Enhancement of near-cloaking using multilayer structures

Mikyoung LIM
(KAIST)

June 23, 2012

This talk is based on the joint work with
Habib Ammari (Ecole Normale Superieure), Josselin Garnier (Université Paris VII), Vincent Jugnon (MIT), Hyeonbae Kang (Inha Univ.), Hyundae Lee (Inha Univ.).

Outline

- Cloaking and near cloaking
- Generalized Polarization Tensors (GPT)
- GPT vanishing structures and near cloaking
- Helmholtz Equation
- Scattering Coefficients vanishing structures

Transformation of PDE

- Let $\Lambda[\sigma]$ be the Dirichlet-to-Neumann map corresponding to the conductivity distribution σ, *i.e.*,

$$\Lambda[\sigma](\phi) = \sigma \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}$$

where u is the solution to

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0, & \text{in } \Omega, \\ u = \phi, & \text{on } \partial\Omega. \end{cases}$$

- If F is a diffeomorphism of Ω which is identity on $\partial\Omega$, then

$$\Lambda[\sigma] = \Lambda[F_*\sigma]$$

where $F_*\sigma$ is the push-forward of σ by F:

$$F_*\sigma(y) = \frac{DF(x)\sigma(x)DF(x)^t}{\det(DF(x))}, \quad x = F^{-1}(y).$$

Singular transformation

by Greenleaf-Lassas-Uhlmann (2003)

- Define $F : \{x : 0 < |x| < 2\} \to \{x : 1 < |x| < 2\}$ by

$$F(x) := \left(1 + \frac{|x|}{2}\right) \frac{x}{|x|}.$$

- Then, $\Lambda[1] = \Lambda[F_* 1]$.
- Things inside $\{|x| < 1\}$ are cloaked by the DtN map.
- Pendry et al (2006) used exactly the same transformation for electromagnetic cloaking (transformation optics).
- Further development toward acoustic and electromagnetic cloaking: Greenleaf-Kurylev-Lassas-Uhlmann (2009).
- $F_* 1$ is singular on $|x| = 1$ (0 in the normal direction, ∞ in tangential direction, 2D)

Near cloaking

Blowing-up a small ball (Kohn-Shen-Vogelius-Weinstein (2008))

- For a small number ρ, let

$$\sigma = \begin{cases} \gamma & \text{if } |x| < \rho, \\ 1 & \text{if } \rho \leq |x| \leq 2. \end{cases}$$

  (γ can be 0 (the core is insulated) or ∞ (perfect conductor))

- Let

$$F(x) = \begin{cases} \left(\dfrac{2-2\rho}{2-\rho} + \dfrac{1}{2-\rho}|x| \right) \dfrac{x}{|x|} & \text{if } \rho \leq |x| \leq 2, \\ \dfrac{x}{\rho} & \text{if } |x| \leq \rho. \end{cases}$$

  Then F maps B_2 onto B_2 and blows up B_ρ onto B_1.

- Then,
$$\|\Lambda[F_*\sigma] - \Lambda[1]\| \leq C\rho^2.$$
- Further development toward acoustic cloaking: Kohn-Onofrei-Vogelius-Weinstein, Liu, Nguyen.

- $\Lambda[F_*\sigma] = \Lambda[\sigma]$ and

$$\Lambda[\sigma](\phi)(x) = \Lambda[1](\phi)(x) + \nabla U(0) \cdot M \frac{\partial}{\partial \nu_x} \nabla_y G(x,0) + \text{h.o.t}, \quad x \in \partial\Omega,$$

  where U is the solution to

$$\begin{cases} \Delta U = 0 & \text{in } \Omega, \\ U = \phi & \text{on } \partial\Omega, \end{cases}$$

  M is the polarization tensor of B_ρ, and $G(x,y)$ is the Green function for Ω.

- PT for a ball B_ρ (with conductivity γ): $M = \dfrac{2(\gamma-1)}{\gamma+1}|B_\rho| I$.

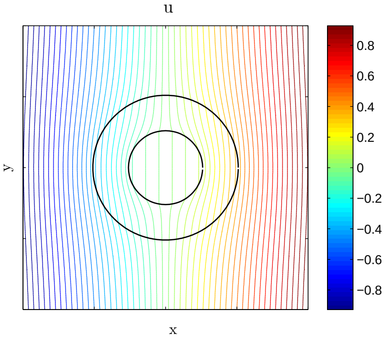
- Is it possible to make PT vanish by taking a shape other than a circle? (If so, we may achieve an enhanced near cloaking.)
- Not possible by simply connected shape with constant conductivity because of Hashin-Shtrikman bounds for PT (proved by Lipton (93), Capdeboscq-Vogelius (03)): Let $M = M(\gamma, D)$ be the PT for D. Then

$$\mathrm{Tr}(M) \leq |D|(\gamma - 1)(1 + \frac{1}{\gamma}),$$

and

$$|D|\mathrm{Tr}(M^{-1}) \leq \frac{1+\gamma}{\gamma - 1}.$$

Neutral inclusion of Hashine



Neutral inclusion does not perturb the uniform fields outside the inclusion.

Generalized Polarization Tensors

Conductivity distribution:

$$\sigma = \chi(\mathbb{R}^d \setminus \overline{\Omega}) + \gamma\chi(\Omega).$$

Suppose Ω is a single inclusion or multiple inclusions and consider

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0 \quad \text{in } \mathbb{R}^d, \\ u(x) - a \cdot x = O(|x|^{1-d}) \quad \text{as } |x| \to \infty. \end{cases}$$

The dipolar asymptotic expansion at infinity:

$$u(x) = a \cdot x - \frac{1}{\omega_d} \frac{\langle a, Mx \rangle}{|x|^d} + O(|x|^{-d}), \quad \text{as } |x| \to \infty.$$

$M = M(k, \Omega) = (m_{ij})$: the Polarization Tensor associated with Ω (or more precisely σ).

For a given entire harmonic function H, consider

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0 & \text{in } \mathbb{R}^d, \\ u(x) - H(x) = O(|x|^{1-d}) & \text{as } |x| \to \infty. \end{cases}$$

Multipolar expansions:

$$u(x) = H(x) + \sum_\alpha \sum_\beta \frac{(-1)^{|\beta|}}{\alpha!\beta!} \partial^\alpha H(0) m_{\alpha\beta} \partial^\beta \Gamma(x), \quad |x| \to \infty.$$

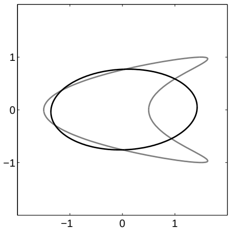
$\{m_{\alpha\beta}\}$: *Generalized* Polarization Tensors (GPT).
($\Gamma(x)$: the fundamental solution for the Laplacian.)

Equivalent ellipse

If γ is constant, then there is a canonical correspondence between the class of ellipses (ellipsoids) and the class of PTs:
If Ω is an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1$, then

$$M(\gamma,\Omega)=(\gamma-1)|\Omega|\begin{bmatrix}\dfrac{a+b}{a+\gamma b} & 0\\ 0 & \dfrac{a+b}{b+\gamma a}\end{bmatrix}.$$

Equivalent ellipse (ellipsoid)= ellipse with the same PT:



GPT and Imaging

by Ammari-Kang-L-Zribi

Aim: Make use of $\displaystyle\sum_{|\alpha|+|\beta|\leq K} a_\alpha b_\beta m_{\alpha\beta}$ for a fixed $K \geq 2$ to image finer details of the shape of the inclusion.

- If $K = 2$, it is imaging by PT (equivalent ellipse).

Optimization Problem: Let Ω be the target domain. Minimize over D

$$J[D] := \frac{1}{2} \sum_{|\alpha|+|\beta|\leq K} w_{|\alpha|+|\beta|} \left| \sum_{\alpha,\beta} a_\alpha b_\beta m_{\alpha\beta}(\gamma, D) - \sum_{\alpha,\beta} a_\alpha b_\beta m_{\alpha\beta}(\gamma, \Omega) \right|^2 .$$

- $w_{|\alpha|+|\beta|}$ are binary weights: $w_{|\alpha|+|\beta|} = 1$(on) or 0(off).
- A good choice for the initial guess: the equivalent ellipse.

Gradient Descent Method

To get a minimum of $F : \mathbb{R}^m \to \mathbb{R}$, one starts with an initial guess x_0 and modify it as

$$x_{n+1} = x_n - \gamma_n \nabla F(x_n),$$

where γ_n is a positive real number.
Note that $\nabla F(x) = \sum_{j=1}^{m} \frac{d}{dt} F(x + t\mathbf{e}_j)\Big|_{t=0} \mathbf{e}_j$.

To approximate the inclusion, modify $D^{(0)}$ (initial guess) as

$$\partial D^{(n+1)} = \partial D^{(n)} - \gamma_n \left(\sum_j \langle d_S J[D^{(n)}], \psi_j \rangle \psi_j \right) \nu,$$

where ν is the outward normal direction to $\partial D^{(n)}$ and $d_S J$ is the shape derivative.

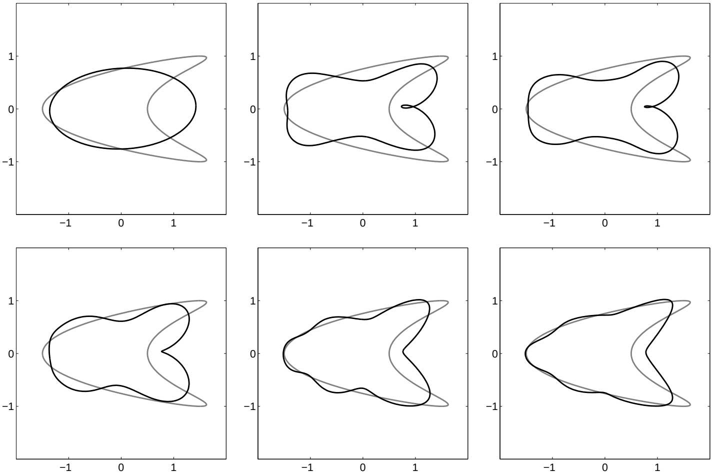


Figure: $K = 6$, 6 iterations.

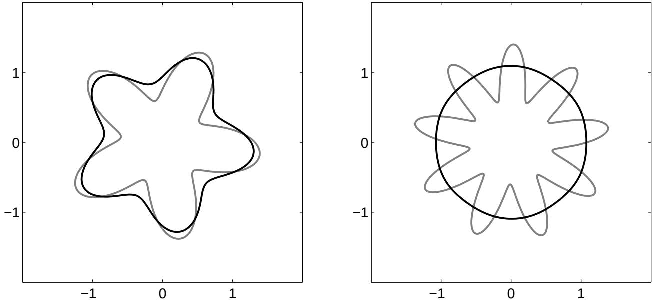
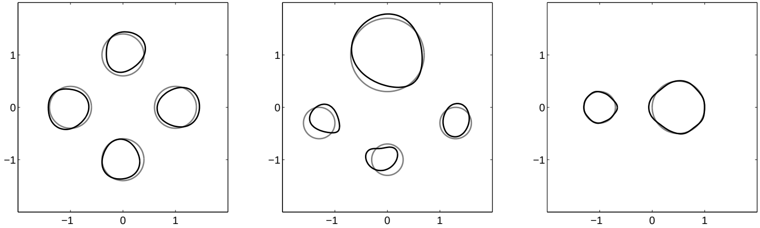


Figure: After 6 iterations

Level-set framework

by Ammari-Garnier-Kang-L-Yu



Harmonic Sums

Let u be the solution to

$$\begin{cases} \nabla \cdot \chi(\mathbb{R}^d \setminus \overline{\Omega}) \nabla u = 0 & \text{in } \mathbb{R}^2, \\ u(x) - H(x) = O(|x|^{-1}) & \text{as } |x| \to \infty. \end{cases}$$

Multipolar expansions:

$$(u - H)(x) = \sum_{\alpha} \sum_{\beta} \frac{(-1)^{|\beta|}}{\alpha! \beta!} \partial^{\alpha} H(0) m_{\alpha\beta} \partial^{\beta} \Gamma(x), \quad |x| \to \infty.$$

- Harmonic sums: $\sum_{\alpha,\beta} a_\alpha b_\beta m_{\alpha\beta}(\gamma, \Omega)$ with $\sum_\alpha a_\alpha x^\alpha$ and $\sum_\beta b_\beta x^\beta$ are harmonic polynomials.
- Denote the harmonic sums as $M^{cc}_{mn}, M^{cs}_{mn}, M^{sc}_{mn}, M^{ss}_{mn}$.
- As $|x| \to \infty$,

$$(u - H)(x) = -\sum_{m,n=1}^{\infty} \left[\frac{\cos m\theta}{2\pi m |x|^m} (M^{cc}_{mn} a^c_n + M^{cs}_{mn} a^s_n) + \frac{\sin m\theta}{2\pi m |x|^m} (M^{sc}_{mn} a^c_n + M^{ss}_{mn} a^s_n) \right]$$

where $H(x) = H(0) + \sum_{n=1}^{\infty} |x|^n (a^c_n \cos n\theta + a^s_n \sin n\theta)$.

If γ is radial, then

- Because of the symmetry of the disc,

$$M^{cs}_{mn}[\sigma] = M^{sc}_{mn}[\sigma] = 0 \quad \text{for all } m, n,$$

$$M^{cc}_{mn}[\sigma] = M^{ss}_{mn}[\sigma] = 0 \quad \text{if } m \neq n,$$

  and

$$M^{cc}_{nn}[\sigma] = M^{ss}_{nn}[\sigma] \quad \text{for all } n.$$

- Let $M_n = M^{cc}_{nn} (= M^{ss}_{nn})$, $n = 1, 2, \ldots$.

Two important lemmas: Let

$$\sigma = \begin{cases} \gamma_0 \text{ (const)}, & |x| < 1, \\ \gamma & 1 \leq |x| < 2, \\ 1 & 2 \leq |x|. \end{cases}$$

where γ is radial.

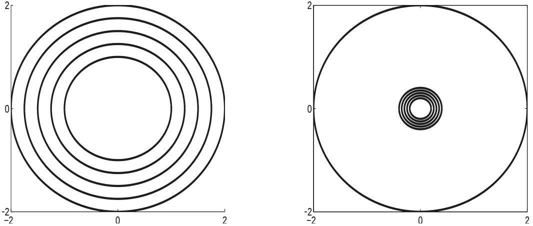


Figure: $\sigma(\frac{1}{\rho}x)$ for $|x| \leq 1$

- Then

$$\left(\Lambda[\sigma(\frac{1}{\rho}\mathbf{x})] - \Lambda[1]\right)(f) = \sum_{k=-\infty}^{\infty} \frac{2|k|\rho^{2|k|}M_{|k|}[\sigma]}{2\pi|k| - \rho^{2|k|}M_{|k|}[\sigma]} f_k e^{ik\theta}.$$

- $|M_k[\sigma]| \leq 2\pi k 2^{2k}$ for all k.

Enhancement of near cloaking

GPT vanishing structure: σ (or γ) is called a GPT vanishing structure of order N if $M_k = 0$ for $k \leq N$.

Let γ be a GPT vanishing structure of order N.

- Using the transformation blowing up a small ball, we can get a near-cloaking structure such that

$$\| \Lambda[\sigma^N] - \Lambda[1] \| = \| \Lambda[F_* \sigma^N] - \Lambda[1] \| \leq C\rho^{2N+2}.$$

Multiply layered structure

- For a positive integer N, let $1 = r_{N+1} < r_N < \ldots < r_1 = 2$ and define

$$A_j := \{r_{j+1} < r \leq r_j\}, \quad j = 1, 2, \ldots, N.$$

- $A_0 = \mathbb{R}^2 \setminus B_2$, $A_{N+1} = B_1$.
- Set σ_j to be the conductivity of A_j for $j = 1, 2, \ldots, N+1$, and $\sigma_0 = 1$. Let

$$\sigma = \chi(A_0) + \sum_{j=1}^{N} \sigma_j \chi(A_j) + \sigma_{N+1}\chi(A_{N+1}).$$

(σ_{N+1} may (or may not) be fixed: σ_{N+1} is fixed to be 0 if the core is insulated.)

- The transmission conditions on the interface $\{r = r_j\}$:

$$\begin{bmatrix} a_j^{(k)} \\ b_j^{(k)} \end{bmatrix} = \frac{1}{2\sigma_j} \begin{bmatrix} \sigma_j + \sigma_{j-1} & (\sigma_j - \sigma_{j-1}) r_j^{-2k} \\ (\sigma_j - \sigma_{j-1}) r_j^{2k} & \sigma_j + \sigma_{j-1} \end{bmatrix} \begin{bmatrix} a_{j-1}^{(k)} \\ b_{j-1}^{(k)} \end{bmatrix},$$

  and hence

$$\begin{aligned} \begin{bmatrix} a_{N+1}^{(k)} \\ b_{N+1}^{(k)} \end{bmatrix} &= \prod_{j=1}^{N+1} \frac{1}{2\sigma_j} \begin{bmatrix} \sigma_j + \sigma_{j-1} & (\sigma_j - \sigma_{j-1}) r_j^{-2k} \\ (\sigma_j - \sigma_{j-1}) r_j^{2k} & \sigma_j + \sigma_{j-1} \end{bmatrix} \begin{bmatrix} a_0^{(k)} \\ b_0^{(k)} \end{bmatrix} \\ &=: \begin{bmatrix} p_{11}^{(k)} & p_{12}^{(k)} \\ p_{21}^{(k)} & p_{22}^{(k)} \end{bmatrix} \begin{bmatrix} a_0^{(k)} \\ b_0^{(k)} \end{bmatrix}. \end{aligned}$$

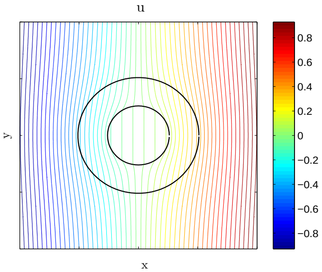
- Since $b_{N+1}^{(k)} = 0$ (in the inner disk),

$$M_k = -2\pi k \frac{b_0^{(k)}}{a_0^{(k)}} = 2\pi k \frac{p_{21}^{(k)}}{p_{22}^{(k)}}.$$

Optimization

- We fix N and $r_j = 2 - \frac{j-1}{N}$. We iteratively modify $\boldsymbol{\sigma}^{(i)} = (\sigma_1^{(i)}, \ldots, \sigma_{N+1}^{(i)})$ as
$$\boldsymbol{\sigma}^{(i+1)} = \boldsymbol{\sigma}^{(i)} - A_i^{\dagger}\mathbf{b}^{(i)},$$
where $A_i^{\dagger}$ is the pseudoinverse of $A_i := \left.\frac{\partial(M_1,\ldots,M_N)}{\partial\boldsymbol{\sigma}}\right|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}^{(i)}}$, and $\mathbf{b}^{(i)} = [M_1 \ \cdots \ M_N]^T|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}^{(i)}}$.

$M = 0$ (GPT vanishing structure of order 1) = the neutral inclusion of Hashine



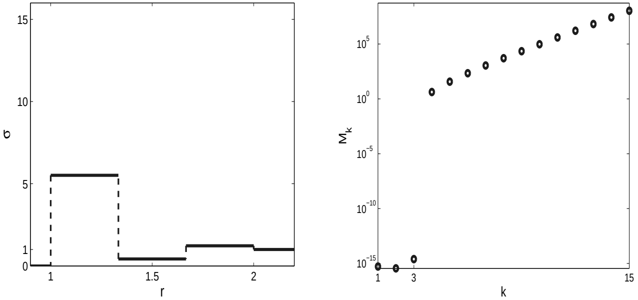


Figure: The conductivity of the core is fixed to be 0. $N = 3$

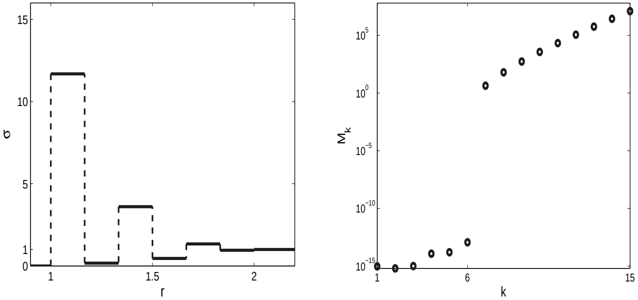


Figure: The conductivity of the core is fixed to be 0. $N = 6$

Numerical Test

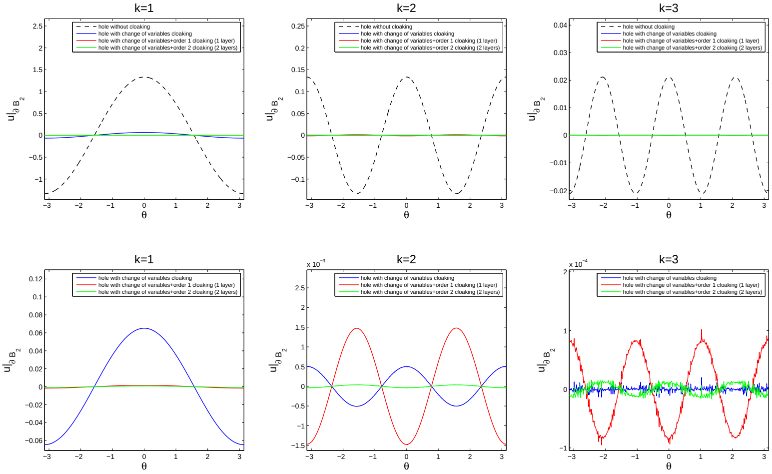
Compare $\Lambda[F_* \sigma]$ for three cases

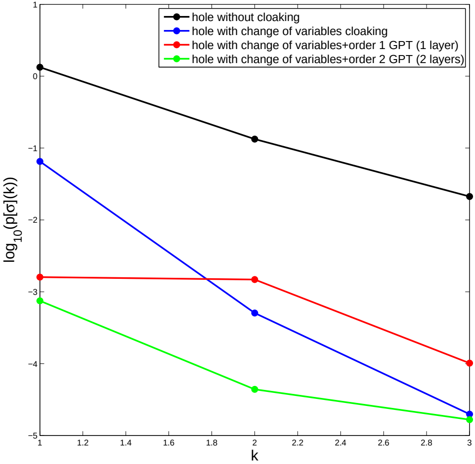
- (blue) Small hole (Kohn-Shen-Vogelius-Weinstein (2008))
- (red) Small hole + one-layer coating
- (green) Small hole + two-layer coating

FEM with Anisotropic Conductivity.

- $\rho = 0.25$
- $e_k = \cos k\theta$
- $p[\sigma](k) = \|\Lambda[F_*\sigma](e_k) - \Lambda[1](e_k)\|_\infty$

Numerical Test




hole without cloaking
hole with change of variables cloaking
hole with change of variables+order 1 GPT (1 layer)
hole with change of variables+order 2 GPT (2 layers)
log10(p[σ](k))
k
1
0
−1
−2
−3
−4
−5
1
1.2
1.4
1.6
1.8
2
2.2
2.4
2.6
2.8
3

Helmholtz equation

Consider the solution u to

$$\begin{cases} \nabla \cdot \dfrac{1}{\mu} \nabla u + \omega^2 \epsilon u = 0 \quad \text{in } \mathbb{R}^2, \\ (u - U) \text{ satisfies the outgoing condition.} \end{cases}$$

where (ϵ, μ) is the pair of electromagnetic parameters (permittivity and permeability) and U is an incident field.

- Let $A_\infty[\epsilon, \mu]$ be the far-field pattern, *i.e.*,

$$(u - U)(\mathbf{x}) = -ie^{-\frac{\pi i}{4}} \frac{e^{ik_0|\mathbf{x}|}}{\sqrt{8\pi k_0 |\mathbf{x}|}} A_\infty[\epsilon, \mu, \omega](\theta_k, \theta_\mathbf{x}) + o(|\mathbf{x}|^{-\frac{1}{2}}) \quad \text{as } |\mathbf{x}| \to \infty.$$

Let $\mathcal{S}_D^k[\varphi]$ be the single layer potential:

$$\mathcal{S}_D^k[\varphi](\mathbf{x}) = \int_{\partial D} \Gamma_k(\mathbf{x}-\mathbf{y})\varphi(\mathbf{y})d\sigma(\mathbf{y}).$$

where

$$\Gamma_k(\mathbf{x}) = -\frac{i}{4}H_0^{(1)}(k|\mathbf{x}|),$$

and $H_0^{(1)}$ is the Hankel function of the first kind of order zero.

Scattering Coefficients of an inclusion

$$\begin{cases} \nabla\cdot\Big(\frac{1}{\mu_0}\chi(\mathbb{R}^2\setminus\bar{D})+\frac{1}{\mu_1}\chi(D)\Big)\nabla u+\omega^2\Big(\epsilon_0\chi(\mathbb{R}^2\setminus\bar{D})+\epsilon_1\chi(D)\Big)u=0 \quad \text{in } \mathbb{R}^2, \\ (u-U) \text{ satisfies the outgoing condition.} \end{cases}$$

- For $U(\mathbf{x})=J_m(k_0|\mathbf{x}|)e^{im\theta_{\mathbf{x}}}$,

$$u(\mathbf{x})=\begin{cases} U(\mathbf{x})+\mathcal{S}_D^{k_0}[\psi](\mathbf{x}), & \mathbf{x}\in\mathbb{R}^2\setminus\bar{D}, \\ \mathcal{S}_D^{k}[\varphi](\mathbf{x}), & \mathbf{x}\in D, \end{cases}$$

where $(\varphi,\psi)\in L^2(\partial D)\times L^2(\partial D)$ is the unique solution to

$$\begin{cases} \mathcal{S}_D^{k}[\varphi]-\mathcal{S}_D^{k_0}[\psi]=U \\ \frac{1}{\mu}\frac{\partial(\mathcal{S}_D^{k}[\varphi])}{\partial\nu}\Big|_{-}-\frac{1}{\mu_0}\frac{\partial(\mathcal{S}_D^{k_0}[\psi])}{\partial\nu}\Big|_{+}=\frac{1}{\mu_0}\frac{\partial U}{\partial\nu} \end{cases} \quad \text{on } \partial D.$$

Define

$$W_{nm}=W_{nm}[\epsilon,\mu,\omega]:=\int_{\partial D}J_n(k_0|\mathbf{y}|)e^{-in\theta_{\mathbf{y}}}\psi_m(\mathbf{y})d\sigma(\mathbf{y}).$$

- Since
$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{m\in\mathbb{Z}} e^{im(\frac{\pi}{2}-\theta_{\mathbf{k}})} J_m(k|\mathbf{x}|)e^{im\theta_{\mathbf{x}}},$$
where J_m is the bessel function of order m, we have
$$u(\mathbf{x}) - e^{i\mathbf{k}\cdot\mathbf{x}} = -\frac{i}{4}\sum_{n\in\mathbb{Z}} H_n^{(1)}(k_0|\mathbf{x}|)e^{in\theta_{\mathbf{x}}} \sum_{m\in\mathbb{Z}} W_{nm} e^{im(\frac{\pi}{2}-\theta_{\mathbf{k}})} \quad \text{as } |\mathbf{x}| \to \infty.$$

- Let θ and θ' be respectively the incident and scattered direction. Then we have

$$A_\infty[\epsilon,\mu,\omega](\theta,\theta') = \sum_{n,m\in\mathbb{Z}} (-i)^n i^m e^{in\theta'} W_{nm}[\epsilon,\mu,\omega] e^{-im\theta}.$$

- $$A_\infty\left[\mu\circ\Psi_{\frac{1}{\rho}},\epsilon\circ\Psi_{\frac{1}{\rho}},\omega\right] = A_\infty[\mu,\epsilon,\rho\omega],$$

  where

$$\Psi_{\frac{1}{\rho}}(\mathbf{x}) = \frac{1}{\rho}\mathbf{x},\quad \mathbf{x}\in\mathbb{R}^2.$$

S-vanishing structure of order N at low frequencies.

There is a constant C depending on (ϵ, μ, ω) and ρ_0 such that

$$|W_{nm}[\epsilon, \mu, \rho\omega]| \leq \frac{C^{n+m}}{|n|^{|n|}|m|^{|m|}} \rho^{|n|+|m|} \quad \text{for all } n, m \in \mathbb{Z}.$$

for all $\rho \leq \rho_0$ where the constant C depends on (ϵ, μ, ω) but is independent of ρ as long as $\rho \leq \rho_0$.

We look for a structure such that

$$W_n[\mu, \epsilon, \rho\omega] = o(\rho^{2N}) \quad \text{for all } |n| \leq N \text{ and } \rho \to 0.$$

Transformation: Blow up a small ball

Let the parameter distributions ϵ and μ associated to a S-vanishing structure of order N. Let

$$(F)_*(\mu \circ \Psi_{\frac{1}{\rho}}) = \frac{(DF)(\mu \circ \Psi_{\frac{1}{\rho}})(DF)^T}{|\det(DF)|} \circ F^{-1},$$

and

$$(F)_*(\epsilon \circ \Psi_{\frac{1}{\rho}}) = \frac{(DF)(\epsilon \circ \Psi_{\frac{1}{\rho}})(DF)^T}{|\det(DF)|} \circ F^{-1}.$$

For ρ small enough, we have

$$A_\infty\left[(F)_*(\mu \circ \Psi_{\frac{1}{\rho}}), (F)_*(\epsilon \circ \Psi_{\frac{1}{\rho}}), \omega\right](\theta, \theta') = o(\rho^{2N}).$$

Multilayer structures

- For a positive integer N, let $1 = r_{N+1} < r_N < \ldots < r_1 = 2$ and define

$$A_j := \{r_{j+1} < r \leq r_j\}, \quad j = 1, 2, \ldots, N.$$

- $A_0 = \mathbb{R}^2 \setminus B_2$, $A_{N+1} = B_1$.
- Set $\mu_0 = 1$ and $\epsilon_0 = 1$. Let

$$\mu = \sum_{j=0}^{L+1} \mu_j \chi(A_j) \quad \text{and} \quad \epsilon = \sum_{j=0}^{L+1} \epsilon_j \chi(A_j).$$

Scattering Coefficients

- We look for solutions u_n of the form

$$u_n(\mathbf{x}) = a_j^{(n)} J_n(k_j r) e^{in\theta} + b_j^{(n)} H_n^{(1)}(k_j r) e^{in\theta}, \quad \mathbf{x} \in A_j, \quad j = 0, \ldots, L,$$

with $a_0^{(n)} = 1$.

- From the transmission conditions,

$$\begin{bmatrix} J_n(k_j r_j) & H_n^{(1)}(k_j r_j) \\ \sqrt{\frac{\epsilon_j}{\mu_j}} J_n'(k_j r_j) & \sqrt{\frac{\epsilon_j}{\mu_j}} \left(H_n^{(1)}\right)'(k_j r_j) \end{bmatrix} \begin{bmatrix} a_j^{(n)} \\ b_j^{(n)} \end{bmatrix}$$
$$= \begin{bmatrix} J_n(k_{j-1} r_j) & H_n^{(1)}(k_{j-1} r_j) \\ \sqrt{\frac{\epsilon_{j-1}}{\mu_{j-1}}} J_n'(k_{j-1} r_j) & \sqrt{\frac{\epsilon_{j-1}}{\mu_{j-1}}} \left(H_n^{(1)}\right)'(k_{j-1} r_j) \end{bmatrix} \begin{bmatrix} a_{j-1}^{(n)} \\ b_{j-1}^{(n)} \end{bmatrix}.$$

- The Neumann condition $\frac{\partial u_n}{\partial \nu}|_+ = 0$ on $|\mathbf{x}| = r_{L+1}$ amounts to

$$\begin{bmatrix} 0 & 0 \\ J_n'(k_L) & \left(H_n^{(1)}\right)'(k_L) \end{bmatrix} \begin{bmatrix} a_L^{(n)} \\ b_L^{(n)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

- We have

$$W_n = 4ib_0^{(n)}.$$

- $$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = P^{(n)}[\epsilon, \mu, \omega] \begin{bmatrix} a_0^{(n)} \\ b_0^{(n)} \end{bmatrix},$$

  where

$$P^{(n)}[\epsilon, \mu, \omega] := \begin{bmatrix} 0 & 0 \\ p_{21}^{(n)} & p_{22}^{(n)} \end{bmatrix} = (-\frac{\pi}{2} i\omega)^L \left(\prod_{j=1}^{L} \mu_j r_j \right) \begin{bmatrix} 0 & 0 \\ J_n'(k_L) & \left(H_n^{(1)}\right)'(k_L) \end{bmatrix}$$

$$\times \prod_{j=1}^{L} \begin{bmatrix} \sqrt{\frac{\epsilon_j}{\mu_j}} \left(H_n^{(1)}\right)'(k_j r_j) & -H_n^{(1)}(k_j r_j) \\ -\sqrt{\frac{\epsilon_j}{\mu_j}} J_n'(k_j r_j) & J_n(k_j r_j) \end{bmatrix} \begin{bmatrix} J_n(k_{j-1} r_j) & H_n^{(1)}(k_{j-1} r_j) \\ \sqrt{\frac{\epsilon_{j-1}}{\mu_{j-1}}} J_n'(k_{j-1} r_j) & \sqrt{\frac{\epsilon_{j-1}}{\mu_{j-1}}} \left(H_n^{(1)}\right)'(k_{j-1} r_j) \end{bmatrix}.$$

Behavior of $W_n[\mu, \epsilon, t]$, $t = \rho\omega$, as $\rho \to 0$

- Behavior of Bessel functions for small arguments:
  As $t \to 0$, we have

$$J_n(t) = \frac{t^n}{2^n}\left(\frac{1}{\Gamma(n+1)} - \frac{\frac{1}{4}t^2}{\Gamma(n+2)} + \frac{(\frac{1}{4}t^2)^2}{2!\Gamma(n+3)} - \frac{(\frac{1}{4}t^2)^3}{3!\Gamma(n+4)} + \cdots\right),$$

$$Y_n(t) = -\frac{(\frac{1}{2}t)^{-n}}{\pi}\sum_{l=0}^{n-1}\frac{(n-l-1)!}{l!}(\frac{1}{4}t^2)^l + \frac{2}{\pi}\ln(\frac{1}{2}t)J_n(t)$$
$$-\frac{(\frac{1}{2}t)^n}{\pi}\sum_{l=0}^{\infty}(\psi(l+1)+\psi(n+l+1))\frac{(-\frac{1}{4}t^2)^l}{l!(n+l)!},$$

where $\psi(1) = -\gamma$ and $\psi(n) = -\gamma + \sum_{l=1}^{n-1} 1/l$ for $n \geq 2$ with γ being the Euler constant.

- For $n \geq 0$, we have

$$W_n[\mu, \epsilon, t] = t^{2n} \left(\tilde{W}_n^0[\mu, \epsilon] + \sum_{k=1}^{(N-n)} \sum_{j=0}^{M_{n,k}} \tilde{W}_n^{k,j}[\mu, \epsilon] t^{2k} (\ln t)^j \right) + o(t^{2N+1}),$$

where $M_{n,k} \in \mathbb{N}$ and $\tilde{W}_n^{k,j}[\mu, \epsilon]$ are independent of t.

- We look for (μ, ϵ) satisfying that

$$\tilde{W}_n^0[\mu, \epsilon] = 0 \text{ and } \tilde{W}_n^{k,j}[\mu, \epsilon] = 0, \quad \text{for } 0 \leq n \leq N.$$

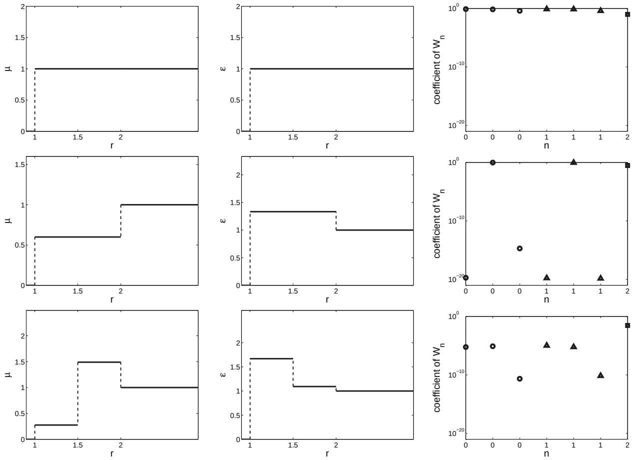
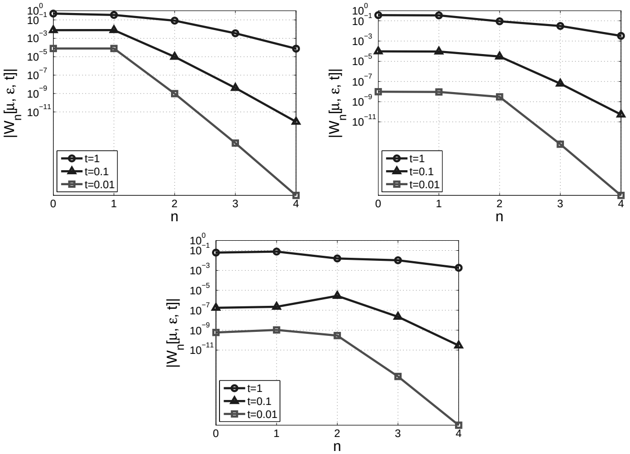


Figure: Graphs on the first and second column show the permeability profile μ and the permittivity profile ϵ. The right column show the components $[t^2, t^4, t^4 \log t]$ of W_0 and W_1, and $[t^4]$ of W_2.


$|W_n[\mu, \varepsilon, t]|$
n
t=1
t=0.1
t=0.01

Happy Birthday!