Inverse boundary problems for magnetic Schrödinger operators with bounded magnetic potentials

Katya Krupchyk

Department of Mathematics and Statistics, University of Helsinki, Finland

Joint work with Gunther Uhlmann
Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set with Lipschitz boundary. Consider the magnetic Schrödinger operator,

$$L_{A,q}(x, D) = \sum_{j=1}^{n} (D_j + A_j(x))^2 + q(x),$$

where $D_j = i^{-1} \partial_{x_j}$, $A = (A_1, \ldots, A_n) \in L^\infty(\Omega, \mathbb{C}^n)$ is the magnetic potential, and $q \in L^\infty(\Omega, \mathbb{C})$ is the electric potential.

Let $u \in C_0^\infty(\Omega)$ and let us write

$$L_{A,q}u = -\Delta u + A \cdot Du + D \cdot (Au) + (A^2 + q)u.$$

Since $Au \in L^\infty(\Omega) \cap \mathcal{E}'(\Omega) \subset L^2(\mathbb{R}^n) \cap \mathcal{E}'(\Omega)$, we see that

$$L_{A,q} : C_0^\infty(\Omega) \to H^{-1}(\mathbb{R}^n) \cap \mathcal{E}'(\Omega)$$

is a bounded operator. Here

$$\mathcal{E}'(\Omega) := \{ \nu \in \mathcal{D}'(\Omega) : \text{supp}(\nu) \text{ is compact} \}.$$
Let $u \in H^1(\Omega)$ be a solution to

$$L_{A,q}u = 0 \quad \text{in} \quad \Omega.$$ 

The set of the Cauchy data is given by

$$C_{A,q} := \{(u|_{\partial \Omega}, (\partial_\nu u + i(A \cdot \nu)u)|_{\partial \Omega}) : u \in H^1(\Omega) \text{ and } L_{A,q}u = 0 \text{ in } \Omega\}.$$ 

Here $u|_{\partial \Omega} \in H^{1/2}(\partial \Omega)$, and we set

$$\langle \partial_\nu u + i(A \cdot \nu)u, g \rangle_{\partial \Omega} := \int_{\Omega} (\nabla u \cdot \nabla G + iA \cdot (u \nabla G - G \nabla u) + (A^2 + q)uG) \, dx,$$

where $g \in H^{1/2}(\partial \Omega)$ and $G \in H^1(\Omega)$ is such that $G|_{\partial \Omega} = g$.

It follows that $(\partial_\nu u + i(A \cdot \nu)u)|_{\partial \Omega} \in H^{-1/2}(\partial \Omega)$.

**Inverse boundary value problem**: Determine $A$ and $q$ in $\Omega$ from the set of the Cauchy data $C_{A,q}$. 
Z. Sun (1993) : There is an obstruction to uniqueness.

Let $\psi \in W^{1,\infty}(\Omega)$. Then

$$e^{-i\psi} L_{A,q} e^{i\psi} = L_{A+\nabla \psi,q},$$

and

$$e^{-i\psi} (\partial_\nu + i(A \cdot \nu)) e^{i\psi} u = (\partial_\nu + i(A + \nabla \psi) \cdot \nu) u \text{ on } \partial \Omega.$$

Thus, if $\psi|_{\partial \Omega} = 0$, then

$$C_{A,q} = C_{A+\nabla \psi,q}.$$

Hence, given $C_{A,q}$, we may only hope to recover the magnetic field $dA$ in $\Omega$, which is defined by

$$dA = \sum_{1 \leq j < k \leq n} (\partial_{x_j} A_k - \partial_{x_k} A_j) dx_j \wedge dx_k.$$

Here $A = \sum_{j=1}^{n} A_j dx_j$. 
Indeed, $C_{A,q}$ determines $dA$ and $q$ in $\Omega$ under some regularity assumptions on $A$ and $q$.

- Z. Sun (1993):
  - $A \in W^{2,\infty}$ with $\|dA\|_{L^\infty}$ is small, and $q \in L^\infty$.

  - $A \in C^\infty$ and $q \in C^\infty$;
  - $A \in C^2$ and $q \in L^\infty$.

- C. Tolmasky (1998):
  - $A \in C^1$ and $q \in L^\infty$.

  - $A \in L^\infty$, with some additional assumptions, and in particular, a smallness condition, and $q \in L^\infty$.

- M. Salo (2004):
  - $A$ Dini continuous and $q \in L^\infty$. 
**Theorem** (K., Uhlmann, 2012). Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set with Lipschitz boundary, and assume that $A_1, A_2 \in L^\infty(\Omega, \mathbb{C}^n)$ and $q_1, q_2 \in L^\infty(\Omega, \mathbb{C})$. If $C_{A_1,q_1} = C_{A_2,q_2}$, then $dA_1 = dA_2$ and $q_1 = q_2$ in $\Omega$. 
Sketch of proof

- **Step 1.** Construction of complex geometric optics solutions (CGO solutions) to the equation $L_{A,q}u = 0$ in $\Omega$ with $A \in L^\infty$ and $q \in L^\infty$.

The use of such solutions in inverse boundary value problems has a long and distinguished tradition going back to the fundamental works of A. Calderón (1980), J. Sylvester and G. Uhlmann (1987), ...

CGO solutions are solutions of the form,

$$u(x, \zeta; h) = e^{x \cdot \zeta/h}(a(x, \zeta; h) + r(x, \zeta; h)),$$

where $\zeta \in \mathbb{C}^n$, $\zeta \cdot \zeta = 0$, $|\zeta| \sim 1$, $0 < h \leq h_0$, $a$ is a smooth amplitude, and $r$ is a remainder, which tends to zero as $h \to 0$. 
Our approach to the construction of CGO solutions is based on the method of Carleman estimates.

To construct CGO solutions using this method, one wants the conjugated operator

$$e^{-\varphi/h} h^2 L_{A,q} e^{\varphi/h}, \quad \varphi(x) = x \cdot \text{Re}\zeta,$$

to be solvable in $\Omega$ in the semiclassical sense.

The point of the Carleman estimate is exactly to provide us with the appropriate tool for deducing semiclassical solvability of the conjugated operator.

Our starting point: the Carleman estimate for $-h^2 \Delta$ due to M. Salo and L. Tzou (2009), which is a generalization of the corresponding estimate of C. Kenig, J. Sjöstrand, and G. Uhlmann (2007).
Proposition. (Salo–Tzou, 2009). Let $\varphi(x) = \alpha \cdot x$, $\alpha \in \mathbb{R}^n$, $|\alpha| \sim 1$, and let $\varphi_{\varepsilon} = \varphi + \frac{h}{2\varepsilon} \varphi^2$ be a convexification of $\varphi$. Then for $0 < h \ll \varepsilon \ll 1$ and $s \in \mathbb{R}$, we have

$$\frac{h}{\sqrt{\varepsilon}}\|u\|_{H^{s+2}_{\text{scl}}(\mathbb{R}^n)} \leq C\|e^{\varphi_{\varepsilon}/h}(-h^2\Delta)e^{-\varphi_{\varepsilon}/h}u\|_{H^s_{\text{scl}}(\mathbb{R}^n)}, \quad C > 0, \quad (1.1)$$

for all $u \in C_0^\infty(\Omega)$.

Here

$$\|u\|_{H^s_{\text{scl}}(\mathbb{R}^n)} = \|\langle hD\rangle^s u\|_{L^2(\mathbb{R}^n)}, \quad \langle \xi \rangle = (1 + |\xi|^2)^{1/2},$$

is the natural semiclassical norm in the Sobolev space $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$.

Recalling that

$$L_{A,q} : C_0^\infty(\Omega) \rightarrow H^{-1}(\mathbb{R}^n) \cap \mathcal{E}'(\Omega)$$

is a bounded operator, it will be natural to use (1.1) with $s = -1$ and $\varepsilon > 0$ sufficiently small but fixed.

The estimate (1.1) can be perturbed by lower order terms, and we get
Proposition. Let $A \in L^\infty(\Omega, \mathbb{C}^n)$ and $q \in L^\infty(\Omega, \mathbb{C})$ and let $\varphi(x) = \alpha \cdot x$, $\alpha \in \mathbb{R}^n$, $|\alpha| \sim 1$. For $0 < h \ll 1$, we have

\[ h\|u\|_{H^1_{\text{scl}}(\mathbb{R}^n)} \leq C\|e^{\varphi/h}(h^2 L_{A,q})e^{-\varphi/h}u\|_{H^{-1}_{\text{scl}}(\mathbb{R}^n)}, \]  

(1.2)

for all $u \in C_0^\infty(\Omega)$.

The formal $L^2$ adjoint of the operator

\[ L_\varphi := e^{\varphi/h}(h^2 L_{A,q})e^{-\varphi/h} \]

is of the form

\[ L^*_\varphi := e^{-\varphi/h}(h^2 L_{A,q})e^{\varphi/h}, \]

and therefore, (1.2) holds for the adjoint.

Thus, the Carleman estimate (1.2) for $L^*_\varphi$ can be converted to the following solvability result for $L_\varphi$. 
**Proposition.** Let $A \in L^\infty(\Omega, \mathbb{C}^n)$, $q \in L^\infty(\Omega, \mathbb{C})$, and let $\varphi(x) = \alpha \cdot x$, $\alpha \in \mathbb{R}^n$, $|\alpha| \sim 1$. If $h > 0$ is small enough, then for any $v \in H^{-1}(\Omega)$, there is a solution $u \in H^1(\Omega)$ of the equation

$$e^{\varphi/h}(h^2 L_A, q)e^{-\varphi/h}u = v \quad \text{in} \quad \Omega,$$

which satisfies

$$\|u\|_{H^1_{scl}(\Omega)} \leq \frac{C}{h} \|v\|_{H^{-1}_{scl}(\Omega)}.$$

Here

$$\|u\|^2_{H^1_{scl}(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|hDu\|^2_{L^2(\Omega)}, \quad \|v\|_{H^{-1}_{scl}(\Omega)} = \sup_{0 \neq \psi \in C_0^\infty(\Omega)} \frac{|\langle v, \psi \rangle_\Omega|}{\|\psi\|_{H^1_{scl}(\Omega)}}.$$
The most singular term in
\[ L_{A,q} u = -\Delta u + A \cdot Du + D \cdot (Au) + (A^2 + q)u, \quad u \in C_0^\infty(\Omega), \]
is given by \( D \cdot (Au) \). Associated with this term, we introduce the bounded operator,
\[ m_A : H^1(\Omega) \to H^{-1}(\Omega), \quad m_A(u) = D \cdot (Au), \]
where the distribution \( m_A(u) \) is given by
\[ \langle m_A(u), v \rangle_\Omega = - \int_\Omega Au \cdot Dv \, dx, \quad v \in C_0^\infty(\Omega). \]

Conjugating the operator \( h^2 L_{A,q} \) by \( e^{x \cdot \zeta / h} \), we get
\[ e^{-x \cdot \zeta / h} \circ h^2 L_{A,q} \circ e^{x \cdot \zeta / h} = -h^2 \Delta - 2ih\zeta \cdot D + h^2 A \cdot D - 2hi\zeta \cdot A + h^2 m_A + h^2 (A^2 + q). \]

We would like to find \( a \) and \( r \) so that
\[ e^{-x \cdot \zeta / h} h^2 L_{A,q} (e^{x \cdot \zeta / h} r) \]
\[ = -(-h^2 \Delta - 2ih\zeta \cdot D + h^2 A \cdot D - 2hi\zeta \cdot A + h^2 m_A + h^2 (A^2 + q))a \text{ in } \Omega. \]

To obtain nice remainder estimates for our CGO solutions, we need to get rid of terms of order \( h \) in RHS.
We get the first transport equation,

$$ \zeta \cdot Da + \zeta \cdot Aa = 0 \quad \text{in} \quad \mathbb{R}^n. $$

Here $A$ has been extended to $\mathbb{R}^n \setminus \Omega$ by zero.

$$ A \in L^\infty \implies a \in L^\infty, $$

which is not acceptable. Indeed, when solving the equation for the remainder $r$, we will encounter the term

$$ -h^2 \Delta a, $$

which is too singular to apply our solvability result.

To cope with this difficulty, we shall replace $A$ by its regularization.
To this end, let $\Psi_\tau(x) = \tau^{-n}\Psi(x/\tau)$, $\tau > 0$, be the usual mollifier with $\Psi \in C^\infty_0(\mathbb{R}^n)$, $0 \leq \Psi \leq 1$, and $\int \Psi \, dx = 1$.

Then

$$A^\# = A * \Psi_\tau \in C^\infty_0(\mathbb{R}^n, \mathbb{C}^n),$$

and

$$A = A^\# + (A - A^\#).$$

Since $A \in (L^\infty \cap \mathcal{E}')(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, we have

$$\|A - A^\#\|_{L^2(\mathbb{R}^n)} \to 0, \quad \tau \to 0.$$ 

As $A \in L^\infty(\mathbb{R}^n)$, we get

$$\|\partial^\alpha A^\#\|_{L^\infty(\mathbb{R}^n)} = O(\tau^{-|\alpha|}), \quad \tau \to 0, \quad \text{for all } \alpha, \quad |\alpha| \geq 0.$$
Now we require that $a$ should satisfy the equation,

$$
\zeta \cdot Da + \zeta \cdot A^\# a = 0 \quad \text{in} \quad \mathbb{R}^n.
$$

As $\zeta \cdot \zeta = 0$, and we shall choose $\zeta$ so that $|\text{Re}\zeta| = |\text{Im}\zeta| = 1$, the operator

$$
N_\zeta := \zeta \cdot \nabla
$$

is the $\bar{\partial}$ operator in the plane spanned by $\text{Re}\zeta$ and $\text{Im}\zeta$. We choose a solution of (1.3) in the form $a = e^{\Phi^\#}$ with

$$
\Phi^\#(x, \zeta; \tau) = N_\zeta^{-1}(-i\zeta \cdot A^\#) \in C^\infty(\mathbb{R}^n).
$$

Here

$$
(N_\zeta^{-1} f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{f(x - y_1 \text{Re}\zeta - y_2 \text{Im}\zeta)}{y_1 + iy_2} \, dy_1 \, dy_2, \quad f \in C_0(\mathbb{R}^n).
$$

Using the mapping properties

$$
N_\zeta^{-1} : W^{k,\infty}(\mathbb{R}^n) \cap \mathcal{E}'(\mathbb{R}^n) \to W^{k,\infty}(\mathbb{R}^n), \quad k \geq 0,
$$

we get

$$
\|\partial^\alpha \Phi^\#\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha \tau^{-|\alpha|}, \quad \text{for all} \quad \alpha, \quad |\alpha| \geq 0.
$$
Define
\[ \Phi(\cdot, \zeta) := N^{-1}_\zeta(-i\zeta \cdot A) \in L^\infty(\mathbb{R}^n). \]

**Proposition.** (J. Sylvester and G. Uhlmann (1987).) Let \(-1 < \delta < 0\) and let \(f \in L^2_{\delta+1}(\mathbb{R}^n)\). Then there exists a constant \(C > 0\), independent of \(\zeta\), such that
\[ \|N^{-1}_\zeta f\|_{L^2_\delta(\mathbb{R}^n)} \leq C \|f\|_{L^2_{\delta+1}(\mathbb{R}^n)}. \]

Here
\[ \|f\|_{L^2_\delta(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} (1 + |x|^2)\delta |f(x)|^2 dx. \]

Since
\[ \|A - A^\#\|_{L^2(\mathbb{R}^n)} \to 0, \quad \tau \to 0, \]
we have
\[ \Phi^\#(\cdot, \zeta; \tau) \to \Phi(\cdot, \zeta) \]
in \(L^2_{loc}(\mathbb{R}^n)\) as \(\tau \to 0\).
Determination of the remainder $r$:

$$e^{-x \cdot \zeta / h^2} L_{A,q} e^{x \cdot \zeta / h} r = -h^2 L_{A,q} a + 2hi\zeta \cdot (A - A^\#)a := g \quad \text{in} \quad \Omega.$$  

To apply the solvability result, we need to estimate the norm $\|g\|_{H^{-1}_{scl}(\Omega)}$.

The most interesting term to estimate is $\|h^2 m_A(a)\|_{H^{-1}_{scl}(\Omega)}$. When estimating this term, we argue by duality and consider, for $0 \neq \psi \in C_0^\infty(\Omega)$,

$$|\langle h^2 m_A(a), \psi \rangle_\Omega| = \left| \int_\Omega h^2 Aa \cdot D\psi \, dx \right|$$

$$\leq \left| \int_\Omega h^2 A^# a \cdot D\psi \, dx \right| + \left| \int_\Omega h^2 (A - A^#)a \cdot D\psi \, dx \right|$$

$$\leq \left| \int_\Omega h^2 (D \cdot (A^#a))\psi \, dx \right| + O(h)\|A - A^#\|_{L^2(\Omega)}\|hD\psi\|_{L^2(\Omega)}$$

$$\leq (O(h^2 / \tau) + O(h)o_{\tau \to 0}(1))\|\psi\|_{H^1_{scl}(\Omega)}.$$  

Altogether we find

$$\|g\|_{H^{-1}_{scl}(\Omega)} \leq O(h^2 / \tau^2) + O(h)o_{\tau \to 0}(1).$$
Choosing now $\tau = h^\sigma$ with $0 < \sigma < 1/2$, we get

$$\|g\|_{H^{-1}_{scl}(\Omega)} = o(h) \quad \text{as} \quad h \to 0.$$ 

Thanks to the solvability result, for $h > 0$ small enough, there exists a solution $r \in H^1(\Omega)$ of the equation

$$e^{-x \cdot \text{Re} \zeta / h} h^2 L_{A,q} e^{x \cdot \text{Re} \zeta / h} (e^{i x \cdot \text{Im} \zeta / h} r) = e^{i x \cdot \text{Im} \zeta / h} g \quad \text{in} \quad \Omega,$$

such that $\|r\|_{H^1_{scl}(\Omega)} = o(1)$ as $h \to 0$. 
Summarizing, we obtain the following result.

**Proposition.** Let $A \in L^\infty(\Omega, \mathbb{C}^n)$, $q \in L^\infty(\Omega, \mathbb{C})$, and let $\zeta \in \mathbb{C}^n$ be such that $\zeta \cdot \zeta = 0$, $|\text{Re}\zeta| = |\text{Im}\zeta| = 1$. Then for all $h > 0$ small enough, there exists a solution $u(x, \zeta; h) \in H^1(\Omega)$ to the magnetic Schrödinger equation $L_A, q u = 0$ in $\Omega$, of the form

$$u(x, \zeta; h) = e^{x \cdot \zeta / h}(e^{\Phi^\#(x, \zeta; h)} + r(x, \zeta; h)).$$

The function $\Phi^\#(\cdot, \zeta; h) \in C^\infty(\mathbb{R}^n)$ satisfies $\|\partial^\alpha \Phi^\#\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha h^{-\sigma |\alpha|}$, $0 < \sigma < 1/2$, for all $\alpha$, $|\alpha| \geq 0$, and $\Phi^\#(\cdot, \zeta; h)$ converges to $\Phi(\cdot, \zeta) := N_\zeta^{-1}(-i\zeta \cdot A) \in L^\infty(\mathbb{R}^n)$ in $L^2_{\text{loc}}(\mathbb{R}^n)$ as $h \to 0$. Here we have extended $A$ by zero to $\mathbb{R}^n \setminus \Omega$. The remainder $r$ is such that $\|r\|_{H^1_{\text{scl}}(\Omega)} = o(1)$ as $h \to 0$. 
Step 2. Converting $C_{A_1,q_1} = C_{A_2,q_2}$ into the following integral identity,

$$
\int_{\Omega} i(A_1 - A_2) \cdot (u_1 \nabla u_2 - u_2 \nabla u_1) \, dx + \int_{\Omega} (A_1^2 - A_2^2 + q_1 - q_2) u_1 u_2 \, dx = 0,
$$

which holds for any $u_1, u_2 \in H^1(\Omega)$ satisfying $L_{A_1,q_1} u_1 = 0$ in $\Omega$ and $L_{A_2,q_2} u_2 = 0$ in $\Omega$, respectively.

Step 3. Testing the integral identity against the following family of CGO solutions,

$$
\begin{align*}
    u_1(x, \zeta_1; h) &= e^{x \cdot \zeta_1/h}(e^{\Phi_1^\#(x,\mu_1+i\mu_2;h)} + r_1(x, \zeta_1; h)), \\
    u_2(x, \zeta_2; h) &= e^{x \cdot \zeta_2/h}(e^{\Phi_2^\#(x,-\mu_1+i\mu_2;h)} + r_2(x, \zeta_2; h)),
\end{align*}
$$

where

$$
\begin{align*}
    \zeta_j \cdot \zeta_j &= 0, \quad j = 1, 2, \quad (\zeta_1 + \overline{\zeta_2})/h = i\xi, \\
    \zeta_1 &= \mu_1 + i\mu_2 + O(h), \quad \zeta_2 = -\mu_1 + i\mu_2 + O(h), \quad \text{as} \quad h \to 0, \\
    \xi, \mu_1, \mu_2 &\in \mathbb{R}^n, \quad |\mu_1| = |\mu_2| = 1, \quad \mu_1 \cdot \mu_2 = \mu_1 \cdot \xi = \mu_2 \cdot \xi = 0.
\end{align*}
$$
Step 4. Multiplying the integral identity by $h$ and letting $h \to 0$, we get

$$\lim_{h \to 0} (\mu_1 + i\mu_2) \cdot \int_{\Omega} (A_1 - A_2)e^{ix \cdot \xi} e^{\Phi^1 + \Phi^2} dx = 0.$$  

Since

$$\Phi^1_1(\cdot, \mu_1 + i\mu_2; h) \to \Phi_1(\cdot, \mu_1 + i\mu_2) := N_{\mu_1 + i\mu_2}^{-1}(-i(\mu_1 + i\mu_2) \cdot A_1),$$

$$\Phi^1_2(\cdot, -\mu_1 + i\mu_2; h) \to \Phi_2(\cdot, -\mu_1 + i\mu_2) := N_{-\mu_1 + i\mu_2}^{-1}(-i(-\mu_1 + i\mu_2) \cdot \overline{A_2}),$$

in $L^2_{\text{loc}}(\mathbb{R}^n)$ as $h \to 0$, and $\|\Phi^j\|_{L^\infty} = O(1)$, we see that

$$(\mu_1 + i\mu_2) \cdot \int_{\mathbb{R}^n} (A_1 - A_2)e^{ix \cdot \xi} e^{\Phi_1 + \Phi_2} dx = 0.$$  

Step 5. Conclude that

$$(\mu_1 + i\mu_2) \cdot \int_{\mathbb{R}^n} (A_1 - A_2)e^{ix \cdot \xi} dx = 0.$$  

(G. Eskin, J. Ralston (1995) for $A_j$ smooth; it can be extended to $A_j \in (L^\infty \cap \mathcal{E}')(\mathbb{R}^n)$ by regularization)
The conclusion that $dA_1 = dA_2$ in $\Omega$ and $q_1 = q_2$ in $\Omega$ now follows easily.