Gunther 60

Inverse Source Problem

Collaborators:
Steve Kusiak
Roland Griesmaier
Martin Hanke

John Sylvester - U W
Forward Source Problem

\[(\Delta + k^2) u^+ = f \text{ in } \mathbb{R}^n\]

**Theorem** \[\exists! \text{ outgoing } u^+\]

For \( k \in \mathbb{C}^+ \)
\[\hat{u} = \frac{\hat{f}(\xi)}{k^2 - \xi^2}\]

For \( k \in \mathbb{R} \)
\[\hat{u} = \lim_{\varepsilon \to 0} \frac{\hat{f}(\xi)}{(k+i\varepsilon)^2 - \xi^2}\]

Outgoing \(\Longleftrightarrow\) Sommerfeld Radiation Condition

\(\Longleftrightarrow\) Fourier Transform \((t \leftrightarrow k)\) in time
of solution to the wave equation
that vanishes in the past
Far Field = Restricted Fourier Transform

\[(\Delta + k^2)u = F \quad [\text{compactly supported}]\]

\[u = \frac{\hat{F}(\xi)}{(k + i\alpha)^2 - \xi^2}\]

as \( r \to \infty \)

\[u \sim \frac{e^{ikn}}{(kn)^{\frac{1}{2}}} \hat{u}(\Omega) = \frac{e^{ikn}}{(kn)^{\frac{1}{2}}} \hat{F}(k\Omega)\]

Rellich's Lemma: If \( u \) decays faster than \( \frac{1}{r^{\gamma_2}} \) at \( \infty \), \( u = 0 \) outside \( \text{ch}(\text{supp} F) \).

Unique Continuation: \( u = 0 \) on \( \text{supp}_\infty F \)

\( \text{supp}_\infty F = \text{unbounded connected component of } \mathbb{R}^2 \setminus (\text{supp} F) \)
The Inverse Source Problem
\((\Delta + k^2) u^+ = f\)

**Linear but Non-Uniqueness**

**Non-Radiating Sources** \(f \mapsto \Phi\) has a big kernel

**NR Volume Sources** \([F \in L^2(\text{compact set})]\)

\(f \in \text{NR} \iff f = (\Delta + k^2) \Phi_{oo}\)

\(\Phi_{oo}\) means \(H^2_0(\text{compact set})\)

**Proof**

\((\Delta + k^2) \Phi_{oo} = 0 \quad \text{on} \quad \{\varphi^2 = k^2\}\)

\(\Rightarrow\) Rellich + unique continuation \(\Rightarrow u^+ = \Phi_{oo}\)
**Free Sources**

**Corollary**  Every source supported in \( \Omega \) has a unique equivalent \( \Omega \)-Free source.

Equivalent sources radiate the same far field
A free source satisfies \((\Delta + k^2) F = 0\) in \( \Omega \)

**Proof**  Solve the clamped plate equation

\[
(N^2 + \Delta + k^2)^2 \Phi_{oo} = (\Delta + k^2) \Phi
\]

\[
\Phi = f - (\Delta + k^2) \Phi_{oo} \text{ is equivalent to } F \text{ and } (\Delta + k^2) F = 0
\]

**Pseudo Inverse**  The unique free source is also the source with minimal \( L^2(\Omega) \) norm
Criticisms of the $\Omega$-Free Source

1. We need to start by choosing $\Omega$.

2. No matter how small the support of the true source is, the support of the free source is all of $\Omega$.

3. But it does have the smallest $L^2$-norm.
The $L^2$-norm of the Free Source

$$(\Delta + k^2) u^+ = F \quad (\Delta + k^2) \varphi^0 = 0$$

$$\varphi^0 = \sum_{n=0}^{\infty} \frac{J_n(kr)}{n!} e^{in\theta}$$

**Standard Far Field Calculation**

$$\int_{\mathbb{R}^n} f n^0 = \int_{\mathbb{R}^n} \left( \Delta + k^2 \right) u^+ \varphi^0 = \lim_{R \to \infty} \int_{B_R} \left( \Delta + k^2 \right) u^+ \varphi^0$$

$$= \lim_{R \to \infty} \int_{\partial B_R} \frac{\partial u^+}{\partial n} \varphi^0 - u^+ \frac{\partial \varphi^0}{\partial n} = \int_{S^2} u^+ \varphi^0 ds$$

If $\varphi^0 = e^{i n \theta}$ then $\varphi^0 = J_n(kr) e^{i n \theta}$, so

If $u^+ = e^{i n \theta}$ then $F_{\text{free}} = \frac{J_n(kr) e^{i n \theta}}{\|J_n(kr)\|_2(L^2(\mathbb{R}))} \cdot \chi_{S^2}$
The $\mathcal{S}_L$-free source (which has minimal $L^2$ norm)

$$f = \sum f_n e^{in\theta} \frac{J_n(kr)}{||J_n(kr)||_{L^2}}$$

radiates

$$\sum f_n e^{in\theta}$$

$$||f|| = \sum \frac{|f_n|}{||J_n(kr)||_{L^2}}$$

For $\mathcal{S} = \mathcal{B}_R(0)$, $||f||_{L^2}^2 = \sum \frac{|f_n|^2}{||J_n(kr)||_{L^2}^2}$

\[
\|
\| J_n \|_{L^2(B_r)}^2
\]
Test for finding supp $f$

**Theorem**

\[ \exists f \text{ supported in } B_{r}(0) \iff \| f_{n} \|^2 \leq \frac{\| f_{n}(x) \|^2_{L^{2}(B_{r}(0))}}{\| J_{n}(x) \|^2_{L^{2}(B_{r}(0))}} \text{ converges} \]

**Translation**

\[ f^{p} := f(x - p) \Rightarrow \tilde{f}^{p}(\xi) = e^{2\pi i p \cdot \xi} \tilde{f}(\xi) \]

\[ |\tilde{f}| \leq |f^{p}| \leq e^{\| p \|^2} \]

**Theorem**

\[ \exists f \text{ supported in } B_{r}(p) \iff \| f_{n} \|^2 \leq \frac{\| f_{n}(x) \|^2_{L^{2}(B_{r}(p))}}{\| J_{n}(x) \|^2_{L^{2}(B_{r}(p))}} \text{ converges} \]
Combining the Two Tests

Ω carries 1 if ∀ε > 0 ∃ F such that

\[ \text{supp } F \subset \text{N}_{\epsilon}\]
\[ |F| = \delta \]

Question: If Ω₁ and Ω₂ each carry 1, does \( (\Omega_1 \cap \Omega_2) \) carry 1?

In general, No. If Ω₁ and Ω₂ are convex, Yes.
Finding Branch Cuts
(An analogy)

\[ \sqrt{z(z-1)} = \sqrt{1-\frac{1}{z^2}} \sim z - \frac{1}{2z} \ldots \text{as } z \to \infty \]

\[(\Delta + k^2) u = f\]

\[ \overline{\omega}u = [u]dz|_{\gamma_i} \]

for any \( \gamma \) which joins \( 0 \) and \( 1 \)

1. Any curve with the correct endpoints carries the far field.

2. Once \( \gamma \) is fixed, \( f \) is unique (as long as \( \mathcal{C} \setminus \gamma \) is connected)

3. \( \gamma_1 \cap \gamma_2 \) doesn't carry the far field
exists convex branch cut $\Sigma_3$

and $\Sigma_3 \subseteq$ convex hull of any other $\Sigma$

$\Sigma_3 =$ smallest convex set that carries the "far field"
Lemma

If \( \Omega_1 \) carries \( \lambda \) and \( \Omega_2 \) carries \( \lambda \), then \( \Omega_1 \cap \Omega_2 \) carries \( \lambda \)

\[ (\Delta + \kappa^2) u_1 = F_1 \quad (\Delta + \kappa^2) u_2 = F_2 \]

Rellich's Lemma and Unique Continuation guarantee that

\[ u_1 \equiv u_2 \quad \text{on} \quad (\mathbb{R}^n \setminus \Omega_1) \cap (\mathbb{R}^n \setminus \Omega_2) \]

so that

\[ \nu^i := \begin{cases} 
    a u_1 & \text{on} \quad \mathbb{R}^n \setminus \Omega_1 \\
    a u_2 & \text{on} \quad \mathbb{R}^n \setminus \Omega_2 \\
    0 & \text{on} \quad \Omega_1 \cap \Omega_2 \end{cases} \]

is well-defined and

\[ F_3 := (\Delta + \kappa^2) \nu^i \quad \text{is supported in} \quad \mathbb{R}^n \setminus (\Omega_1 \cap \Omega_2) \]
Theorem
Every Far Field [with a compactly supported source] has a unique minimal convex carrier.

Theorem
Every Far Field [with a compactly supported source] has a unique minimal WSCS carrier.

Union of Well Separated Convex Sets -

distance between > diameter of any component convex components

Proof
Define $\text{csupp} = \bigcap \Omega \text{ such that } \Omega \text{ carries a convex }$

Does it carry the Far Field?
Lemma plus compactness says yes.
Unions of Well Separated Convex Sets [compact]

Satisfy

1. Closed under intersection
2. $\mathbb{R}^n \setminus (S_1 \cup S_2)$ connected

which guarantees some lemma.

Moral: It makes theoretical sense to look for collections of sources that are small compared to the distance between them.
How small is the c-support of a far field

\[ f = \chi_{\text{Ellipses}} \]

\[ f = \chi_{\text{Box}} \]

\[ f = \chi_{\text{Point}} \]

\[ f = \chi_{\text{Sum of Points}} \]

\[ c\text{-support} = \text{Point} \]

\[ c\text{-support} = \text{Convex Hull} \]

\[ \text{UWSCE-support} = \text{Union of Points} \]

\[ c\text{-support} = \text{Box} \quad \left[ \text{because of } f \right] \]

\[ c\text{-support} = \text{Box} \quad \text{if Taylor Expansion of } \Phi \text{ at corners starts with harmonic poly} \]

\[ c\text{-support} = \text{Line connecting Foci} \]
Wavelength - Distinguishing well-separated Sources

Point Sources

\[ \hat{\delta}_o = 1 \]

\[ \hat{\delta}_R = e^{i k R \cos \theta} \]

I can distinguish them reliably if the cosine of the angle between them < 1

\[ \frac{\hat{\delta}_o \cdot \hat{\delta}_R}{\sqrt{||\delta_o|| \cdot ||\delta_R||}} = \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{i k R \cos \theta}{\sqrt{2\pi}}} = J_0(kR) < 1 \]

Since \[ |J_0(kR)| < \frac{1}{\sqrt{kR}} \], \( kR < 1 \) suffices.

They need to be more than a wavelength apart.

Increasing \( k \) increases well-posedness.
Distinguishing Bigger Sources

\[ \Theta(\theta) = \text{Far Field carried by } B_1 \]
\[ \phi(\theta) = \text{Far Field carried by } B_2 \]

Theorem

Conclusion

\[ \int \sum_{\theta} \frac{d\theta}{\| \Delta \| \cdot \| \beta \|} \leq \text{const. } \frac{kr^2}{R} \]

But

\[ \text{Span} \{ a_i \} \text{ is dense and Span} \{ B_j \} \text{ is dense} \]

Hypothesis

\[ \frac{\| \phi_i \|_2^2}{\| \Delta \|_2^2} \leq M \]
\[ \frac{\| \phi_j \|_2^2}{\| \beta \|_2^2} \leq M \]

\[ M = \text{Power/sensitivity ratio} \]

Ratio of transmitted power necessary to generate received power
For $R > 3r$, \[ \frac{\text{d}n R \text{d} \omega}{\| \mathbf{n} \| \cdot \| \mathbf{b} \|} \leq \text{const.} \frac{kra^2}{R} \]

and for $\frac{kra}{R}$ small, it can be that big.

**Corollary** If you fix the geometry, and increase $k$, well-posedness get worse.

**Worst Case**

The far fields that are the most similar are the far fields of approximate plane waves perpendicular to the line connecting the centers.
Happy Birthday Gunther!!

I wish you many future collaborators
No more tenants!!
Some Examples to Illustrate the Need For Convexity

We expect minimal carriers to be small sets.

**Theorem** IF $S$ carries $d$, then $S$ carries $d$

\[\{0\} \quad \text{inside} \quad N_S(\Omega)\]

Replace $u^+$ by $\alpha u^+$

and replace $F$ by $(\Delta + k^2)(\alpha u^+)$
Thin Sources (Double and Single Layers)

A thin source is defined if:
1. \( f \in H^{s}(\mathbb{R}) \) for \( s > -2 \).
2. The measure \( (\text{supp} f) = 0 \) and compact.

Theorem
A thin source is NR \( \iff f = \mathcal{C} \psi_{0} \) [linear span]

\( \psi_{0} \) is a free wave \( (\Delta + k^{2}) \psi_{0} = 0 \) in \( \mathbb{R}^{2} \) and \( \psi_{0} \in L^{2}(\mathbb{R}^{2}) \)

\[ \mathcal{C} \psi_{0} \bigg|_{\partial \Omega} = \frac{\partial \psi_{0}}{\partial n} \sigma_{\partial \Omega} + \psi_{0} \delta_{\partial \Omega} \] [Cauchy Data restricted to boundary of \( \Omega \)]

\[ \langle \phi, \mathcal{C} \psi_{0} \rangle_{\mathbb{L}^{2}(\Omega)} = \int_{\Omega} \psi_{0} (\Delta + k^{2}) \phi \] [For any bounded open set \( \Omega \) and \( \psi_{0} \in L^{2}(\mathbb{R}^{2}) \)]
A thin source is \( NR \iff f = C_0 e^{\alpha z} \)

\[ \text{Proof} \]

\[ u^\pm = 0 \]

Start with \( N^0 \) from

Define \( u = \int_{N^0} \) inside \( \Omega \)

\[ (\Delta + k^2) u^\pm = C_0 e^{\alpha z} \]

\[ (\Delta + k^2) u^+ = 0 \] Rellich Lemma

\[ u^+ = 0 \] Unique Continuation

\[ \text{Rellich Lemma} \]

\[ (\Delta + k^2) u^+ = 0 \] here

\[ \int_{\partial \Omega} u^+ = 0 \]

\[ \int_{\partial \Omega} u^+ = 0 \] so \( f = 0 \) here

\[ \frac{\partial u^+}{\partial z} = \text{limit from inside here} \]
A arcs don't contain boundaries.

An arc is a thin source with support $\mathcal{R}$ such that $\mathbb{R}^n \setminus \mathcal{R}$ is connected. $\mathcal{S}$ is not empty.

Theorem

A non-radiating thin source supported on an arc is zero.

Proof

Corollary

Arcs are minimal carriers.

If support of a thin source is an arc, then no subset of the arc carries the same far field.
Example  \[ \mathcal{N} = \text{Free Set} \]

\[ \mathcal{N}_1 \cup \mathcal{N}_2 = \Omega \]

\[ \mathcal{N}_0 \]

\[ \mathcal{N}_1 \cup \mathcal{N}_0 + \mathcal{N}_2 \] is Non-Radiating

\[ \mathcal{N}_2 \]

\[ \mathcal{N}_0 \] and \[ \mathcal{N}_2 \] are equivalent

Both are arcs - so they are minimal

Moral

There is no unique minimal carrier

But

\[ \mathcal{F}_3 \] is the unique smallest convex set that carried the Far Field
Example \( \gamma^0 \) is non-radiating \( R = \sigma_1 \cup \sigma_2 \)

\( \gamma^0 \) and \(-\gamma^0\) radiate the same far field but only \( \gamma_1 \) is well-separated.