# Adiabatic limits and eigenvalues

## Gunther Uhlmann's 60th birthday meeting

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# Outline

I have not really worked on inverse problems since Gunther and I last collaborated in a project on backscattering. So I thought I would describe some results on *adiabatic limits* in various settings and finish with some related questions

- Basic structure of an adiabatic problem
- Inversion of operators
- Spectrum of adiabatic operators
- An adiabatic inverse problem

First, let me remind you of a core inverse problem – one that I am would very much like to solve or see solved. This is Kac's problem.

## Problem

*Do the Dirichlet (and/or Neumann) eigenvalues for a (smooth) bounded strictly convex domain in the plane determine the domain?*

- If one drops the smoothness *and* convexity assumptions then there are counterexamples (but very rigid ones)
- Unfortunately I have nothing new to say about this problem!
- You should talk to Hamid Hezari and Steve Zelditch about their recent work on perturbation of ellipses

- The notion of an adiabatic limit in Physics really arose in thermodynamics but the use of the term in differential analysis/geometry follows a paper by Witten
- Witten discusses the adiabatic limit of the eta invariant for a manifold fibred over a circle
- More generally one can think of a fibre bundle

$$\begin{array}{ccc} Z & \text{——} & M \\ & & \Big\downarrow \phi \\ & & B. \end{array} \quad (1)$$

- For compact manifolds this is the same notion as a submersion, i.e. just a smooth map with surjective differential from each point of the domain.

- The inverse image of a small neighbourhood $U$ of each point in $B$ under $\phi : M \longrightarrow B$ is diffeomorphic to the product $Z \times U$ for a fixed compact manifold $Z$ with smooth transitions between overlaps.
- Thus $M$ comes equipped with an exhaustion by disjoint smooth fibres looking like $Z$
- One can give $M$ an 'adiabatic' metric, meaning a family of metrics depending on a parameter $\epsilon$ of the form

$$g + \epsilon^{-2}\phi^* h \tag{2}$$

- Here $g$ is some metric (maybe only strictly positive on the fibres) on $M$ and $h$ is a metric on the base, $B$.
- Thus a fixed tangent vector on $M$ becomes 'long' in the base direction as $\epsilon \downarrow 0$

- Near a point of $M$ there are coordinates $z$ along the fibres and $y$ in the base – these are constant on the fibres
- The vector fields of 'bounded length' with respect to an adiabatic metric are then the combinations of $\partial_{z_j}$ and $\epsilon\partial_{y_l}$.
- Commutators of these behave sensibly so one can form 'adiabatic differential operators' as locally looking like

$$P = \sum_{|\alpha|+|\beta|\le m} p_{\alpha,\beta}(\epsilon, z, y)\partial_z^\alpha(\epsilon\partial_y)^\beta. \tag{3}$$

- Adiabatic ellipticity means that the polynomial

$$p_m = \sum_{|\alpha|+|\beta|=m} p_{\alpha,\beta}(\epsilon, z, y)\zeta^\alpha\eta^\beta \tag{4}$$

should have no real zeros.

- The symbol here is defined for all $\epsilon \geq 0$
- There is an *adiabatic model operator* well defined at $\epsilon = 0$, given locally by

$$A(P) = \sum_{|\alpha|+|\beta|\leq m} p_{\alpha,\beta}(0, z, y)\partial_z^\alpha \zeta^\beta \tag{5}$$

- This is a family of operator on the fibres with conormal parameters from the base
- Notice that this is like a partial semiclassical limit with non-commutativity remaining along the fibres.

- A relatively easy result is

**Theorem**

*If $P$ is an elliptic adiabatic operator and $A(P)$ is invertible (for all values of the parameters) then $P$ is invertible for small $\epsilon > 0$.*

- This invertibility comes with precise uniformity down to $\epsilon = 0$.

- The Laplacian, $\Delta$, for any adiabatic metric is an example of an elliptic adiabatic family
- $A(\Delta) = \Delta_{Z_b} + |\zeta|_b^2$.
- Thus the Theorem above applies to $\Delta - z$ for $z \notin [0, \infty)$
- If one thinks about the eigenvalues of Laplacian for an adiabatic metric one can be guided *to some extent* by the product case for which the eigenvalues are

$$\begin{aligned} M = Z \times B,\ g = g_Z + \epsilon^{-2} h_B \\ \Delta_g = \Delta_Z + \epsilon^2 \Delta_B \\ \lambda(\Delta_g) = \lambda_j(Z) + \epsilon^2 \lambda_k(B) \end{aligned} \tag{6}$$

- In general of course this does not make sense, since the $\lambda_j(Z_b)$ will be functions on $B$, which parameterizes the fibres, and the $\lambda_k(B)$ do not make sense at all since there is no obvious base operator.
- The product case is a reasonable guide provided there is a $\lambda_j$ which is constant – independent of $b \in B$.
- Generally there are no such constant eigenvalues

- One such case that Rafe Mazzeo and I looked at some years ago is the Laplacian on forms – for which $A(P)$ is generally not invertible at $\zeta = 0$.
- We considered what happens to the Hodge cohomology – the harmonic forms – on $M$ as $\epsilon \downarrow 0$.
- The dimension of the harmonic forms of fixed degree is independent of $\epsilon$, being the corresponding Betti number.

**Theorem**

*For any adiabatic metric there is a smooth basis, $u_j(\epsilon)$ of harmonic forms, down to $\epsilon = 0$.*

- The limits $u_j(0)$ of these smooth forms consist of harmonic forms on the fibres $Z_b$ which ‘depend in a harmonic way’ on the base variables.
- However, not *all* such forms occur as the limits of truly harmonic forms.
- Which harmonic sections occur in the limit can be worked out from the Taylor series in $\epsilon$
- This construction implements the Leray-Serre spectral sequence for the cohomology of the total space.

- I want to emphasize here that it is very significant that the fibre Laplacians have smoothly varying null space – the harmonic forms on $Z$ (for varying metrics)
- Suppose one considers a fibration with fibres which are manifolds with boundary, thus $M$ is also a manifold with boundary
- For an adiabatic metric consider the Laplacian on $M$ with Dirichlet boundary conditions
- Then the fibre Laplacians are invertible and (a small extension) of the Theorem above shows that $\Delta$ is uniformly invertible down to $\epsilon = 0$

- What then happens to the eigenvalues of $\Delta$ as $\epsilon \downarrow 0$?
- The lowest fibre eigenvalue for $\lambda_1(Z_b)$ is simple and hence smooth in $b$.
- As $\epsilon \downarrow 0$ the lowest eigenvalues of $\Delta$ are close to $I = \inf_{b\in B} \lambda_1(Z_b)$ and concentrate above the point or points in $B$ where this is assumed.
- If all the minima are non-degenerate the lowest eigenvalue corresponds to a rescaled harmonic oscillator in the base variable near each of these points and are of the form

$$I + \epsilon^2 t_j + O(\epsilon^3)$$

If we pass from the realm of manifolds with boundary to those with corners there is a natural weakening of the notion of a fibration to a b-fibration.

Picture!

These sorts of considerations can be extended to somewhat more singular settings.

1. The fibration with singular fibres given by a Morse function on a compact manifold – there is an extension of Witten's theorem on the eta invariant to this case (a question of M. Atiyah)
2. Gluing constructions corresponding to blowing up points in manifolds – for instance the construction of kähler metrics with constant scalar curvature (with M. Singer)
3. The eigenvalues of planar triangles as functions on the moduli space – so corresponding to all collapse modes (with D. Grieser)

- Now, let me come back to the planar domain problem I mentioned at the beginning.
- Let's replace the domain by an adiabatic one, or if you like a race track or a wave guide:

$$\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2; R(\theta) - \epsilon \leq r \leq R(\theta)\}. \tag{7}$$

- Here $0 < R \in \mathcal{C}^\infty(\mathbb{S})$ is a smooth function which is periodic of period $2\pi$ and $r$, $\theta$ are standard polar coordinates. So this domain need not be strictly convex, but is certainly star-shaped around the origin.

- To make this look like an adiabatic problem, we can introduce polar coordinates and then rescale, to get coordinates $(t, \theta)$ where

$$t = \frac{R(\theta) - r}{\epsilon} \in [0, 1]. \tag{8}$$

- Since we have rescaled it, the $t$ variable, forming the fibre, is being shrunk while the base variable, $\theta \in \mathbb{S}$, is of fixed size.

- The adiabatic vector fields can now be seen

$$\partial_r = -\frac{1}{\epsilon}\partial_t,\ \partial_\theta = \partial_\theta + \frac{R'}{\epsilon}\partial_t. \tag{9}$$

- Thus the Euclidean Laplacian becomes an elliptic adiabatic operator

$$(\partial_x^2 + \partial_y^2) = \epsilon^{-2}P(\epsilon, t, \theta, \epsilon\partial_t, \partial_\theta) \tag{10}$$

- We can then ask – what can we recover from knowledge of the eigenvalues for small $\epsilon$?

- Clearly we need to add boundary conditions.
- The Dirichlet condition will mean that

$$\min \lambda(\Delta_D) > C\epsilon^{-2},\ C > 0 \tag{11}$$

  and in particular the family is invertible.
- The Neumann condition lead to small eigenvalues.
- Perhaps unfortunately the leading terms here are very simple:

$$\lambda_k(\Delta_D) = ck^2 + \epsilon F(\epsilon, k)$$

  where $c$ is fixed.
- Question: Is the problem behind the small eigenvalues for the Neumann problem integrable – are there invariants which can be recovered from them?
- In particular of course can one recover $R$ (up to rotation) from these small eigenvalues?

- Instead of scaling the domain one could force the width to be constant in the sense that one could look at the region

$$\{z \in \mathbb{R}^2; d(z, C) \le \epsilon\}$$

where $C$ is the fixed bounding curve.

- What happens to the eigenvalues then?

Best wishes Gunther for many more years and theorems!