Singular Monge-Ampère equations in geometry

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Let Ω be a domain in $\mathbb{C}^n$. If $\phi \in \mathcal{C}^2(\Omega)$, then a typical complex Monge-Ampére (CMA) equation is a fully nonlinear partial differential equation of the form

$$\det\left(\mathrm{Id} + \sqrt{-1}\,\frac{\partial^2 \phi}{\partial z_i \overline{\partial z_j}}\right) = F(z, \phi, \nabla\phi),$$

where

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\partial_{x_j} - \sqrt{-1}\partial_{y_j}\right), \quad \frac{\partial}{\partial \overline{z_j}} = \frac{1}{2}\left(\partial_{x_j} + \sqrt{-1}\partial_{y_j}\right).$$

This is elliptic precisely when the matrix $\mathrm{Id} + \mathrm{Hess}_{\mathbb{C}}(\phi)$ is a positive definite Hermitian matrix.

There are many variants, and also analogous *real* Monge-Ampère equations.

Such equations can be phrased intrinsically when the domain Ω is replaced by a Kähler manifold (M, g).

Recall that a Hermitian metric is Kähler if the 2-form $\omega = \sum g_{i\bar{j}} dz_i \wedge \overline{dz_j}$ is closed.

This is equivalent to the fact that it is possible to choose a *holomorphic* change of variables so that the this metric, pulled back in this new coordinate chart, satisfies

$$g_{i\bar{j}} = \delta_{ij} + \mathcal{O}(|z|^2).$$

If g is a Kähler metric and $\phi \in \mathcal{C}^2(M)$, then we define a Hermitian $(1,1)$ tensor g_ϕ by

$$(g_\phi)_{i\bar{\jmath}} = g_{i\bar{\jmath}} + \sqrt{-1}\frac{\partial^2}{\partial z_i \overline{\partial z_j}} = g_{i\bar{j}} + \sqrt{-1}\phi_{i\bar{\jmath}}.$$

This is a metric precisely if the matrix on the right is Hermitian positive definite, and if this is the case, then we write $\phi \in \mathcal{H}_g$.

Any such metric g_ϕ is said to be in the same Kähler class as g.

Kähler classes are the replacement for conformal classes in this setting.

The canonical metric problem:

Given a Kähler manifold (M, g), find a 'better' metric g_ϕ in the same Kähler class. Or, if possible, find a 'best' one!

Applications: higher dimensional uniformization, fundamental to classification problems in complex and algebraic geometry, etc.

Improvement of metric $\rightsquigarrow$ Kähler-Ricci flow
Best metric $\rightsquigarrow$ Kähler-Einstein metrics.

Definition: A Kähler metric g is called Kähler-Einstein (KE) if the Ricci tensor of g is a scalar multiple of g.

Using the complex structure, convert Ric into a $(1,1)$ form

$$\rho_g = \sum_{i,\bar{\jmath}} \mathrm{Ric}_{i\bar{\jmath}} dz_i \wedge \overline{dz_j}.$$

Thus g is KE if and only if $\rho_g = \mu\omega_g$ for some $\mu \in \mathbb{R}$.

Standard facts: $d\rho_g = 0$, and its de Rham (or rather, Dolbeault) cohomology class is determined only in terms of the complex structure,

$$\frac{1}{2\pi i}\left[\rho_g\right] = c_1(M),$$

the first Chern class of M, but is otherwise independent of g.

This presents an obstruction to the existence of KE metrics in a given Kähler class: a necessary condition is whether the class $c_1(M)$ admits a representative γ such that $2\pi i\gamma$ is positive definite ($\mu > 0$) or negative definite ($\mu < 0$). The case $\mu = 0$ corresponds to $c_1(M) = 0$, which contains the representative $\gamma \equiv 0$.

Calabi's Conjecture: Is this obstruction the only one? More precisely:

Given (M, g) compact, Kähler, and suppose that $c_1(M) < 0$ or $c_1(M) > 0$. Then is it possible to find a function ϕ on M such that $(g_\phi)_{i\bar{j}}$ remains positive definite and such that $\rho_{g_\phi} = \mu\omega_{g_\phi}$ where $\mu < 0$ or $\mu > 0$, respectively?
If $c_1(M) = 0$ and β is any $(\bar{\partial})$ exact $(1,1)$ form, can one find ϕ so that $\rho_{g_\phi} = \beta$?

This question has been one of the central foci of research in complex geometry for the past 30-40 years.

As a PDE, this amounts to solving the complex Monge-Ampère equation

$$\frac{\det\left(g_{i\bar{j}}+\sqrt{-1}\phi_{i\bar{j}}\right)}{\det\left(g_{i\bar{j}}\right)}=e^{F-\mu\phi}.$$

Here $F \in \mathcal{C}^{\infty}$ is the error term, and measures the discrepancy from g itself being Kähler-Einstein.

Major results:

Aubin, Yau (mid-'70's): The case $\mu < 0$

Yau (mid 1970's): The case $\mu = 0$

There are known obstructions for existence when $\mu > 0$

Tian (late 1980's): $\dim_{\mathbb{C}} M = 2$, $\mu > 0$ (assuming that known obstruction vanishes).

A huge amount of work since that time.

Ultimate goal: give precise algebro-geometric conditions which are necessary and equivalent for existence when $\mu > 0$.

Now suppose that (M, g) is Kähler as before, and that $D \subset M$ is a (possibly reducible) divisor, so $D = D_1 \cup \ldots \cup D_N$ where each D_j is a smooth complex codimension one submanifold, and such that D has simple normal crossings.

In coordinates this means that locally each D_i can be described by an equation $\{z_i = 0\}$ for some choice of complex coordinates $(z_1, \ldots, z_n)$, and that near intersections,

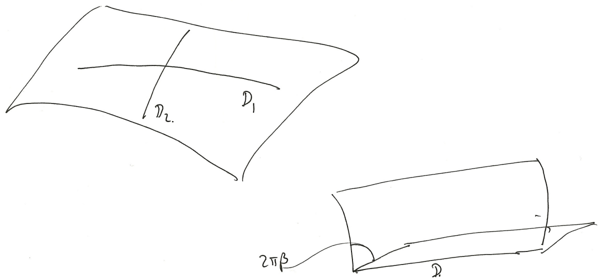
$$D_{i_1} \cap \ldots \cap D_{i_\ell} = \{z_{i_1} = \ldots = z_{i_\ell} = 0\}.$$

We also assume that the D_i are orthogonal to one another at the intersection loci.

Problem (proposed by Tian in the early '90's, and more recently by Donaldson about 5 years ago):

Assume that $c_1(M) - \sum_{j=1}^N (1-\beta_j)c_1(L_{D_j}) = \mu[\omega]$, where $[\omega]$ is the Kähler class, for some choice of constants $\beta_1, \ldots, \beta_N \in (0,1)$ and $\mu \in \mathbb{R}$. Can one then find a Kähler-Einstein metric with $\rho' = \mu\omega'$ in the same Kähler class as g and which is 'bent' with angle $2\pi\beta_j$ along D_j for every j?

This adds a small amount of flexibility to the problem.


D_2
D_1
$2\pi\beta$
D

Donaldson's program: prove the existence of KE edge metrics with $\beta \ll 1$; then study what happens as β increases up to 1. Either this succeeds and one can take a limit and obtain a *smooth* KE metric at $\beta = 1$, or else there is some breakdown, which hopefully can be analyzed and connected to algebraic geometry.

Thus what would remain is a very delicate compactness theorem: find the precise conditions under which this family of KE metrics does not 'blow up'.

Progress on this question:

- Jeffres, mid '90's, uniqueness (for a given β);
- An announcement from late '90's (Jeffres-M), covered existence when $\mu < 0$, $\beta \leq 1/2$ (details never appeared).
- Campagna-Guenancia-Paun, 2011; general D, $\mu \leq 0$, $\beta \leq 1/2$. Smooth approximation technique which gives little information about geometry.
- Donaldson, 2011; D smooth, local deformation theory, $\beta \in (0, 1)$, all μ.
- Brendle, 2011; existence when D smooth, $\mu = 0$ and $\beta \leq 1/2$.

The case $\beta \leq 1/2$ contains all the orbifold cases. It turns out to be significantly easier, for reasons I will describe.

- Jeffres-M-Rubinstein, 2011; existence when D smooth, $\beta < 1$.
- M-Rubinstein, 2012. Existence in general case and resolution of Tian-Donaldson conjectures; general D, $\beta < 1$.

The classical (Aubin-Yau) method:

Consider the family of equations

$$\frac{\det\left(g_{i\bar{j}} + \sqrt{-1}\phi_{i\bar{j}}\right)}{\det\left(g_{i\bar{j}}\right)} = e^{tF-\mu\phi}, \qquad (\star)$$

and, as usual, the set $J = \{t \in [0,1] : \exists \text{ a solution to } (\star)\}$.

- J is nonempty ($0 \in J$ trivially).
- J is open
- J is closed.

For the openness argument, simply invoke the inverse function theorem using that the linearization of $(\star)$ at a point $t_0 \in J$ is

$$L_{t_0} = \Delta_{g_{t_0}} + \mu.$$

Here g_{t_0} is the metric corresponding to ϕ_{t_0}. Note that if M compact and smooth and $\mu < 0$, this is an isomorphism (say, between Hölder spaces), while if $\mu = 0$ it is invertible on the complement of the constants. For $\mu > 0$ it may fail to be invertible.

As for closedness, these require the famous a priori estimates developed by Aubin and Yau. Briefly, if $\mu < 0$, then $||\phi_t||_{\mathcal{C}^0}$ follows immediately from the maximum principle; if $\mu = 0$, this $\mathcal{C}^0$ bound is more subtle and relies on Moser iteration.

The $\mathcal{C}^2$ estimate relies on a *lower* bound for the bisectional curvature of the initial metric g. Recall, if X and Y are orthonormal, then

$$\mathrm{Bisec}(X, Y) = \mathrm{Riem}(X, \overline{X}, Y, \overline{Y}).$$

The $\mathcal{C}^3$ estimate is technically difficult, but we can now invoke the theory developed by Evans and Krylov to say that the a priori $\mathcal{C}^0$ and $\mathcal{C}^2$ bounds imply an a priori $\mathcal{C}^{2,\alpha}$ bound.

We wish to implement exactly the same strategy to find KE edge metrics.

Step 1: Find an initial Kähler metric g which has the correct geometric structure, i.e. makes an edge with angle $2\pi\beta_j$ along D_j.

Step 2: Define a continuity path, and prove both openness and closedness along this path.

For Step 1, the ‘model’ example (for the flat ambient space $\mathbb{C}^n$ with $D = \{z_1 = \ldots = z_k = 0\}$ is

$$\omega_\beta = \frac{1}{2}\sqrt{-1}\sum_{j=1}^{k} |z_j|^{2\beta_j - 2}|dz_j|^2 + \sum_{\ell=k+1}^{n} |dz_\ell|^2.$$

For the actual problem, choose a holomorphic section s_j on L_{D_j} and a Hermitian metric h_j on each of these line bundles, and set

$$\omega_\beta = \omega + \epsilon \sum_{j=1}^{k} \sqrt{-1}\partial\bar{\partial}|s_j|_{h_j}^{2\beta_j}$$

Here ω is the ambient smooth metric and $\epsilon > 0$ is sufficiently small.

Finding such an initial approximate solution is precisely where the cohomological condition $c_1(M) - \sum(1-\beta_j)c_1(L_{D_j}) = \mu[\omega]$ enters.

The big problem: if any $\beta_j > \frac{1}{2}$, then Bisec_g is almost certainly *NOT* bounded below!

In fact, there is (probably) a cohomological condition on the component divisors D_j which obstructs the existence of a Kähler edge metric with lower bisectional curvature bounds.

The challenges ahead:

- Find a new continuity path
- Study the linearized operator at any solution along the continuity path. This is a linear elliptic *edge operator*. There is a very complete theory of pseudodifferential edge operators in which to carry out parametrix constructions to investigate regularity. Use this to prove openness. (Accomplished using direct arguments by Donaldson, but on function spaces not well suited for other aspects of the problem when $\beta > 1/2$.)
- Find new a priori estimates which do not require a lower bound on bisectional curvature!

We use the new continuity path

$$\frac{\det\left(g_{i\bar{\jmath}} + \sqrt{-1}\phi_{i\bar{\jmath}}\right)}{\det\left(g_{i\bar{\jmath}}\right)} = e^{F - s\phi}, \qquad (\star\star)$$

where $-\infty < s \le \mu$, or even, combining these,

$$\frac{\det\left(g_{i\bar{\jmath}} + \sqrt{-1}\phi_{i\bar{\jmath}}\right)}{\det\left(g_{i\bar{\jmath}}\right)} = e^{tF - s\phi}, \qquad (\star\star')$$

for $-\infty < s \le \mu$ and $0 \le t \le 1$.

This new continuity path was introduced by Rubinstein in his work on Kähler-Ricci iteration, which can be regarded as a type of discretization of Kähler-Ricci flow.

Define

$$J = \{(s,t) : \exists \text{ solution to } \star\star'\}$$

First issue, why is J nonempty?

Straightforward perturbation argument when $s \ll 0$.

To begin to discuss openness and closedness, need to decide on function spaces. Consider the simple edge case first (D smooth). Recall, metric

$$g \sim |z_1|^{2\beta}|dz_1|^2 + \ldots + |dz_n|^2.$$

Choose coordinates $z_1 = \rho e^{i\tilde{\theta}}$, $z' = (z_2, \ldots, z_n)$ and $y = (\mathrm{Re}\,(z'), \mathrm{Im}\,(z'))$.

Finally, set

$$r = \frac{\rho^{1+\beta}}{1+\beta}, \;\; \theta = (1+\beta)\tilde{\theta}$$

and we use (r, θ, y).

In these coordinates

$$\Delta_g \sim \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\beta^2}{r^2}\frac{\partial^2}{\partial \theta^2} + \Delta_y.$$

Find function spaces on which Δ_g has good mapping and regularity properties.

There are (at least) two reasonable choices of function spaces:

$$\mathcal{C}_w^{k,\alpha}(M,D) \quad \text{based on differentiating by } \frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial y_j}$$

the wedge Hölder spaces, used by Donaldson, Brendle, and

$$\mathcal{C}_e^{k,\alpha}(M,D) \quad \text{based on differentiating by } r\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, r\frac{\partial}{\partial y_j}$$

the edge Hölder spaces, used by us.

Both behave well with respect to dilations $(r,\theta,y) \longmapsto (\lambda r, \theta, \lambda y + y_0)$ (homogeneous of degrees -1 and 0, respectively).

Define the Hölder-Friedrichs domain:

$$\mathcal{D}^{k,\alpha}_{w/e} = \{u \in \mathcal{C}^{2,\alpha}_{w/e} : \Delta u \in \mathcal{C}^{k,\alpha}_{w/e}\}$$

Note: if $u \in \mathcal{C}^{2,\alpha}_{e}$, then we expect that $\Delta u = \mathcal{O}(r^{-2})$, so if $u \in \mathcal{D}^{0,\alpha}_{e}$, then it has at least some extra regularity properties near the edge which allow the cancellation to happen.

The usual mechanism:

$$u \sim a_{01}(y)\log r + a_{00}(y) + r^{\frac{1}{\beta}}(a_{11}(y)\cos\theta + a_{12}(y)\sin\theta) + \tilde{u}(r,\theta,y).$$

The indicial roots of this problem are $\frac{k}{\beta}$, $k \in \mathbb{Z}$.

Friedrichs extension $\Longrightarrow$ the coefficent $a_{01}(y) \equiv 0$.

Note the big change: if $\beta < 1/2$, then $1/\beta > 2$ so we only need to worry about the leading terms.

A key difficulty is regularity of the coefficients a_{ij} in y (this was what held up my old approach with Jeffres many years ago).

Donaldson's idea: consider the L^2 Friedrichs extension of the Laplacian and its Green function G. He proved 'by hand' that

$$\partial \circ G, \ \partial\overline{\partial} \circ G$$

are bounded on $\mathcal{C}^{0,\alpha}_w$.

In other words, although the 'real' derivatives may give problematic terms, the complex (z and $\bar{z}$) derivatives do not, and this is sufficient to understand issues related to the Laplacian of a Kähler metric, which is built out of these complex derivatives.

This observation by Donaldson and the estimates he proves turn out to be enough to deal with all issues related to existence when $\beta \leq 1/2$ (subsequently carried out by Brendle).

Namely, openness via inverse function theorem on $\mathcal{D}_{w}^{0,\alpha}$ and closedness by an adaptation of the Aubin-Yau estimates since one has full curvature bounds when $\beta \leq 1/2$.

Theorem (Jeffres-M-Rubinstein)

For all $\beta < 1$, the 'Riesz potential operators'

$$\frac{\partial}{\partial z_i} \circ G, \frac{\partial}{\partial \overline{z_j}} \circ G, \frac{\partial^2}{\partial z_i \partial \overline{z_j}} \circ G$$

are all bounded on $\mathcal{C}_e^{0,\alpha}$. If $\beta \le 1/2$, then

$$\frac{\partial^2}{\partial r^2} \circ G, \frac{1}{r}\frac{\partial}{\partial r} \circ G, \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \circ G, \frac{1}{r}\frac{\partial^2}{\partial r \partial y_j} \circ G, \frac{\partial^2}{\partial y_i \partial y_j} \circ G$$

are all bounded on $\mathcal{C}_e^{0,\alpha}$.

$\mathcal{C}^0$ estimates:

- $\mu < 0$: can use Aubin's maximum principle arguments, as adapted by Jeffres to the edge setting
- $\mu = 0$: there is a Sobolev inequality on manifolds with edges (easiest to see using results of Saloff-Coste and others on equivalence with heat kernel asymptotics), and one can also make the Moser iteration argument work, so Yau's estimates adapt.
- $\mu > 0$: one can obtain a $\mathcal{C}^0$ estimate only when the twisted Mabuchi energy is proper. This is the exact analogue of the situation in the smooth case, and is known to be true in a number of circumstances.

$\mathcal{C}^2$ estimate:

This is a new estimate based on an old inequality due to Chern and Lu, itself essentially a generalized Schwarz Lemma.

This uses an upper bound on the bisectional curvature of the initial Kähler metric g and a lower bound on the Ricci curvature of the metric $g_{t,s}$ along the continuity path. However, this lower bound is trivial precisely because $g_{t,s}$ is a solution to a complex Monge-Ampère equation which states that its Ricci curvature is $s\omega_t - tF$.

Remarkable fact: the initial Kähler edge metric g *does* have an upper bound on its bisectional curvature for all $\beta \leq 1$!

For D smooth this is based on some calculations by C. Li and worked out by Li and Rubinstein. A difficult calculation.

For general D, this is still true, and unfortunately an even more difficult calculation.

$\mathcal{C}_e^{2,\alpha}$ estimate:

By the definition of these spaces, it suffices to prove this estimate in Whitney cubes

$$B_{\epsilon,y_0} = \{(r,\theta,y) : \epsilon/2 \le r \le 2\epsilon,\ |y - y_0| \le 2\epsilon,\ \theta \in S^1\},$$

but these edge Hölder spaces are homogeneous with respect to dilation (in r and y), so it is actually enough to prove the estimate in cubes B_{1,y_0}, where it reduces to a now standard *local* version of the Evans-Krylov estimate.

A subtlety:

We can now take a limit of solutions in $\mathcal{C}_e^{2,\alpha'}$ for any $0 < \alpha' < \alpha$.

However, the openness argument does *not* work for solutions/metrics in these spaces.

The way out: prove a regularity theorem. The limiting solution $u = u_{t_0,s_0}$ solves a complex Monge-Ampère equation. One can then prove that it is necessarily polyhomogeneous along D, i.e.

$$u \sim \sum_{i,\ell=0}^{\infty} \left(r^{\frac{i}{\beta}+\ell} a_{i\ell 1} \cos(j\beta\theta) + a_{j\ell 2} \sin(j\beta\theta) \right)$$

with all $a_{i\ell j}(y) \in \mathcal{C}^\infty(D)$.

Hence this regularity theorem serves as a crucial intermediary, rather than a cosmetic afterthought, since it is what allows us to cycle back from the closedness to the openness argument.

Remainder of talk:

- The Chern-Lu inequality
- Some ideas about the proof of boundedness of these Riesz potential type operators (both in the case where D is smooth and where D has simple normal crossings), as well as the regularity theorem.
- Crossing edges ....

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- Crossing edges ....

Chern-Lu inequality:

Let (M, ω), (N, η) be compact Kähler manifolds and let $f : M \to N$ be a holomorphic map with $\partial f \neq 0$. Then

$$\Delta \log |\partial f|^2 \geq \frac{(\mathrm{Ric}\,\omega \otimes \eta)(\partial f, \bar{\partial} f)}{|\partial f|^2} - \frac{\omega \otimes R^N(\partial f, \bar{\partial} f, \partial f, \bar{\partial} f)}{|\partial f|^2}.$$

In particular, if f is the identity map and $\omega = \eta + \sqrt{-1}\partial\bar{\partial}\phi$ then

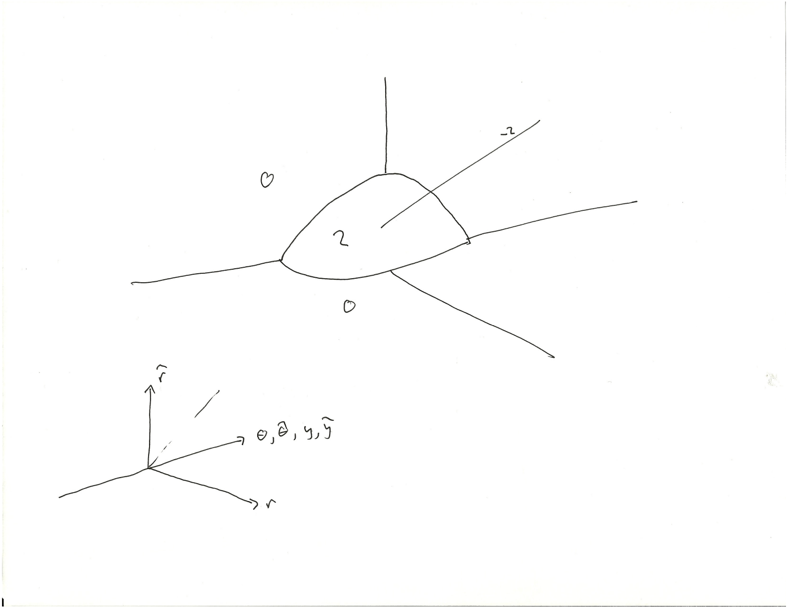
$$\Delta_\omega \big(\log \mathrm{tr}_\omega \eta - (C_2 + 2C_3 + 1)\phi \big) \geq -C_1 - (C_2 + 2C_3 + 1)n + \mathrm{tr}_\omega \eta$$

The constants depend on upper bounds of bisectional curvature of (N, η) and lower bound for Ricci curvature of (M, ω) and $\sup |\phi|$.

Structure of the Friedrichs Green function $G(r,\theta,y,\tilde{r},\tilde{\theta},\tilde{y})$. (First, when D is smooth).

This is an elliptic pseudodifferential edge operator, and in fact an element of the space $\Psi_e^{-2,2,0,0}([M;D])$ defined by (M, 1991).

The superscripts denote orders of various filtrations: the initial -2 indicates that it is a pseudodifferential operator of order -2; the $+2$ indicates its "front face" behaviour, namely that the symbol of this operator decays like r^2 as $r \to 0$, which matches the fact that the symbol of Δ itself blows up (in an appropriate sense) like r^{-2}; the remaining 0 and 0 are the orders of the expansion of the Schwartz kernel of this operator along the side faces. These reflect the fact that we are using the Friedrichs inverse, which chooses the solution that omits the $\log r$ term in its expansion. These indices are equal because G is symmetric.


-2
0
2
0
$\tilde{r}$
$\theta, \tilde{\theta}, y, \tilde{y}$
r

Using this, we may now examine the structure of $X \circ G$ and $XX' \circ G$ where X, X' are chosen amongst

$$\frac{\partial}{\partial r},\ r^{-1},\ r^{-1}\frac{\partial}{\partial_\theta},\ \frac{\partial}{\partial_{y_j}}.$$

The lifts of each of these to the blown up space $\widetilde{M}^2_e$ blow up to order 1 at the front face and sometimes also the left face.

We observe that each $XX' \circ G \in \Psi_e^{0,0,0,0}$, and that every pseudodifferential operator of this type is bounded on $\mathcal{C}_e^{0,\alpha}$, when $\beta < 1/2$.

When $1/2 < \beta < 1$, we can still show that $\partial_{z_i} \circ G$, $\partial^2_{z_i\overline{z_j}} \circ G$; this uses special cancellations along the left face because of the known structure of the expansion of the Schwartz kernel of G there.

Finally, a (very) brief indication of what needs to be done when D has simple normal crossings.

First, blow up M along each of the D_j: $\widetilde{M} = \left[M; D_1; D_2; \ldots D_N\right]$. This is independent of order since the D_j are transverse to one another.

This is a manifold with corners, with coordinates $(r_1, \ldots, r_N, \theta_1, \ldots, \theta_N, y_1, \ldots, y_{2n-2N})$.

Construct Green function on a blowup of $\widetilde{M} \times \widetilde{M}$. Must incorporate the correct double-space for the simple edge case ($N = 1$) as well as for all the cases where N is smaller. Thus, work inductively.

The new feature: one must also blow up the intersections where any subcollection of the r's vanish, $\{r_{i_1} = \ldots = r_{i_\ell} = 0\}$, $\ell \leq N$ – not in $\widetilde{M}$ but in the double-space.

The effect of this is that solutions $u(r_1, \ldots, r_N, \theta_1, \ldots, \theta_N, y)$ may not be product polyhomogeneous anymore at these corners.

Work inductively!

This blow-up picture turns also out to be a convenient one for understanding the computation that shows the bisectional curvature of the reference metric is bounded above.

HAPPY BIRTHDAY GUNTHER!!!!