepsilon/h}(-h^2\Delta)e^{-\varphi_\varepsilon/h}u\|_{H^s_{\mathrm{scl}}(\mathbb{R}^n)}, \quad C > 0, \qquad (1.1)$$

for all $u \in C_0^\infty(\Omega)$.

Here

$$\|u\|_{H^s_{\mathrm{scl}}(\mathbb{R}^n)} = \|\langle hD\rangle^s u\|_{L^2(\mathbb{R}^n)}, \quad \langle\xi\rangle = (1 + |\xi|^2)^{1/2},$$

is the natural semiclassical norm in the Sobolev space $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$.

Recalling that

$$L_{A,q} : C_0^\infty(\Omega) \to H^{-1}(\mathbb{R}^n) \cap \mathcal{E}'(\Omega)$$

is a bounded operator, it will be natural to use (1.1) with $s = -1$ and $\varepsilon > 0$ sufficiently small but fixed.

The estimate (1.1) can be perturbed by lower order terms, and we get

Proposition. Let $A \in L^\infty(\Omega, \mathbb{C}^n)$ and $q \in L^\infty(\Omega, \mathbb{C})$ and let $\varphi(x) = \alpha \cdot x$, $\alpha \in \mathbb{R}^n$, $|\alpha| \sim 1$. For $0 < h \ll 1$, we have

$$h\|u\|_{H^1_{\mathrm{scl}}(\mathbb{R}^n)} \le C\|e^{\varphi/h}(h^2 L_{A,q})e^{-\varphi/h}u\|_{H^{-1}_{\mathrm{scl}}(\mathbb{R}^n)}, \tag{1.2}$$

for all $u \in C_0^\infty(\Omega)$.

The formal L^2 adjoint of the operator

$$L_\varphi := e^{\varphi/h}(h^2 L_{A,q})e^{-\varphi/h}$$

is of the form

$$L_\varphi^* := e^{-\varphi/h}(h^2 L_{\overline{A},\overline{q}})e^{\varphi/h},$$

and therefore, (1.2) holds for the adjoint.

Thus, the Carleman estimate (1.2) for L_φ^* can be converted to the following solvability result for L_φ.

Proposition. Let $A \in L^\infty(\Omega, \mathbb{C}^n)$, $q \in L^\infty(\Omega, \mathbb{C})$, and let $\varphi(x) = \alpha \cdot x$, $\alpha \in \mathbb{R}^n$, $|\alpha| \sim 1$. If $h > 0$ is small enough, then for any $v \in H^{-1}(\Omega)$, there is a solution $u \in H^1(\Omega)$ of the equation

$$e^{\varphi/h}(h^2 L_{A,q})e^{-\varphi/h}u = v \quad \text{in} \quad \Omega,$$

which satisfies

$$\|u\|_{H^1_{\mathrm{scl}}(\Omega)} \le \frac{C}{h}\|v\|_{H^{-1}_{\mathrm{scl}}(\Omega)}.$$

Here

$$\|u\|^2_{H^1_{\mathrm{scl}}(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|hDu\|^2_{L^2(\Omega)}, \quad \|v\|_{H^{-1}_{\mathrm{scl}}(\Omega)} = \sup_{0 \ne \psi \in C_0^\infty(\Omega)} \frac{|\langle v, \psi\rangle_\Omega|}{\|\psi\|_{H^1_{\mathrm{scl}}(\Omega)}}.$$

The most singular term in

$$L_{A,q}u = -\Delta u + A \cdot Du + D \cdot (Au) + (A^2 + q)u, \quad u \in C_0^\infty(\Omega),$$

is given by $D \cdot (Au)$. Associated with this term, we introduce the bounded operator,

$$m_A : H^1(\Omega) \to H^{-1}(\Omega), \quad m_A(u) = D \cdot (Au),$$

where the distribution $m_A(u)$ is given by

$$\langle m_A(u), v\rangle_\Omega = -\int_\Omega Au \cdot Dvdx, \quad v \in C_0^\infty(\Omega).$$

Conjugating the operator $h^2 L_{A,q}$ by $e^{x\cdot\zeta/h}$, we get

$$e^{-x\cdot\zeta/h} \circ h^2 L_{A,q} \circ e^{x\cdot\zeta/h} = -h^2\Delta - 2ih\zeta\cdot D + h^2 A\cdot D - 2hi\zeta\cdot A + h^2 m_A + h^2(A^2+q).$$

We would like to find a and r so that

$$e^{-x\cdot\zeta/h} h^2 L_{A,q}(e^{x\cdot\zeta/h} r)$$
$$= -(-h^2\Delta - 2ih\zeta \cdot D + h^2 A \cdot D - 2hi\zeta \cdot A + h^2 m_A + h^2(A^2 + q))a \text{ in } \Omega.$$

To obtain nice remainder estimates for our CGO solutions, we need to get rid of terms of order h in RHS.

We get the first transport equation,

$$\zeta \cdot Da + \zeta \cdot Aa = 0 \quad \text{in} \quad \mathbb{R}^n.$$

Here A has been extended to $\mathbb{R}^n \setminus \Omega$ by zero.

$$A \in L^\infty \quad \Longrightarrow \quad a \in L^\infty,$$

which is not acceptable. Indeed, when solving the equation for the remainder r, we will encounter the term

$$-h^2 \Delta a,$$

which is too singular to apply our solvability result.

To cope with this difficulty, we shall replace A by its regularization.

To this end, let $\Psi_\tau(x) = \tau^{-n}\Psi(x/\tau)$, $\tau > 0$, be the usual mollifier with $\Psi \in C_0^\infty(\mathbb{R}^n)$, $0 \le \Psi \le 1$, and $\int \Psi dx = 1$.

Then

$$A^\sharp = A * \Psi_\tau \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^n),$$

and

$$A = A^\sharp + (A - A^\sharp).$$

Since $A \in (L^\infty \cap \mathcal{E}')(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, we have

$$\|A - A^\sharp\|_{L^2(\mathbb{R}^n)} \to 0, \quad \tau \to 0.$$

As $A \in L^\infty(\mathbb{R}^n)$, we get

$$\|\partial^\alpha A^\sharp\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}(\tau^{-|\alpha|}), \quad \tau \to 0, \quad \text{for all} \quad \alpha, \quad |\alpha| \ge 0.$$

Now we require that a should satisfy the equation,

$$\zeta \cdot Da + \zeta \cdot A^{\sharp} a = 0 \quad \text{in} \quad \mathbb{R}^n. \tag{1.3}$$

As $\zeta \cdot \zeta = 0$, and we shall choose ζ so that $|\mathrm{Re}\zeta| = |\mathrm{Im}\zeta| = 1$, the operator

$$N_\zeta := \zeta \cdot \nabla$$

is the $\bar{\partial}$ operator in the plane spanned by $\mathrm{Re}\zeta$ and $\mathrm{Im}\zeta$.
We choose a solution of (1.3) in the form $a = e^{\Phi^{\sharp}}$ with

$$\Phi^{\sharp}(x, \zeta; \tau) = N_\zeta^{-1}(-i\zeta \cdot A^{\sharp}) \in C^\infty(\mathbb{R}^n).$$

Here

$$(N_\zeta^{-1} f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{f(x - y_1 \mathrm{Re}\zeta - y_2 \mathrm{Im}\zeta)}{y_1 + i y_2} dy_1 dy_2, \quad f \in C_0(\mathbb{R}^n).$$

Using the mapping properties

$$N_\zeta^{-1} : W^{k,\infty}(\mathbb{R}^n) \cap \mathcal{E}'(\mathbb{R}^n) \to W^{k,\infty}(\mathbb{R}^n), \quad k \geq 0,$$

we get

$$\|\partial^\alpha \Phi^{\sharp}\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha \tau^{-|\alpha|}, \quad \text{for all} \quad \alpha, \quad |\alpha| \geq 0.$$

Define

$$\Phi(\cdot,\zeta) := N_\zeta^{-1}(-i\zeta\cdot A) \in L^\infty(\mathbb{R}^n).$$

Proposition. (J. Sylvester and G. Uhlmann (1987).) Let $-1 < \delta < 0$ and let $f \in L^2_{\delta+1}(\mathbb{R}^n)$. Then there exists a constant $C > 0$, independent of ζ, such that

$$\|N_\zeta^{-1} f\|_{L^2_\delta(\mathbb{R}^n)} \le C\|f\|_{L^2_{\delta+1}(\mathbb{R}^n)}.$$

Here

$$\|f\|^2_{L^2_\delta(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|x|^2)^\delta |f(x)|^2 dx.$$

Since

$$\|A - A^\sharp\|_{L^2(\mathbb{R}^n)} \to 0, \quad \tau \to 0,$$

we have

$$\Phi^\sharp(\cdot,\zeta;\tau) \to \Phi(\cdot,\zeta)$$

in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ as $\tau \to 0$.

Determination of the remainder r :

$$e^{-x\cdot\zeta/h}h^2L_{A,q}e^{x\cdot\zeta/h}r=-h^2L_{A,q}a+2hi\zeta\cdot(A-A^\sharp)a:=g\quad\text{in}\quad\Omega.$$

To apply the solvability result, we need to estimate the norm $\|g\|_{H^{-1}_{\mathrm{scl}}(\Omega)}$. The most interesting term to estimate is $\|h^2m_A(a)\|_{H^{-1}_{\mathrm{scl}}(\Omega)}$. When estimating this term, we argue by duality and consider, for $0\neq\psi\in C_0^\infty(\Omega)$,

$$\begin{aligned}|\langle h^2m_A(a),\psi\rangle_\Omega|&=\left|\int_\Omega h^2Aa\cdot D\psi dx\right|\\&\le\left|\int_\Omega h^2A^\sharp a\cdot D\psi dx\right|+\left|\int_\Omega h^2(A-A^\sharp)a\cdot D\psi dx\right|\\&\le\left|\int_\Omega h^2(D\cdot(A^\sharp a))\psi dx\right|+\mathcal{O}(h)\|A-A^\sharp\|_{L^2(\Omega)}\|hD\psi\|_{L^2(\Omega)}\\&\le(\mathcal{O}(h^2/\tau)+\mathcal{O}(h)o_{\tau\to0}(1))\|\psi\|_{H^1_{\mathrm{scl}}(\Omega)}.\end{aligned}$$

Altogether we find

$$\|g\|_{H^{-1}_{\mathrm{scl}}(\Omega)}\le\mathcal{O}(h^2/\tau^2)+\mathcal{O}(h)o_{\tau\to0}(1).$$

Choosing now $\tau = h^\sigma$ with $0 < \sigma < 1/2$, we get

$$\|g\|_{H^{-1}_{\mathrm{scl}}(\Omega)} = o(h) \quad \text{as} \quad h \to 0.$$

Thanks to the solvability result, for $h > 0$ small enough, there exists a solution $r \in H^1(\Omega)$ of the equation

$$e^{-x\cdot \mathrm{Re}\zeta/h} h^2 L_{A,q} e^{x\cdot \mathrm{Re}\zeta/h}(e^{ix\cdot \mathrm{Im}\zeta/h} r) = e^{ix\cdot \mathrm{Im}\zeta/h} g \quad \text{in} \quad \Omega,$$

such that $\|r\|_{H^1_{\mathrm{scl}}(\Omega)} = o(1)$ as $h \to 0$.

Summarizing, we obtain the following result.

Proposition. Let $A \in L^\infty(\Omega, \mathbb{C}^n)$, $q \in L^\infty(\Omega, \mathbb{C})$, and let $\zeta \in \mathbb{C}^n$ be such that $\zeta \cdot \zeta = 0$, $|\text{Re}\zeta| = |\text{Im}\zeta| = 1$. Then for all $h > 0$ small enough, there exists a solution $u(x, \zeta; h) \in H^1(\Omega)$ to the magnetic Schrödinger equation $L_{A,q} u = 0$ in Ω, of the form

$$u(x, \zeta; h) = e^{x \cdot \zeta / h}(e^{\Phi^\sharp(x,\zeta;h)} + r(x, \zeta; h)).$$

The function $\Phi^\sharp(\cdot, \zeta; h) \in C^\infty(\mathbb{R}^n)$ satisfies $\|\partial^\alpha \Phi^\sharp\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha h^{-\sigma|\alpha|}$, $0 < \sigma < 1/2$, for all α, $|\alpha| \geq 0$, and $\Phi^\sharp(\cdot, \zeta; h)$ converges to $\Phi(\cdot, \zeta) := N_\zeta^{-1}(-i\zeta \cdot A) \in L^\infty(\mathbb{R}^n)$ in $L^2_{\text{loc}}(\mathbb{R}^n)$ as $h \to 0$. Here we have extended A by zero to $\mathbb{R}^n \setminus \Omega$. The remainder r is such that $\|r\|_{H^1_{\text{scl}}(\Omega)} = o(1)$ as $h \to 0$.

- Step 2. Converting $C_{A_1,q_1} = C_{A_2,q_2}$ into the following integral identity,

$$\int_\Omega i(A_1 - A_2)\cdot(u_1\nabla\overline{u_2} - \overline{u_2}\nabla u_1)dx + \int_\Omega (A_1^2 - A_2^2 + q_1 - q_2)u_1\overline{u_2}dx = 0,$$

which holds for any $u_1, u_2 \in H^1(\Omega)$ satisfying $L_{A_1,q_1}u_1 = 0$ in Ω and $L_{\overline{A_2},\overline{q_2}}u_2 = 0$ in Ω, respectively.

- Step 3. Testing the integral identity against the following family of CGO solutions,

$$u_1(x,\zeta_1;h) = e^{x\cdot\zeta_1/h}(e^{\Phi_1^\sharp(x,\mu_1+i\mu_2;h)} + r_1(x,\zeta_1;h)),$$
$$u_2(x,\zeta_2;h) = e^{x\cdot\zeta_2/h}(e^{\Phi_2^\sharp(x,-\mu_1+i\mu_2;h)} + r_2(x,\zeta_2;h)),$$

where

$$\zeta_j\cdot\zeta_j = 0,\quad j=1,2,\quad (\zeta_1+\overline{\zeta_2})/h = i\xi,$$
$$\zeta_1 = \mu_1 + i\mu_2 + \mathcal{O}(h),\quad \zeta_2 = -\mu_1 + i\mu_2 + \mathcal{O}(h),\quad \text{as}\quad h\to 0,$$
$$\xi,\mu_1,\mu_2 \in \mathbb{R}^n,\quad |\mu_1| = |\mu_2| = 1,\quad \mu_1\cdot\mu_2 = \mu_1\cdot\xi = \mu_2\cdot\xi = 0.$$

- Step 4. Multiplying the integral identity by h and letting $h \to 0$, we get

$$\lim_{h\to 0}(\mu_1 + i\mu_2) \cdot \int_\Omega (A_1 - A_2) e^{ix\cdot\xi} e^{\Phi_1^\sharp + \overline{\Phi_2^\sharp}} dx = 0.$$

Since

$$\Phi_1^\sharp(\cdot, \mu_1 + i\mu_2; h) \to \Phi_1(\cdot, \mu_1 + i\mu_2) := N^{-1}_{\mu_1+i\mu_2}(-i(\mu_1 + i\mu_2) \cdot A_1),$$

$$\Phi_2^\sharp(\cdot, -\mu_1 + i\mu_2; h) \to \Phi_2(\cdot, -\mu_1 + i\mu_2) := N^{-1}_{-\mu_1+i\mu_2}(-i(-\mu_1 + i\mu_2) \cdot \overline{A_2}),$$

in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ as $h \to 0$, and $\|\Phi_j^\sharp\|_{L^\infty} = \mathcal{O}(1)$, we see that

$$(\mu_1 + i\mu_2) \cdot \int_{\mathbb{R}^n} (A_1 - A_2) e^{ix\cdot\xi} e^{\Phi_1 + \overline{\Phi_2}} dx = 0.$$

- Step 5. Conclude that

$$(\mu_1 + i\mu_2) \cdot \int_{\mathbb{R}^n} (A_1 - A_2) e^{ix\cdot\xi} dx = 0.$$

(G. Eskin, J. Ralston (1995) for A_j smooth ; it can be extended to $A_j \in (L^\infty \cap \mathcal{E}')(\mathbb{R}^n)$ by regularization)

The conclusion that $dA_1 = dA_2$ in Ω and $q_1 = q_2$ in Ω now follows easily.