Geodesic ray transforms and tensor tomography

Mikko Salo
University of Jyväskylä

Joint with Gabriel Paternain (Cambridge) and Gunther Uhlmann (UCI / UW)

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X-ray transform

X-ray transform for $f \in C_c(\mathbb{R}^n)$:

$$If(x, \theta) = \int_{-\infty}^{\infty} f(x + t\theta)\, dt, \quad x \in \mathbb{R}^n, \theta \in S^{n-1}.$$

Inverse problem: Recover f from its X-ray transform If.

- coincides with Radon transform if $n = 2$, first inversion formula by Radon (1917)
- basis for medical imaging methods CT and PET
- Cormack, Hounsfield (1979): Nobel prize in medicine for development of CT

X-ray transform

We will consider more general ray transforms that may involve

- weight factors
- integration over more general families of curves
- integration of tensor fields

Weighted transforms

Ray transform with attenuation $a \in C_c(\mathbb{R}^n)$:

$$I^a f(x,\theta) = \int_{-\infty}^{\infty} f(x+t\theta) e^{\int_0^\infty a(x+t\theta+s\theta)\,ds}\,dt, \quad x \in \mathbb{R}^n, \theta \in S^{n-1}.$$

Arises in the imaging method SPECT and in inverse transport with attenuation:

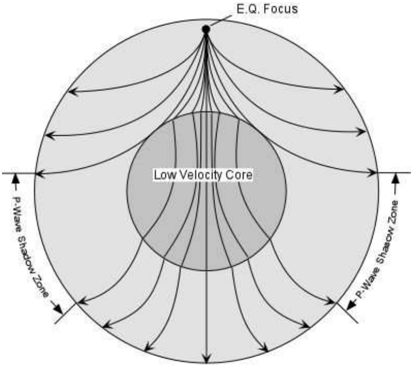
$$Xu + au = -f$$

where $Xu(x,\theta) = \theta \cdot \nabla_x u(x,\theta)$ is the geodesic vector field.

Injectivity ($n = 2$): Arbuzov-Bukhgeim-Kazantsev (1998).

Boundary rigidity

Travel time tomography: recover the sound speed of Earth from travel times of earthquakes.



Boundary rigidity

Model the Earth as a compact Riemannian manifold (M, g) with boundary. A scalar sound speed $c(x)$ corresponds to

$$g(x) = \frac{1}{c(x)^2}\, dx^2.$$

A general metric g corresponds to anisotropic sound speed.

Inverse problem: determine the metric g from travel times $d_g(x, y)$ for $x, y \in \partial M$.

By coordinate invariance can only recover g up to isometry. Easy counterexamples: region of low velocity, hemisphere.

Boundary rigidity

Definition

A compact manifold (M, g) with boundary is *simple* if any two points are joined by a unique geodesic depending smoothly on the endpoints, and ∂M is strictly convex.

Conjecture (Michél 1981)

A simple manifold (M, g) is determined by d_g up to isometry.

- Herglotz (1905), Wiechert (1905): recover $c(r)$ if

$$\frac{d}{dr}\left(\frac{r}{c(r)}\right) > 0$$

- Pestov-Uhlmann (2005): recover g on simple surfaces

Geodesic ray transform

Let (M, g) be compact with smooth boundary. Linearizing $g \mapsto d_g$ in a fixed conformal class leads to the *ray transform*

$$If(x, v) = \int_0^{\tau(x,v)} f(\gamma(t, x, v))\, dt$$

where $x \in \partial M$ and $v \in S_x M = \{v \in T_x M\,;\, |v| = 1\}$.

Here $\gamma(t, x, v)$ is the geodesic starting from point x in direction v, and $\tau(x, v)$ is the time when γ exits M. We assume that (M, g) is *nontrapping*, i.e. τ is always finite.

Tensor tomography

Applications of tomography for m-tensors:

- $m = 0$: deformation boundary rigidity in a conformal class, seismic and ultrasound imaging
- $m = 1$: Doppler ultrasound tomography
- $m = 2$: deformation boundary rigidity
- $m = 4$: travel time tomography in elastic media

Tensor tomography

Let $f = f_{i_1 \cdots i_m} \, dx^{i_1} \otimes \cdots \otimes dx^{i_m}$ be a symmetric m-tensor in M. Define $f(x,v) = f_{i_1 \cdots i_m}(x) v^{i_1} \cdots v^{i_m}$. The *ray transform* of f is

$$I_m f(x,v) = \int_0^{\tau(x,v)} f(\varphi_t(x,v)) \, dt, \quad x \in \partial M, v \in S_x M,$$

where φ_t is the geodesic flow,

$$\varphi_t(x,v) = (\gamma(t,x,v), \dot{\gamma}(t,x,v)).$$

In coordinates

$$I_m f(x,v) = \int_0^{\tau(x,v)} f_{i_1 \cdots i_m}(\gamma(t)) \dot{\gamma}^{i_1}(t) \cdots \dot{\gamma}^{i_m}(t) \, dt.$$

Tensor tomography

Recall the Helmholtz decomposition of $F : \mathbb{R}^n \to \mathbb{R}^n$,

$$F = F^s + \nabla h, \quad \nabla \cdot F^s = 0.$$

Any symmetric m-tensor f admits a *solenoidal decomposition*

$$f = f^s + dh, \quad \delta f^s = 0, \quad h|_{\partial M} = 0$$

where h is a symmetric $(m-1)$-tensor, $d = \sigma \nabla$ is the inner derivative (σ is symmetrization), and $\delta = d^*$ is divergence.

By fundamental theorem of calculus, $I_m(dh) = 0$ if $h|_{\partial M} = 0$. I_m is said to be *s-injective* if it is injective on solenoidal tensors.

Tensor tomography

Conjecture (Pestov-Sharafutdinov 1988)

If (M, g) is simple, then I_m is s-injective for any $m \geq 0$.

Positive results on simple manifolds:

- Mukhometov (1977): $m = 0$
- Anikonov (1978): $m = 1$
- Pestov-Sharafutdinov (1988): $m \geq 2$, negative curvature
- Sharafutdinov-Skokan-Uhlmann (2005): $m \geq 2$, recovery of singularities
- Stefanov-Uhlmann (2005): $m = 2$, simple real-analytic g
- Sharafutdinov (2007): $m = 2$, simple 2D manifolds

Tensor tomography

Theorem (Paternain-S-Uhlmann 2011)

If (M, g) is a simple surface, then I_m is s-injective for any m.

More generally:

Theorem (Paternain-S-Uhlmann 2011)

Let (M, g) be a nontrapping surface with convex boundary, and assume that I_0 and I_1 are s-injective and I_0^* is surjective. Then I_m is s-injective for $m \geq 2$.

Wave equation

Let $\Omega \subset \mathbb{R}^n$ bounded domain, $q \in C(\overline{\Omega})$.

$$(\partial_t^2 - \Delta + q)u = 0 \text{ in } \Omega \times [0, T], \quad u(0) = \partial_t u(0) = 0.$$

Boundary measurements

$$\Lambda_q^{Hyp} : u|_{\partial\Omega \times [0,T]} \mapsto \partial_\nu u|_{\partial\Omega \times [0,T]}.$$

Inverse problem: recover q from Λ_q^{Hyp}.

- scattering measurements related to X-ray transform (Lax-Phillips, . . .)
- recover X-ray transform of q from Λ_q^{Hyp} by geometrical optics solutions (Rakesh-Symes 1988)

Anisotropic Calderón problem

Medical imaging, Electrical Impedance Tomography:

$$\begin{cases} \Delta_g u = 0 & \text{in } M, \\ u = f & \text{on } \partial M. \end{cases}$$

Here g models the electrical resistivity of the domain M, and Δ_g is the Laplace-Beltrami operator. Boundary measurements

$$\Lambda_g : f \mapsto \partial_\nu u|_{\partial M}.$$

Inverse problem: given Λ_g, determine g up to isometry.

Known in 2D (Nachman, Lassas-Uhlmann), open in 3D.

Anisotropic Calderón problem

Dos Santos-Kenig-S-Uhlmann (2009): complex geometrical optics solutions

$$\Delta_g u = 0 \text{ in } M, \quad u = e^{\tau x_1}(v + r), \ \ \tau \gg 1.$$

Need that $(M, g) \subset\subset (\mathbb{R} \times M_0, g)$ where (M_0, g_0) is compact with boundary, and g is conformal to $e \oplus g_0$.

Here v is related to a high frequency quasimode on (M_0, g_0). Concentration on geodesics allows to use Fourier transform in the Euclidean part $\mathbb{R}$ and attenuated geodesic ray transform in (M_0, g_0).

Transport equation

Let (M, g) be a simple surface, and suppose that f is an m-tensor on M with $I_m f = 0$. Want to show that $f = dh$.

The function

$$u(x, v) = \int_0^{\tau(x,v)} f(\varphi_t(x, v))\, dt, \quad (x, v) \in SM$$

solves the transport equation

$$Xu = -f \;\text{ in } SM, \quad u|_{\partial(SM)} = 0.$$

Here $Xu(x, v) = \frac{\partial}{\partial t} u(\varphi_t(x, v))|_{t=0}$ is the *geodesic vector field*. Enough to show that $u = 0$.

Second order equation

Isothermal coordinates allow to identify

$$SM = \{(x,\theta)\,;\, x \in \overline{\mathbb{D}},\ \theta \in [0,2\pi)\}.$$

The *vertical vector field* on SM is $V = \frac{\partial}{\partial\theta}$. Want to show

$$\begin{cases} Xu = -f \\ u|_{\partial(SM)} = 0 \end{cases} \implies u = 0.$$

If f is a 0-tensor, $f = f(x)$, then $Vf = 0$. Enough to show

$$\begin{cases} VXu = 0 \\ u|_{\partial(SM)} = 0 \end{cases} \implies u = 0.$$

Second order equation

Need a uniqueness result for $P = VX$, where

$$P = e^{-\lambda} \frac{\partial}{\partial \theta} \left(\cos\theta \frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial x_2} + h(x,\theta) \frac{\partial}{\partial \theta} \right).$$

Facts about P:

- second order operator on 3D manifold SM
- has multiple characteristics
- $P + W$ has compactly supported solutions for some first order perturbation W
- subelliptic estimate $\|u\|_{H^1(SM)} \leq C\|Pu\|_{L^2(SM)}$

Uniqueness

Pestov identity in $L^2(SM)$ inner product when $u|_{\partial(SM)} = 0$:

$$\|Pu\|^2 = \|Au\|^2 + \|Bu\|^2 + (i[A, B]u, u)$$

where $P = A + iB$, $A^* = A$, $B^* = B$.

Computing the commutator gives (with K the Gaussian curvature of (M, g))

$$\|Pu\|^2 = \underbrace{\|XVu\|^2 - (KVu, Vu)}_{\geq 0 \text{ on simple manifolds}} + \|Xu\|^2$$

Thus $Pu = 0$ implies $u = 0$, showing injectivity of I_0.

Tensor tomography

Let $Xu = -f$ in SM, $u|_{\partial(SM)} = 0$ where f is an m-tensor. Interpret u and f as sections of trivial bundle $E = SM \times \mathbb{C}$, get

$$D_X^0 u = -f$$

where $D_X^0 = d$ is the flat connection.

This equation has gauge group via multiplication by functions c on M (preserves m-tensors). Gauge equivalent equations

$$D_X^A(cu) = -cf$$

where $D^A = d + A$ and $A = -c^{-1}dc$.

Tensor tomography

Pestov identity with a connection (in $L^2(SM)$ norms):

$$\|V(X+A)u\|^2 = \|(X+A)Vu\|^2 - (KVu, Vu) + \|(X+A)u\|^2 + (*F_A Vu, u)$$

Here $*$ is Hodge star and

$$F_A = dA + A \wedge A$$

is the curvature of the connection $D^A = d + A$.

If the curvature $*F_A$ and the expression (Vu, u) have suitable signs, gain a positive term in the energy estimate.

Tensor tomography

Problem: if D^A is gauge equivalent to D^0, then $F_A = F_0 = 0$.

Need a *generalized gauge transformation* that arranges a sign for F_A. This breaks the m-tensor structure of the equation, but is manageable if the gauge transform is *holomorphic*.

Fourier analysis in θ (Guillemin-Kazhdan 1978):

$$L^2(SM) = \bigoplus_{k=-\infty}^{\infty} H_k, \quad u = \sum_{k=-\infty}^{\infty} u_k$$

where H_k is the eigenspace of $-iV$ with eigenvalue k. A function $u \in L^2(SM)$ is *holomorphic* if $u_k = 0$ for $k < 0$.

Tensor tomography

Theorem (Holomorphic gauge transformation)

If A is a 1-form on a simple surface, there is a holomorphic $w \in C^\infty(SM)$ such that $X + A = e^w \circ X \circ e^{-w}$.

Related to injectivity of attenuated ray transform on simple surfaces (S-Uhlmann 2011).

Tensor tomography

Let $f = \sum_{k=-m}^{m} f_k$ be an m-tensor, and let

$$Xu = -f, \quad u|_{\partial(SM)} = 0.$$

Choose a primitive φ of the volume form ω_g of (M, g), so $d\varphi = \omega_g$. Let $s > 0$ be large, let $A_s = -is\varphi$, and choose a holomorphic w with $X + A_s = e^{sw} \circ X \circ e^{-sw}$.

The equation becomes

$$(X + A_s)(e^{sw} u) = -e^{sw} f, \quad e^{sw} u|_{\partial(SM)} = 0.$$

Here the curvature of A_s has a sign and one has information on Fourier coefficients of $e^{sw} f$. The Pestov identity with connection allows to control Fourier coefficients of $e^{sw} u$, eventually proving s-injectivity of I_m.

Relation to Carleman estimates

Pestov identity with connection A_s resembles a Carleman estimate:

$$s^{1/2}\|u\|_{L^2_x \dot{H}^{1/2}_\theta} \lesssim \|e^{sw}X(e^{-sw}u)\|_{L^2_x \dot{H}^1_\theta}.$$

Positivity comes from $\mathrm{Im}\,(w)$! This is enough to

- absorb large attenuation (even for systems)
- absorb error terms coming from m-tensors

This may not be enough to

- localize in space
- absorb error terms coming from curvature of M

Open questions

Conjecture

I_m is s-injective on simple manifolds when $\dim(M) \geq 3$ and $m \geq 2$.

Conjecture

I_m is s-injective on any compact nontrapping manifold with strictly convex boundary.