Analysis of the anomalous localized resonance

Hyundae Lee (Inha University, Korea)
Joint work with Habib Ammari, Giulio Ciraolo, Hyeonbae Kang, Graeme Milton.

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Outline

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• Integral operators and its symmetry
• Spectral analysis of ALR
• ALR in annulus region
Surface plasmon

Let

\[ \epsilon = \begin{cases} 
1 & \text{in } \{(x, y) : y \geq 0\}, \\
-1 & \text{in } \{(x, y) : y < 0\}. 
\end{cases} \]

Consider

\[ \nabla \cdot \epsilon \nabla u = 0 \quad \text{in } \mathbb{R}^2. \]

Then one solution is

\[ u = \begin{cases} 
e^{-y+ix} & \text{in } \{(x, y) : y \geq 0\}, \\
-e^{y-ix} & \text{in } \{(x, y) : y < 0\}. 
\end{cases} \]
Let $\Omega$ be a smooth domain in $\mathbb{R}^2$ and let $D \subset \Omega$. The permittivity distribution in $\mathbb{R}^2$ is given by

$$
\epsilon_\delta = \begin{cases} 
1 & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\
-1 + i\delta & \text{in } \Omega \setminus \overline{D}, \\
1 & \text{in } D.
\end{cases}
$$
Problem

For a given function $f$ compactly supported in $\mathbb{R}^2$ satisfying

$$\int_{\mathbb{R}^2} f \, dx = 0,$$

we consider the following equation:

$$\nabla \cdot \epsilon_\delta \nabla V_\delta = f \quad \text{in} \quad \mathbb{R}^2,$$

with decaying condition $V_\delta(x) \to 0$ as $|x| \to \infty$.

Since the equation degenerates as $\delta \to 0$, we can expect some singular behavior of the solution, depending on the source term $f$. 

Figure: Anomalous resonance, Milton et al (2006).

- Energy concentration near interfaces, depending on the location of source.
- Associated with the cloaking effect of polarizable dipole.
- Generalized to a small inclusion with a specific boundary condition by Bouchitté and B. Schweizer(2010).

- There is some cloaking effect even in the presence of a small dielectric inclusion, not perfect.
- Blow-up may not depend on the location of the source in a layer of general shape.
A fundamental problem is to find a region $\Omega^*$ containing $\Omega$ such that if $f$ is supported in $\Omega^* \setminus \overline{\Omega}$, then
\[
\int_{\Omega \setminus \overline{D}} \delta |\nabla V_\delta|^2 dx \to \infty \quad \text{as } \delta \to 0.
\]

- Such a region $\Omega^* \setminus \overline{\Omega}$ is called the anomalous resonance region or cloaking region. The quantity $\int_{\Omega \setminus D} \delta |\nabla V_\delta|^2 dx$ is a part of the absorbed energy.

- The blow-up of the energy may or may not occur depending on $f$. So the problem is not only finding the anomalous resonance region $\Omega^* \setminus \overline{\Omega}$ but also characterizing those source terms $f$ which actually make the anomalous resonance happen.
Relation to cloaking

Suppose $f$ is a polarizable dipole at $x_0$, i.e.,

$$V_\delta(x) = U_\delta(x) + A_\delta \cdot \nabla G(x - x_0), \quad A_\delta = k \nabla U_\delta(x_0),$$

for some given coefficient $k$.

If ALR happens, then we should have

$$A_\delta \to 0 \quad \text{as} \quad \delta \to 0.$$ 

Otherwise $\int_{\Omega \setminus D} \delta |\nabla V_\delta|^2 dx$ blows up, which is not physical.
Let $F$ be the Newtonian potential of $f$, i.e.,

$$F(x) = \int_{\mathbb{R}^2} G(x - y)f(y)dy, \quad x \in \mathbb{R}^2.$$ 

Then $F$ satisfies $\Delta F = f$ in $\mathbb{R}^2$, and the solution $V_\delta$ may be represented as

$$V_\delta(x) = F(x) + S_{\Gamma_i}[\varphi_i](x) + S_{\Gamma_e}[\varphi_e](x)$$

for some functions $\varphi_i \in L^2_0(\Gamma_i)$ and $\varphi_e \in L^2_0(\Gamma_e)$ ($L^2_0$ is the collection of all square integrable functions with the integral zero).

The transmission conditions along the interfaces $\Gamma_e$ and $\Gamma_i$ satisfied by $V_\delta$ read

$$(-1 + i\delta) \frac{\partial V_\delta}{\partial \nu} \bigg|_+ = \frac{\partial V_\delta}{\partial \nu} \bigg|_- \quad \text{on } \Gamma_i$$

$$\frac{\partial V_\delta}{\partial \nu} \bigg|_+ = (-1 + i\delta) \frac{\partial V_\delta}{\partial \nu} \bigg|_- \quad \text{on } \Gamma_e.$$
Using the jump formula for the normal derivative of the single layer potentials, the pair of potentials \((\varphi_i, \varphi_e)\) is the solution to

\[
\begin{bmatrix}
z_\delta l - \mathcal{K}_{\Gamma_i}^* & - \frac{\partial}{\partial \nu_i} S_{\Gamma_e} \\
\frac{\partial}{\partial \nu_e} S_{\Gamma_i} & z_\delta l + \mathcal{K}_{\Gamma_e}^*
\end{bmatrix}
\begin{bmatrix}
\varphi_i \\
\varphi_e
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial F}{\partial \nu_i} \\
- \frac{\partial F}{\partial \nu_e}
\end{bmatrix}.
\]

on \(L^2_0(\Gamma_i) \times L^2_0(\Gamma_e)\), where we set

\[
z_\delta = \frac{i \delta}{2(2 - i \delta)}.
\]

Note that the operator can be viewed as a compact perturbation of the operator

\[
\begin{bmatrix}
z_\delta l - \mathcal{K}_{\Gamma_i}^* & 0 \\
0 & z_\delta l + \mathcal{K}_{\Gamma_e}^*
\end{bmatrix}.
\]
• We now recall Kellogg's result on the spectrums of $\mathcal{K}_{\Gamma_i}^*$ and $\mathcal{K}_{\Gamma_e}^*$. The eigenvalues of $\mathcal{K}_{\Gamma_i}^*$ and $\mathcal{K}_{\Gamma_e}^*$ lie in the interval [$-\frac{1}{2}, \frac{1}{2}$].

• Observe that $z_\delta \to 0$ as $\delta \to 0$ and that there are sequences of eigenvalues of $\mathcal{K}_{\Gamma_i}^*$ and $\mathcal{K}_{\Gamma_e}^*$ approaching to 0 since $\mathcal{K}_{\Gamma_i}^*$ and $\mathcal{K}_{\Gamma_e}^*$ are compact. So 0 is the essential singularity of the operator valued meromorphic function

$$\lambda \in \mathbb{C} \mapsto (\lambda I + \mathcal{K}_{\Gamma_e}^*)^{-1}.$$ 

This causes a serious difficulty in dealing with (11).

• We emphasize that $\mathcal{K}_{\Gamma_e}^*$ is not self-adjoint in general. In fact, $\mathcal{K}_{\Gamma_e}^*$ is self-adjoint only when $\Gamma_e$ is a circle or a sphere.
Properties of $\mathbf{K}^*$

Let $\mathcal{H} = L^2(\Gamma_i) \times L^2(\Gamma_e)$. Let the Neumann-Poincaré-type operator $\mathbf{K}^* : \mathcal{H} \to \mathcal{H}$ be defined by

$$\mathbf{K}^* := \begin{bmatrix}
-\mathbf{K}_{\Gamma_i}^* & -\frac{\partial}{\partial \nu_i} S_{\Gamma_e} \\
\frac{\partial}{\partial \nu_e} S_{\Gamma_i} & \mathbf{K}_{\Gamma_e}^*
\end{bmatrix}.$$

Then the integral equation can be written as

$$(z_\delta \mathbb{I} + \mathbf{K}^*) \Phi_\delta = g$$

and the $L^2$-adjoint of $\mathbf{K}^*$, $\mathbf{K}$, is given by

$$\mathbf{K} = \begin{bmatrix}
-\mathbf{K}_{\Gamma_i} & \mathbf{D}_{\Gamma_e} \\
-\mathbf{D}_{\Gamma_i} & \mathbf{K}_{\Gamma_e}
\end{bmatrix}.$$

We may check that the spectrum of $\mathbf{K}^*$ lies in the interval $[-1/2, 1/2]$. 

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Let $S$ be given by

$$S = \begin{bmatrix}
S_{\Gamma_i} & S_{\Gamma_e} \\
S_{\Gamma_i} & S_{\Gamma_e}
\end{bmatrix}. $$

- The operator $-S$ is self-adjoint and $-S \geq 0$ on $\mathcal{H}$.
- The Calderón’s identity is generalized.

$$SK^* = KS, $$

i.e., $SK^*$ is self-adjoint.
- $K^* \in C_2(\mathcal{H})$, Schatten-von Neumann class of compact operators.
We recall the result of Khavinson et al (2007)
Let $M \in C_p(\mathcal{H})$. If there exists a strictly positive bounded operator $R$ such that $R^2 M$ is self adjoint, then there is a bounded self-adjoint operator $A \in C_p(\mathcal{H})$ such that

$$AR = RM.$$ 

**Theorem**

There exists a bounded self-adjoint operator $A \in C_2(\mathcal{H})$ such that

$$A \sqrt{-S} = \sqrt{-SK^*}.$$
Limiting properties of the solution

- ALR occurs if and only if
  \[
  \int_{\Omega \setminus D} \delta |\nabla V_\delta|^2 \, dx \approx \delta \int_{\Omega \setminus D} |\nabla (S_{\Gamma i} [\varphi^\delta_i] + S_{\Gamma e} [\varphi^\delta_e])|^2 \, dx \to \infty \quad \text{as} \ \delta \to \infty.
  \]

- One can use

\[
A \sqrt{-S} = \sqrt{-SK^*}
\]

to obtain

\[
\int_{\Omega \setminus D} |\nabla (S_{\Gamma i} [\varphi^\delta_i] + S_{\Gamma e} [\varphi^\delta_e])|^2 \, dx = -\frac{1}{2} \langle \Phi_\delta, S\Phi_\delta \rangle + \langle K^* \Phi_\delta, S\Phi_\delta \rangle
\]

\[
= \frac{1}{2} \langle \sqrt{-S}\Phi_\delta, \sqrt{-S}\Phi_\delta \rangle - \langle A \sqrt{-S}\Phi_\delta, \sqrt{-S}\Phi_\delta \rangle.
\]
Since $\mathbb{A}$ is self-adjoint, we have an orthogonal decomposition

$$\mathcal{H} = \text{Ker}\mathbb{A} \oplus (\text{Ker}\mathbb{A})^\perp,$$

and $(\text{Ker}\mathbb{A})^\perp = \overline{\text{Range}\mathbb{A}}$. Let $P$ and $Q = I - P$ be the orthogonal projections from $\mathcal{H}$ onto $\text{Ker}\mathbb{A}$ and $(\text{Ker}\mathbb{A})^\perp$, respectively.

Let $\lambda_1, \lambda_2, \ldots$ with $|\lambda_1| \geq |\lambda_2| \geq \ldots$ be the nonzero eigenvalues of $\mathbb{A}$ and $\Psi_n$ be the corresponding (normalized) eigenfunctions. Since $\mathbb{A} \in C_2(\mathcal{H})$, we have

$$\sum_{n=1}^{\infty} \lambda_n^2 < \infty,$$

and

$$\mathbb{A}\Phi = \sum_{n=1}^{\infty} \lambda_n \langle \Phi, \Psi_n \rangle \Psi_n, \quad \Phi \in \mathcal{H}.$$
We apply $\sqrt{-S}$ to $(z_\delta \mathbb{I} + K^*)\Phi_\delta = g$ to obtain

$$(z_\delta \sqrt{-S} + \sqrt{-SK^*})\Phi_\delta = \sqrt{-S}g.$$  

Then

$$(z_\delta \mathbb{I} + A)\sqrt{-S}\Phi_\delta = \sqrt{-S}g.$$  

Projecting onto $\ker A$ and $(\ker A)^\perp$, we have

$$P\sqrt{-S}\Phi_\delta = \frac{1}{z_\delta} P\sqrt{-S}g,$$

$$Q\sqrt{-S}\Phi_\delta = \sum_n \frac{\langle Q\sqrt{-S}g, \psi_n \rangle}{\lambda_n + z_\delta} \psi_n.$$  

We also get

$$A\sqrt{-S}\Phi_\delta = \sum_n \frac{\lambda_n \langle Q\sqrt{-S}g, \psi_n \rangle}{\lambda_n + z_\delta} \psi_n.$$
We have
\[
\int_{\Omega \setminus D} \left| \nabla (S_{\Gamma_i} [\varphi_i^\delta] + S_{\Gamma_e} [\varphi_e^\delta]) \right|^2 \, dx = \frac{1}{2} \left\langle \sqrt{-S} \Phi_\delta, \sqrt{-S} \Phi_\delta \right\rangle - \left\langle A \sqrt{-S} \Phi_\delta, \sqrt{-S} \Phi_\delta \right\rangle \\
\approx \frac{1}{\delta^2} \| P \sqrt{-S} g \|^2 + \sum_n \frac{|\left\langle Q \sqrt{-S} g, \psi_n \right\rangle|^2}{|\lambda_n|^2 + \delta^2}.
\]

Let \( \Phi_n \) be the (normalized) eigenfunctions of \( K^* \).

**Theorem**

If \( P \sqrt{-S} g \neq 0 \), then LR takes place. If \( \text{Ker}(K^*) = \{0\} \), then ALR takes place if and only if
\[
\delta \sum_n \frac{|\left\langle S g, \Phi_n \right\rangle|^2}{\lambda_n^2 + \delta^2} \to \infty \quad \text{as} \ \delta \to 0.
\]
Anomalous resonance in annulus

The above theorem gives a necessary and sufficient condition on the source term \( f \) for the blow up of the electromagnetic energy in \( \Omega \setminus \overline{D} \). This condition is in terms of the Newton potential of \( f \).

We explicitly compute eigenvalues and eigenfunctions of \( A \) for the case of an annulus configuration. We consider the anomalous resonance when domains \( \Omega \) and \( D \) are concentric disks. We calculate the explicit form of the limiting solution. Throughout this section, we set \( \Omega = B_e = \{ |x| < r_e \} \) and \( D = B_i = \{ |x| < r_i \} \), where \( r_e > r_i \).

**Lemma**

Let \( \rho := \frac{r_i}{r_e} \). Then

\[
\text{Ker} \ K^* = \{0\}
\]

and the eigenvalues of \( A \) are \( \{ \pm \rho |n| \} \).
• Let $\frac{\partial F}{\partial \nu_e} = \sum_{n \neq 0} g_n e^{in\theta}$. There exists $\delta_0$ such that

$$E_\delta := \int_{B_e \setminus B_i} \delta |\nabla V_\delta|^2 \approx \sum_{n \neq 0} \frac{\delta |g_n^e|^2}{|n|(\delta^2 + \rho^2 |n|)}$$

uniformly in $\delta \leq \delta_0$.

• $\limsup_{|n| \to \infty} \frac{|g_n^e|^2}{|n| \rho |n|} = \infty$ implies only $\limsup_{\delta \to 0} E_\delta = \infty$

(pointed out by J. Lu and J. Jorgensen).
GP : There exists a sequence \( \{ n_k \} \) with \(|n_1| < |n_2| < \cdots \) such that

\[
\lim_{k \to \infty} \rho |n_{k+1} - n_k| \frac{|g^n_{e_k}|^2}{|n_k| \rho |n_k|} = \infty.
\]

Lemma

If \( \{ g^n_e \} \) satisfies the condition GP, then

\[
\lim_{\delta \to 0} E_\delta = \infty.
\]

- If \( \lim_{n \to \infty} \frac{|g^n_e|^2}{|n| \rho |n|} = \infty \), then \( \lim_{\delta \to 0} E_\delta = \infty \).
Suppose that the source function is supported inside the radius \( r_* = \sqrt{r_e^3 r_i^{-1}} \). Then its Newtonian potential cannot be extended harmonically in \(|x| < r_*\) in general. So, if \( F \) is given by

\[
F = c - \sum_{n \neq 0} a_n |n| r |e^{in\theta}, \quad r < r_e,
\]

then the radius of convergence is less than \( r_* \). Thus we have

\[
\limsup_{|n| \to \infty} |n| |a_n|^2 r_*^2 |n| = \infty,
\]

and \( \limsup_{|n| \to \infty} \frac{|g_e^n|^2}{|n| |\rho| n|} = \infty \) holds. The GP condition is equivalent to that there exists \( \{n_k\} \) with \(|n_1| < |n_2| < \cdots \) such that

\[
\lim_{k \to \infty} \rho^{n_{k+1}-n_k} |n_k| |a_{n_k}|^2 r_*^2 |n_k| = \infty.
\]
The following is the main theorem.

**Theorem**

Let $f$ be a source function supported in $\mathbb{R}^2 \setminus \overline{B}_e$ and $F$ be the Newtonian potential of $f$.

(i) If $F$ does not extend as a harmonic function in $B_{r^*}$, then weak ALR occurs, i.e.,

$$\limsup_{\delta \to 0} E_\delta = \infty.$$  

(ii) If the Fourier coefficients of $F$ satisfy GP, then ALR occurs, i.e.,

$$\lim_{\delta \to 0} E_\delta = \infty.$$  

(iii) If $F$ extends as a harmonic function in a neighborhood of $\overline{B}_{r^*}$, then ALR does not occur, i.e.,

$$E_\delta < C$$  

for some $C$ independent of $\delta$.  

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Examples

• If $f$ is a dipole source in $B_{r_*} \setminus \overline{B_e}$, i.e., $f(x) = a \cdot \nabla \delta_y(x)$ for a vector $a$ and $y \in B_{r_*} \setminus \overline{B_e}$ where $\delta_y$ is the Dirac delta function at $y$. Then $F(x) = a \cdot \nabla G(x - y)$ and the ALR takes place. This was found by Milton et al.

• If $f$ is a quadrapole, i.e., $f(x) = A : \nabla \nabla \delta_y(x) = \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \delta_y(x)$ for a $2 \times 2$ matrix $A = (a_{ij})$ and $y \in B_{r_*} \setminus \overline{B_e}$. Then $F(x) = \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2 G(x-y)}{\partial x_i \partial x_j}$. Thus the ALR takes place.
If $f$ is supported in $\mathbb{R}^2 \setminus \overline{B}_{r^*}$, then $F$ is harmonic in a neighborhood of $\overline{B}_{r^*}$, and hence the ALR does not occur. In fact, we can say more about the behavior of the solution $V_\delta$ as $\delta \to 0$.

**Theorem**

If $f$ is supported in $\mathbb{R}^2 \setminus \overline{B}_{r^*}$, then

$$\int_{B_e \setminus B_i} \delta |\nabla V_\delta|^2 < C.$$  

Moreover,

$$\sup_{|x| \geq r^*} |V_\delta(x) - F(x)| \to 0 \quad \text{as} \quad \delta \to 0.$$
Problems

- How can we describe the cloaking effect when some inclusion is immersed?
- How can we analyze ALR explicitly in terms of the source term when the given geometry is general?
References

Thank you!