The inverse conductivity problem with power densities in dimension $n \geq 2$

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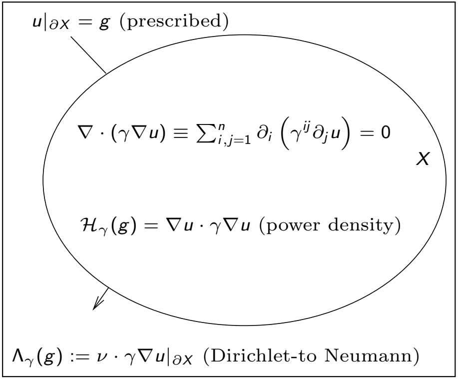
Outline

The inverse conductivity (diffusion) problem

<u>The model:</u> $X \subset \mathbb{R}^n$ bounded domain, $n \geq 2$.



γ is *uniformly elliptic*.

- **Calderón's problem:** Does Λ_γ determine γ uniquely ? stably ?

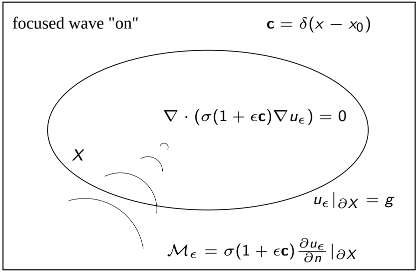
[Calderón '80]

- **Power density problem:** Does $\mathcal{H}_\gamma$ determine γ uniquely ? stably ?

<u>Application:</u> EIT or OT coupled with acoustic waves.

Derivation of power densities - 1/2

By ultrasound modulation



Physical focusing

[Ammari et al. '08]

Synthetic focusing

[Kuchment-Kunyansky '10]

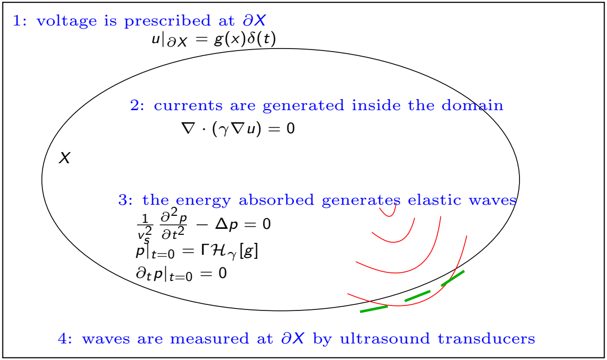
[Bal-Bonnetier-M.-Triki '11]

Small perturbation model:

$\frac{(\mathcal{M}_\epsilon - \mathcal{M}_0)}{\epsilon}$ gives an approximation of $\nabla u_0 \cdot \gamma \nabla u_0$ at x_0.

Derivation of power densities - 2/2

By thermoelastic effects (Impedance-Acoustic CT)



One reconstructs $\Gamma\mathcal{H}_\gamma = \Gamma\nabla u \cdot \gamma \nabla u$ over X (Γ: Grüneisen coefficient)

[Gebauer-Scherzer '09]

Power density measurements - References

Resolution of the power density problem:

- 2D isotropic [Capdeboscq et al. '09].
- 2D-3D isotropic linearized [Kuchment-Kunyansky '11].
- 2D-3D isotropic [Bal-Bonnetier-M.-Triki,IPI '12].
- n-D isotropic and measurements of the form $H_{ij} = \sigma^{2\alpha}\nabla u_i \cdot \nabla u_j$ [M.-Bal,IPI '12].
- 2D anisotropic: reconstruction formulas, stability and numerical implementation [M.-Bal,IP '12].
- Pseudodifferential calculus on the linearized isotropic case [Kuchment-Steinhauer,'12].
- n-D anisotropic [M., Ph.D. thesis '12]

Power density measurements - References

The zero-Laplacian problem: Reconstruct a scalar conductivity γ from knowledge of **one** power density $H = \gamma|\nabla u|^2$. This yields the non-linear PDE

$$\nabla \cdot (H|\nabla u|^{-2}\nabla u) = 0 \quad (X), \quad u|_{\partial X} = g.$$

Hyperbolic equation nicknamed the **zero-Laplacian**.

References:

- Newton-based numerical methods to recover (u, γ) [Ammari et al. '08, Gebauer-Scherzer '09].
- Theoretical work on the Cauchy problem [Bal '11].

Resolution - Overview

Problem: Reconstruct γ from $H_{ij} = \nabla u_i \cdot \gamma \nabla u_j$ with

$$\nabla \cdot \gamma \nabla u_i = 0 \quad (X), \quad u_i|_{\partial X} = g_i, \quad 1 \leq i \leq m.$$

Decompose $\gamma = (\det \gamma)^{\frac{1}{n}} \tilde{\gamma}$ with $\det \tilde{\gamma} = 1$.

We accept *redundancies* of data (no limitation on m a priori).

Outline:

- Local reconstruction algorithms (and their conditions of validity)
  - of $\det \gamma$ from known anisotropic structure $\tilde{\gamma}$
  - of the anisotropic structure $\tilde{\gamma}$
- Global questions:
  - study of **admissible boundary conditions**
  - study of **reconstructible tensors**

Outline

The frame approach, local reconstruction of $\det \gamma$

Differential geometric setup: Euclidean metric and connection $\overline{\nabla}$.

Frame condition: Let n conductivity solutions such that

$(\nabla u_1, \dots, \nabla u_n)$ is a **frame** over some $\Omega \subset X$.

Def: $A := \gamma^{\frac{1}{2}} = (\det A)^{\frac{1}{n}} \widetilde{A}$ with $\det \widetilde{A} = 1$. Set $S_i := A \nabla u_i$.

Data is $H_{ij} = \nabla u_i \cdot \gamma \nabla u_j = S_i \cdot S_j$ and S_i solves:

$$\nabla \cdot (\widetilde{A} S_i) = -F \cdot \widetilde{A} S_i, \quad d(\widetilde{A}^{-1} S_i)^\flat = F^\flat \wedge (\widetilde{A}^{-1} S_i)^\flat, \quad F := \nabla \log (\det A)^{\frac{1}{n}}.$$

We first derive $F = \dfrac{1}{n|H|^{\frac{1}{2}}} \left(\nabla (|H|^{\frac{1}{2}} H^{ij}) \cdot \widetilde{A} S_i \right) \widetilde{A}^{-1} S_j$ by studying

the behavior of the dual frame to $(\widetilde{A}^{-1} S_1, \dots, \widetilde{A}^{-1} S_n)$.

Legend: known data, unknown, anisotropic structure (known here).

Local reconstruction of $\det\gamma$

A **first-order quasi-linear system** is then derived for the frame S

$$\overline{\nabla} S_i = H^{kq} H^{jp} (\overline{\nabla}_{\widetilde{A} S_q} S_i \cdot S_p)\ S_j \otimes (\widetilde{A}^{-1} S_k)^\flat, \quad \text{where}$$

$$2\overline{\nabla}_{\widetilde{A} S_q} S_i \cdot S_p = \widetilde{A} S_q \cdot \nabla H_{ip} + \widetilde{A} S_p \cdot \nabla H_{iq} - \widetilde{A} S_i \cdot \nabla H_{pq} + 2 H_{pq} F \cdot \widetilde{A} S_i - 2 H_{qi} F \cdot \widetilde{A} S_p$$
$$- \mathcal{A}_{\widetilde{A}}(S_q, S_p) \cdot S_i - \mathcal{A}_{\widetilde{A}}(S_i, S_p) \cdot S_q + \mathcal{A}_{\widetilde{A}}(S_q, S_i) \cdot S_p.$$

In short,

$$\overline{\nabla} S_i = \mathcal{S}_i(S, \widetilde{A}, d\widetilde{A}, H, dH), \quad 1 \leq i \leq n,$$

where $\mathcal{S}_i$ is **Lipschitz** w.r.t. $(S_1, \dots, S_n)$. Then,

$$\nabla \log\det\gamma = \mathcal{F}(S, \widetilde{A}, H, dH).$$

- Overdetermined PDEs, solvable for S and $\log\det\gamma$ over $\Omega \subset X$ via ODE's along any characteristic curves.

Local reconstruction of $\det\gamma$

Theorem (Uniqueness and Lipschitz stability in $W^{1,\infty}(\Omega)$)

Over $\Omega \subset X$ where the frame condition is satisfied, $\det\gamma$ is **uniquely** *determined up to a (multiplicative) constant. Moreover,*

$$\|\log\det\gamma - \log\det\gamma'\|_{W^{1,\infty}} \le \varepsilon_0 + C(\|H - H'\|_{W^{1,\infty}} + \|\widetilde{A} - \widetilde{A}'\|_{W^{1,\infty}}),$$

where ε_0 is the error commited at some $x_0 \in \Omega$.

[Capdeboscq et al. '09], [Bal-Bonnetier-M.-Triki, '12], [M.-Bal, IP '12], [M.-Bal, IPI '12]

- **Well-posed** problem if the anisotropy is known.
  - No loss of derivative/resolution on $|\gamma|$.

Anisotropy reconstruction - derivation - 1/2

Goal: Reconstruct $\tilde{\gamma}$ from enough functionals $H_{ij} = \nabla u_i \cdot \gamma \nabla u_j$.

• Start from a frame of conductivity solutions $(\nabla u_1, \dots, \nabla u_n)$ and consider an **additional solution** v.

• Key fact: the decomposition of $A\nabla v$ in the basis $(S_1, \dots, S_n)$ is **known from the power densities**:

$$A\nabla v = \mu_i S_i, \quad \text{with } \mu_i(H) \text{ known.}$$

• Using $\nabla \cdot (AS_i) = 0$ and $d(A^{-1}S_i)^\flat = 0$, we obtain

$$Z_i \cdot \widetilde{A} S_i = 0 \quad \text{and} \quad Z_i^\flat \wedge (\widetilde{A}^{-1} S_i)^\flat = 0, \quad Z_i = \nabla \mu_i.$$

Writing $Z = [Z_1 | \dots | Z_n]$, this is equivalent to $(\langle A, B\rangle := \text{tr}\,(AB^T))$

$$\boxed{\langle \widetilde{A} S, Z\rangle = 0 \quad \text{and} \quad \langle \widetilde{A} S, ZH\Omega\rangle = 0, \quad \Omega \in A_n(\mathbb{R}).}$$

This is $1 + r(n - \frac{r+1}{2})$ linear constraints on $\widetilde{A}S$, where $r = \text{rank } Z$.

Equations: $\nabla \cdot (\gamma \nabla u_i) = 0 \quad (X), \quad u_i|_{\partial X} = g_i, \quad A := \gamma^{\frac{1}{2}}, \quad S_i = A\nabla u_i$

Anisotropy reconstruction - derivation - 2/2

- Hyperplane condition: Assume that $(v_1, \dots, v_\ell)$ are so that $Z_{(1)}, \dots, Z_{(\ell)}$ yield $n^2 - 1$ independent constraints on $\widetilde{A}S$.
- Reconstruct $B = \widetilde{A}S$ via a generalization of the **cross-product** in $\mathcal{M}_n(\mathbb{R})$.
- Reconstruct $\tilde{\gamma} = \widetilde{A}^2 = BH^{-1}B^T$, then $S = \tilde{\gamma}^{-\frac{1}{2}}B$ (then $\det \gamma$).

Theorem (Uniqueness and stability for $\tilde{\gamma}$)

Over $\Omega \subset X$ where the **frame** *condition and the* **hyperplane** *condition are satisfied, $\tilde{\gamma}$ is* **uniquely** *determined, with stability*

$$\|\tilde{\gamma} - \tilde{\gamma}'\|_{L^\infty(\Omega)} \le C\|H - H'\|_{W^{1,\infty}(X)}.$$

[M.-Bal, IP '12] in 2D.

- **Explicit reconstruction**. Loss of **one derivative** on $\tilde{\gamma}$.

Anisotropy reconstruction - remark

In the linearized case, **one** full-rank matrix Z (i.e. **one** well-chosen additional solution) yields a Fredholm inversion (requires the inversion of a strongly coupled elliptic system whose invertibility cannot always be established), although this is only $1 + \frac{n(n-1)}{2}$ constraints.

[Bal-M.-Guo '12], in progress.

Outline

Admissible sets - the frame condition

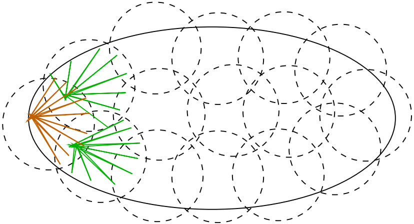
Question: How to fulfill the **frame condition** globally ?

- **Admissibility sets $\mathcal{G}_\gamma^m$, $m \ge n$:**
  - $(g_1, \dots, g_m) \in \mathcal{G}_\gamma^m$ if one can cover X with open sets Ω_p with a frame made of ∇u_i's on each Ω_p.
  - expressible in terms of continuous functionals of the data $\nabla u_i \cdot \gamma \nabla u_j$.

  ▶ $\det \gamma$ is *reconstructible* if $\mathcal{G}_\gamma^m \neq \emptyset$ for some $m \ge n$.

- **Patching local ODE-based reconstructions:**

$$\overline{\nabla} \log \det \gamma = \mathcal{F}(S, H, dH, \widetilde{A}),$$
$$\overline{\nabla} S_i = \mathcal{S}_i(S, H, dH, \widetilde{A}, d\widetilde{A}),$$
$$1 \le i \le n.$$



Admissible sets - the hyperplane condition

Question: How to fulfill the **hyperplane condition** globally ?

- **The admissibility sets $\mathcal{A}_\gamma^{m,\ell}(\mathbf{g})$ for $\mathbf{g} \in \mathcal{G}_\gamma^m$:**
  - $\mathbf{g}$ provides a *support basis* throughout X.
  - $(h_1, \dots, h_\ell) \in \mathcal{A}_\gamma^{m,\ell}(\mathbf{g})$ if the hyperplane condition (expressible in terms of H_{ij} and dH_{ij}) is satisfied throughout X.

    ▶ $\tilde{\gamma}$ is *reconstructible* if $\mathcal{A}_\gamma^{m,\ell}(\mathbf{g}) \neq \emptyset$ for some $\ell \geq 1$.

Admissible sets - Properties

Properties of $\mathcal{G}_\gamma^m$, $m \ge n$ and $\mathcal{A}_\gamma^{m,\ell}(\mathbf{g})$:

- They are open for the topology of $\mathcal{C}^2(\partial X)$
- BC's that work for γ also work for $\mathcal{C}^1$-close perturbations of γ.
- γ is reconstructible if and only if $\Psi_\star\gamma := \left[\frac{D\Psi\ \gamma\ D\Psi^T}{|\det D\Psi|}\right] \circ \Psi^{-1}$ is reconstructible, with $\Psi : X \to \Psi(X)$ a diffeomorphism.

Reconstructibility results:

- If $\gamma = \mathbb{I}_n$, then $\{u_i = x_i\}_{i=1}^n$ provides a support basis and $\{v_j = \frac{1}{2}(x_j^2 - x_{j+1}^2)\}_{j=1}^{n-1}$ allow to reconstruct the anisotropic structure.
- If $\gamma = \sigma\mathbb{I}_n$ with σ smooth, the CGO's provide a support basis to reconstruct σ. [Bal et. al,'12] [M.-Bal, '12]
- As in [Bal-Uhlmann, '12], we expect a reconstructibility result based on the Runge approximation for $\mathcal{C}^1$ conductivities.

A word about the two-dimensional case

- Reconstruction of $\det \gamma$:

The **frame condition** can easily be satisfied globally as soon as two boundary conditions form a homeomorphism of ∂X onto its image. [Alessandrini-Nesi '01]

- Reconstruction of $\tilde{\gamma}$: [M.-Bal, IP '12]

$$\tilde{\gamma} = (JX \cdot Y)^{-1} J(XX^T + YY^T)J, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$
$$Y = \nabla \log \frac{H_{11}H_{22} - H_{12}^2}{H_{22}H_{33} - H_{23}^2} = \nabla \log \frac{\det(\nabla u_1, \nabla u_2)}{\det(\nabla u_2, \nabla u_3)}.$$

With B.C. (g_1, g_2, g_3) chosen linearly independent, the set where Y vanishes (i.e. non reconstructible points) has **empty interior**.

Concluding remarks

Inverse conductivity from power densities:

- **Explicit reconstruction algorithms** under qualitative hypotheses, making the problem **injective**.
- **Stability:**
  - **No loss** of scales for $\det\gamma$.
  - **Loss of one derivative** for $\tilde{\gamma}$.
- Some cases where the **hypotheses are valid**:
  - Near isotropic smooth or anisotropic constant tensors.
  - Push-forwards and C^1-perturbations of the above.

▶ Practical benefits: **Resolution improvements** (compared to boundary measurements) and access to **anisotropic information**.

Extensions of results

- Generalization to **lower regularity** $\gamma \in L^\infty$ or **degenerate cases** ?
- **Potential applications** of the method presented to other hybrid inverse problems
  - CDII, MREIT: $\mathcal{H} = |\gamma \nabla u|$ [Nachman et al.,'10,'11]
  - OT with absorption: $\mathcal{H} = \nabla u \cdot \gamma \nabla u + \sigma_a u^2$, u solves $-\nabla \cdot (\gamma \nabla u) + \sigma_a u = 0$, reconstruct (γ, σ_a) ?
  - Systems: elastography with internal measurements.
- **Adding the coupling coefficient** $(\mathcal{H}_\gamma \leftarrow \Gamma \mathcal{H}_\gamma)$: is Γ reconstructible from enough functionals ?