Analysis of the anomalous localized resonance

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Outline

- Introduction
- Integral operators and its symmetry
- Spectral analysis of ALR
- ALR in annulus region

Surface plasmon

Let

$$\epsilon = \begin{cases} 1 & \text{in } \{(x,y) : y \geq 0\}, \\ -1 & \text{in } \{(x,y) : y < 0\}. \end{cases}$$

Consider

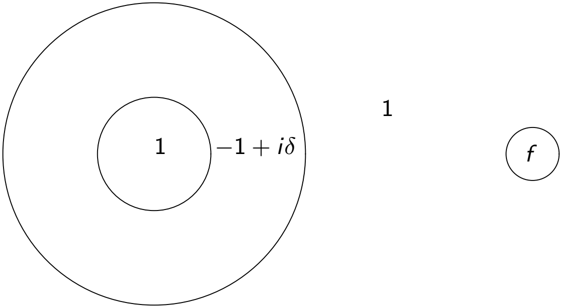
$$\nabla \cdot \epsilon \nabla u = 0 \quad \text{in } \mathbb{R}^2.$$

Then one solution is

$$u = \begin{cases} e^{-y+ix} & \text{in } \{(x,y) : y \geq 0\}, \\ e^{y+ix} & \text{in } \{(x,y) : y < 0\}. \end{cases}$$

Let Ω be a smooth domain in $\mathbb{R}^2$ and let $D \subset \Omega$. The permittivity distribution in $\mathbb{R}^2$ is given by

$$\epsilon_\delta = \begin{cases} 1 & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\ -1 + i\delta & \text{in } \Omega \setminus \overline{D}, \\ 1 & \text{in } D. \end{cases}$$



Problem

For a given function f compactly supported in $\mathbb{R}^2$ satisfying

$$\int_{\mathbb{R}^2} f dx = 0,$$

we consider the following equation:

$$\nabla \cdot \epsilon_\delta \nabla V_\delta = f \quad \text{in } \mathbb{R}^2,$$

with decaying condition $V_\delta(x) \to 0$ as $|x| \to \infty$.

Since the equation degenerates as $\delta \to 0$, we can expect some singular behavior of the solution, depending on the source term f.

Milton-Nicorovici(2006)

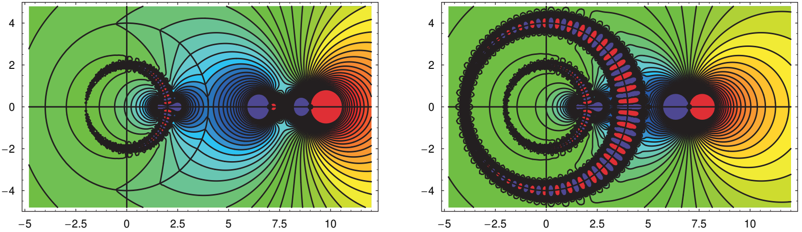


Figure: Anomalous resonance, Milton *et al* (2006).

- Energy concentration near interfaces, depending on the location of source.
- Associated with the cloaking effect of polarizable dipole.
- Generalized to a small inclusion with a specific boundary condition by Bouchitté and B. Schweizer(2010).

Numerical simulation by Bruno-Linter(2007).

- There is some cloaking effect even in the presence of a small dielectric inclusion, not perfect.
- Blow-up may not depend on the location of the source in a layer of general shape.

A fundamental problem is to find a region Ω^* containing Ω such that if f is supported in $\Omega^* \setminus \overline{\Omega}$, then

$$\int_{\Omega\setminus\overline{D}} \delta|\nabla V_\delta|^2 dx \to \infty \quad \text{as } \delta \to 0.$$

- Such a region $\Omega^* \setminus \overline{\Omega}$ is called the anomalous resonance region or cloaking region. The quantity $\int_{\Omega\setminus D} \delta|\nabla V_\delta|^2 dx$ is a part of the absorbed energy.

- The blow-up of the energy may or may not occur depending on f. So the problem is not only finding the anomalous resonance region $\Omega^* \setminus \overline{\Omega}$ but also characterizing those source terms f which actually make the anomalous resonance happen.

Relation to cloaking

Suppose f is a polarizable dipole at x_0, *i.e.*,

$$V_\delta(x) = U_\delta(x) + A_\delta \cdot \nabla G(x - x_0), \quad A_\delta = k \nabla U_\delta(x_0),$$

for some given coefficient k.

If ALR happens, then we should have

$$A_\delta \to 0 \quad \text{as } \delta \to 0.$$

Otherwise $\int_{\Omega \setminus \overline{D}} \delta |\nabla V_\delta|^2 dx$ blows up, which is not physical.

Let F be the Newtonian potential of f, *i.e.*,

$$F(x) = \int_{\mathbb{R}^2} G(x-y)f(y)dy, \quad x \in \mathbb{R}^2.$$

Then F satisfies $\Delta F = f$ in $\mathbb{R}^2$, and the solution V_δ may be represented as

$$V_\delta(x) = F(x) + \mathcal{S}_{\Gamma_i}[\varphi_i](x) + \mathcal{S}_{\Gamma_e}[\varphi_e](x)$$

for some functions $\varphi_i \in L^2_0(\Gamma_i)$ and $\varphi_e \in L^2_0(\Gamma_e)$ (L^2_0 is the collection of all square integrable functions with the integral zero).

The transmission conditions along the interfaces Γ_e and Γ_i satisfied by V_δ read

$$(-1+i\delta)\frac{\partial V_\delta}{\partial \nu}\Big|_+ = \frac{\partial V_\delta}{\partial \nu}\Big|_- \quad \text{on } \Gamma_i$$
$$\frac{\partial V_\delta}{\partial \nu}\Big|_+ = (-1+i\delta)\frac{\partial V_\delta}{\partial \nu}\Big|_- \quad \text{on } \Gamma_e.$$

Using the jump formula for the normal derivative of the single layer potentials, the pair of potentials (φ_i, φ_e) is the solution to

$$\begin{bmatrix} z_\delta I - \mathcal{K}^*_{\Gamma_i} & -\dfrac{\partial}{\partial \nu_i}\mathcal{S}_{\Gamma_e} \\ \dfrac{\partial}{\partial \nu_e}\mathcal{S}_{\Gamma_i} & z_\delta I + \mathcal{K}^*_{\Gamma_e} \end{bmatrix} \begin{bmatrix} \varphi_i \\ \varphi_e \end{bmatrix} = \begin{bmatrix} \dfrac{\partial F}{\partial \nu_i} \\ -\dfrac{\partial F}{\partial \nu_e} \end{bmatrix}.$$

on $L^2_0(\Gamma_i) \times L^2_0(\Gamma_e)$, where we set

$$z_\delta = \frac{i\delta}{2(2 - i\delta)}.$$

Note that the operator can be viewed as a compact perturbation of the operator

$$\begin{bmatrix} z_\delta I - \mathcal{K}^*_{\Gamma_i} & 0 \\ 0 & z_\delta I + \mathcal{K}^*_{\Gamma_e} \end{bmatrix}.$$

- We now recall Kellogg's result on the spectrums of $\mathcal{K}^*_{\Gamma_i}$ and $\mathcal{K}^*_{\Gamma_e}$. The eigenvalues of $\mathcal{K}^*_{\Gamma_i}$ and $\mathcal{K}^*_{\Gamma_e}$ lie in the interval $]-\frac{1}{2}, \frac{1}{2}]$.

- Observe that $z_\delta \to 0$ as $\delta \to 0$ and that there are sequences of eigenvalues of $\mathcal{K}^*_{\Gamma_i}$ and $\mathcal{K}^*_{\Gamma_e}$ approaching to 0 since $\mathcal{K}^*_{\Gamma_i}$ and $\mathcal{K}^*_{\Gamma_e}$ are compact. So 0 is the essential singularity of the operator valued meromorphic function

$$\lambda \in \mathbb{C} \mapsto (\lambda I + \mathcal{K}^*_{\Gamma_e})^{-1}.$$

  This causes a serious difficulty in dealing with (11).

- We emphasize that $\mathcal{K}^*_{\Gamma_e}$ is not self-adjoint in general. In fact, $\mathcal{K}^*_{\Gamma_e}$ is self-adjoint only when Γ_e is a circle or a sphere.

Properties of $\mathbb{K}^*$

Let $\mathcal{H} = L^2(\Gamma_i) \times L^2(\Gamma_e)$. Let the Neumann-Poincaré-type operator $\mathbb{K}^* : \mathcal{H} \to \mathcal{H}$ be defined by

$$\mathbb{K}^* := \begin{bmatrix} -\mathcal{K}^*_{\Gamma_i} & -\dfrac{\partial}{\partial \nu_i}\mathcal{S}_{\Gamma_e} \\ \dfrac{\partial}{\partial \nu_e}\mathcal{S}_{\Gamma_i} & \mathcal{K}^*_{\Gamma_e} \end{bmatrix}.$$

Then the integral equation can be written as

$$(z_\delta \mathbb{I} + \mathbb{K}^*)\Phi_\delta = g$$

and the L^2-adjoint of $\mathbb{K}^*$, $\mathbb{K}$, is given by

$$\mathbb{K} = \begin{bmatrix} -\mathcal{K}_{\Gamma_i} & \mathcal{D}_{\Gamma_e} \\ -\mathcal{D}_{\Gamma_i} & \mathcal{K}_{\Gamma_e} \end{bmatrix}.$$

We may check that the spectrum of $\mathbb{K}^*$ lies in the interval $[-1/2, 1/2]$.

Let $\mathbb{S}$ be given by

$$\mathbb{S} = \begin{bmatrix} \mathcal{S}_{\Gamma_i} & \mathcal{S}_{\Gamma_e} \\ \mathcal{S}_{\Gamma_i} & \mathcal{S}_{\Gamma_e} \end{bmatrix}.$$

- The operator $-\mathbb{S}$ is self-adjoint and $-\mathbb{S} \geq 0$ on $\mathcal{H}$.
- The Calderón's identity is generalized.

$$\mathbb{S}\mathbb{K}^* = \mathbb{K}\mathbb{S},$$

  i.e., $\mathbb{S}\mathbb{K}^*$ is self-adjoint.
- $\mathbb{K}^* \in \mathcal{C}_2(\mathcal{H})$, Schatten-von Neumann class of compact operators.

We recall the result of Khavinson *et al*(2007)
Let $M \in \mathcal{C}_p(\mathcal{H})$. If there exists a strictly positive bounded operator R such that $R^2 M$ is self adjoint, then there is a bounded self-adjoint operator $A \in \mathcal{C}_p(\mathcal{H})$ such that

$$AR = RM.$$

Theorem

There exists a bounded self-adjoint operator $\mathbb{A} \in \mathcal{C}_2(\mathcal{H})$ *such that*

$$\mathbb{A}\sqrt{-\mathbb{S}} = \sqrt{-\mathbb{S}}\mathbb{K}^*.$$

Limiting properties of the solution

- ALR occurs if and only if

$$\int_{\Omega\setminus\overline{D}} \delta|\nabla V_\delta|^2 dx \approx \delta \int_{\Omega\setminus\overline{D}} \left|\nabla(\mathcal{S}_{\Gamma_i}[\varphi_i^\delta] + \mathcal{S}_{\Gamma_e}[\varphi_e^\delta])\right|^2 dx \to \infty \quad \text{as } \delta \to \infty.$$

- One can use

$$\mathbb{A}\sqrt{-\mathbb{S}} = \sqrt{-\mathbb{S}}\mathbb{K}^*$$

to obtain

$$\int_{\Omega\setminus\overline{D}} \left|\nabla(\mathcal{S}_{\Gamma_i}[\varphi_i^\delta] + \mathcal{S}_{\Gamma_e}[\varphi_e^\delta])\right|^2 dx = -\frac{1}{2}\langle \Phi_\delta, \mathbb{S}\Phi_\delta\rangle + \langle \mathbb{K}^*\Phi_\delta, \mathbb{S}\Phi_\delta\rangle$$

$$= \frac{1}{2}\langle \sqrt{-\mathbb{S}}\Phi_\delta, \sqrt{-\mathbb{S}}\Phi_\delta\rangle - \langle \mathbb{A}\sqrt{-\mathbb{S}}\Phi_\delta, \sqrt{-\mathbb{S}}\Phi_\delta\rangle.$$

Since $\mathbb{A}$ is self-adjoint, we have an orthogonal decomposition

$$\mathcal{H} = \mathrm{Ker}\mathbb{A} \oplus (\mathrm{Ker}\mathbb{A})^{\perp},$$

and $(\mathrm{Ker}\mathbb{A})^{\perp} = \overline{\mathrm{Range}\mathbb{A}}$. Let P and $Q = I - P$ be the orthogonal projections from $\mathcal{H}$ onto $\mathrm{Ker}\mathbb{A}$ and $(\mathrm{Ker}\mathbb{A})^{\perp}$, respectively.

Let $\lambda_1, \lambda_2, \ldots$ with $|\lambda_1| \geq |\lambda_2| \geq \ldots$ be the nonzero eigenvalues of $\mathbb{A}$ and Ψ_n be the corresponding (normalized) eigenfunctions. Since $\mathbb{A} \in \mathcal{C}_2(\mathcal{H})$, we have

$$\sum_{n=1}^{\infty} \lambda_n^2 < \infty,$$

and

$$\mathbb{A}\Phi = \sum_{n=1}^{\infty} \lambda_n \langle \Phi, \Psi_n \rangle \Psi_n, \quad \Phi \in \mathcal{H}$$

We apply $\sqrt{-\mathbb{S}}$ to $(z_\delta \mathbb{I} + \mathbb{K}^*)\Phi_\delta = g$ to obtain

$$(z_\delta \sqrt{-\mathbb{S}} + \sqrt{-\mathbb{S}}\mathbb{K}^*)\Phi_\delta = \sqrt{-\mathbb{S}} g.$$

Then

$$(z_\delta \mathbb{I} + \mathbb{A})\sqrt{-\mathbb{S}}\Phi_\delta = \sqrt{-\mathbb{S}} g.$$

Projecting onto $\mathrm{Ker}\mathbb{A}$ and $(\mathrm{Ker}\mathbb{A})^\perp$, we have

$$P\sqrt{-\mathbb{S}}\Phi_\delta = \frac{1}{z_\delta} P\sqrt{-\mathbb{S}} g,$$

$$Q\sqrt{-\mathbb{S}}\Phi_\delta = \sum_n \frac{\langle Q\sqrt{-\mathbb{S}} g, \Psi_n \rangle}{\lambda_n + z_\delta} \Psi_n.$$

We also get

$$\mathbb{A}\sqrt{-\mathbb{S}}\Phi_\delta = \sum_n \frac{\lambda_n \langle Q\sqrt{-\mathbb{S}} g, \Psi_n \rangle}{\lambda_n + z_\delta} \Psi_n.$$

We have

$$\int_{\Omega\setminus\overline{D}} \left|\nabla(\mathcal{S}_{\Gamma_i}[\varphi_i^\delta] + \mathcal{S}_{\Gamma_e}[\varphi_e^\delta])\right|^2 dx = \frac{1}{2}\langle\sqrt{-\mathbb{S}}\Phi_\delta, \sqrt{-\mathbb{S}}\Phi_\delta\rangle - \langle\mathbb{A}\sqrt{-\mathbb{S}}\Phi_\delta, \sqrt{-\mathbb{S}}\Phi_\delta\rangle$$

$$\approx \frac{1}{\delta^2}\|P\sqrt{-\mathbb{S}}g\|^2 + \sum_n \frac{|\langle Q\sqrt{-\mathbb{S}}g, \Psi_n\rangle|^2}{|\lambda_n|^2 + \delta^2}.$$

Let Φ_n be the (normalized) eigenfunctions of $\mathbb{K}^*$.

Theorem

If $P\sqrt{-\mathbb{S}}g \neq 0$, then LR takes place. If $Ker(\mathbb{K}^*) = \{0\}$, then ALR takes place if and only if

$$\delta\sum_n \frac{|\langle\mathbb{S}g, \Phi_n\rangle|^2}{\lambda_n^2 + \delta^2} \to \infty \quad \text{as } \delta \to 0.$$

Anomalous resonance in annulus

The above theorem gives a necessary and sufficient condition on the source term f for the blow up of the electromagnetic energy in $\Omega \setminus \overline{D}$. This condition is in terms of the Newton potential of f.

We explicitly compute eigenvalues and eigenfunctions of $\mathbb{A}$ for the case of an annulus configuration. We consider the anomalous resonance when domains Ω and D are concentric disks. We calculate the explicit form of the limiting solution. Throughout this section, we set $\Omega = B_e = \{|x| < r_e\}$ and $D = B_i = \{|x| < r_i\}$, where $r_e > r_i$.

Lemma

Let $\rho := \frac{r_i}{r_e}$. *Then*

$$\operatorname{Ker} \mathbb{K}^* = \{0\}$$

and the eigenvalues of $\mathbb{A}$ *are* $\{\pm\rho^{|n|}\}$.

- Let $\frac{\partial F}{\partial \nu_e} = \sum_{n\neq 0} g_e^n e^{in\theta}$. There exists δ_0 such that

$$E_\delta := \int_{B_e \setminus \overline{B_i}} \delta |\nabla V_\delta|^2 \approx \sum_{n\neq 0} \frac{\delta |g_e^n|^2}{|n|(\delta^2 + \rho^{2|n|})}$$

uniformly in $\delta \leq \delta_0$.

- $\limsup_{|n|\to\infty} \frac{|g_e^n|^2}{|n|\rho^{|n|}} = \infty$ implies only $\limsup_{\delta\to 0} E_\delta = \infty$ (pointed out by J. Lu and J. Jorgensen).

GP : There exists a sequence $\{n_k\}$ with $|n_1| < |n_2| < \cdots$ such that

$$\lim_{k\to\infty} \rho^{|n_{k+1}|-|n_k|} \frac{|g_e^{n_k}|^2}{|n_k|\rho^{|n_k|}} = \infty.$$

Lemma

If $\{g_e^n\}$ satisfies the condition GP, then

$$\lim_{\delta\to 0} E_\delta = \infty.$$

- If $\lim_{n\to\infty} \frac{|g_e^n|^2}{|n|\rho^{|n|}} = \infty$, then $\lim_{\delta\to 0} E_\delta = \infty$.

Suppose that the source function is supported inside the radius $r_* = \sqrt{r_e^3 r_i^{-1}}$. Then its Newtonian potential cannot be extended harmonically in $|x| < r_*$ in general. So, if F is given by

$$F = c - \sum_{n \neq 0} a_n r^{|n|} e^{in\theta}, \quad r < r_e,$$

then the radius of convergence is less than r_*. Thus we have

$$\limsup_{|n| \to \infty} |n| |a_n|^2 r_*^{2|n|} = \infty,$$

and $\limsup_{|n| \to \infty} \frac{|g_e^n|^2}{|n| \rho^{|n|}} = \infty$ holds. The GP condition is equivalent to that there exists $\{n_k\}$ with $|n_1| < |n_2| < \cdots$ such that

$$\lim_{k \to \infty} \rho^{|n_{k+1}| - |n_k|} |n_k| |a_{n_k}|^2 r_*^{2|n_k|} = \infty.$$

The following is the main theorem.

Theorem

Let f be a source function supported in $\mathbb{R}^2 \setminus \overline{B_e}$ and F be the Newtonian potential of f.

(i) *If F does not extend as a harmonic function in B_{r_*}, then weak ALR occurs, i.e.,*

$$\limsup_{\delta \to 0} E_\delta = \infty.$$

(ii) *If the Fourier coefficients of F satisfy GP, then ALR occurs, i.e.,*

$$\lim_{\delta \to 0} E_\delta = \infty.$$

(iii) *If F extends as a harmonic function in a neighborhood of $\overline{B_{r_*}}$, then ALR does not occur, i.e.,*

$$E_\delta < C$$

for some C independent of δ.

Examples

- If f is a dipole source in $B_{r_*} \setminus \overline{B}_e$, *i.e.*, $f(x) = a \cdot \nabla \delta_y(x)$ for a vector a and $y \in B_{r_*} \setminus \overline{B}_e$ where δ_y is the Dirac delta function at y. Then $F(x) = a \cdot \nabla G(x-y)$ and the ALR takes place. This was found by Milton *et al*.

- If f is a quadrapole, *i.e.*, $f(x) = A : \nabla\nabla\delta_y(x) = \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \delta_y(x)$ for a 2×2 matrix $A = (a_{ij})$ and $y \in B_{r_*} \setminus \overline{B}_e$. Then $F(x) = \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2 G(x-y)}{\partial x_i \partial x_j}$. Thus the ALR takes place.

If f is supported in $\mathbb{R}^2 \setminus \overline{B}_{r_*}$, then F is harmonic in a neighborhood of $\overline{B}_{r_*}$, and hence the ALR does not occur. In fact, we can say more about the behavior of the solution V_δ as $\delta \to 0$.

Theorem

If f is supported in $\mathbb{R}^2 \setminus \overline{B}_{r_*}$, then

$$\int_{B_e \setminus B_i} \delta |\nabla V_\delta|^2 < C.$$

Moreover,

$$\sup_{|x| \geq r_*} |V_\delta(x) - F(x)| \to 0 \quad \textit{as} \quad \delta \to 0.$$

Problems

- How can we describe the cloaking effect when some inclusion is immersed?
- How can we analyze ALR explicitly in terms of the source term when the given geometry is general?

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Thank you!