Active exterior cloaking

Fernando Guevara Vasquez
University of Utah

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Collaborators

Graeme W. Milton (University of Utah)

Daniel Onofrei (University of Houston)

Cloaking types

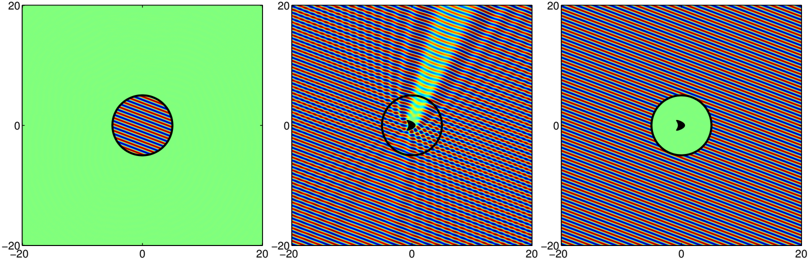
Interior/Exterior: Is the object hidden inside or outside a device?
Passive/Active: Are sources needed to cloak?

- Passive Interior
  - Transformation based cloaking: Leonhardt; Cummer, Pendry, Schurig, Smith; Greenleaf, Kurylev, Lassas, Uhlmann; Farhat, Enoch, Guenneau; Kohn, Onofrei, Shen, Vogelius, Weinstein; Cai, Chettiar, Kildishev, Shalaev; ...
  - Plasmonic cloaking: Alù, Engheta.
- Passive Exterior
  - Anomalous resonances: McPhedran, Milton, Nicorovici.
  - Complementary media: Lai, Chen, Zhang, Chan.
  - Plasmonic cloaking: Alù, Engheta, ...
- Active Interior: Miller
- Active Exterior:
  - Onofrei, Ren: integral equation framework
  - This work: (Laplace and) Helmholtz equations.

Helmholtz equation

$$\Delta u + k^2 u = 0$$

Active interior cloaking



Proposed by Miller 2001, but well known in acoustics since the 60s (Malyuzhinets; Jessel and Mangiante;...)

Green's identity

Let D be a domain in $\mathbb{R}^d$ ($d = 2$ or 3) with Lipschitz boundary.

$$u_d(\mathbf{x}) = \int_{\partial D} dS_{\mathbf{y}} \{-(\mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} u_i(\mathbf{y}))G(\mathbf{x},\mathbf{y}) + u_i(\mathbf{y})\mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} G(\mathbf{x},\mathbf{y})\}$$
$$= \begin{cases} -u_i(\mathbf{x}), & \text{if } \mathbf{x} \in D \\ 0, & \text{otherwise,} \end{cases}$$

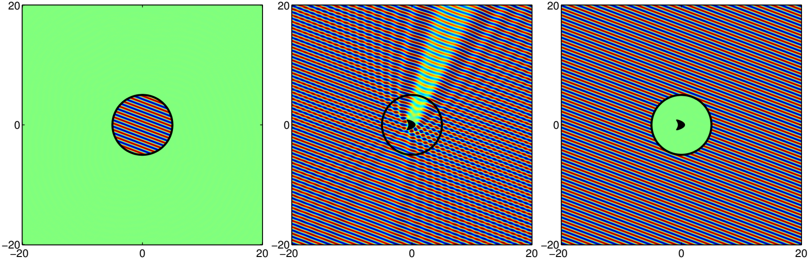
where the Green's function for the Helmholtz equation is

$$G(\mathbf{x},\mathbf{y}) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|) & \text{in 2D} \\ \dfrac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|} & \text{in 3D} \end{cases}$$

⇝ we get a single and double layer potential on ∂D so that

- $u_i + u_d = 0$ in D
- $u_d = 0$ in $\mathbb{R}^d \backslash D$.

Active interior cloaking



- With Green's identities: The object is completely surrounded by the cloak.
- To get exterior cloaking: replace the single and double layer potential in Green's identities by a few devices.

Active exterior cloaking (in 2D)

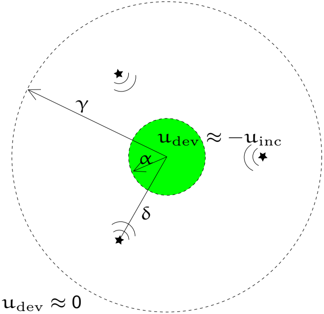
Devices' field u_{dev} must satisfy Helmholtz equation with Sommerfeld radiation condition. For point-like devices located at positions $\mathbf{x}_j$:

$$u_d(\mathbf{x}) = \sum_{j=1}^{n_{dev}} \sum_{m=-\infty}^{\infty} b_{j,m} V_m(\mathbf{x} - \mathbf{x}_j),$$

where the radiating solutions to the Helmholtz equation are

$$V_m(\mathbf{x}) \equiv H_m^{(1)}(k\,|\mathbf{x}|) \exp[im \arg(\mathbf{x})].$$

Designing devices that mimic Green's identities



We need:

(a) $u_{\mathsf{dev}}(\mathbf{x}) \approx -u_{\mathsf{inc}}(\mathbf{x})$ for $|\mathbf{x}| \leqslant \alpha$

(b) $u_{\mathsf{dev}}(\mathbf{x}) \approx 0$ for $|\mathbf{x}| \geqslant \gamma$

Since $u_{\mathrm{tot}} = u_{\mathrm{i}} + u_{\mathrm{d}} + u_{\mathsf{scat}}$,

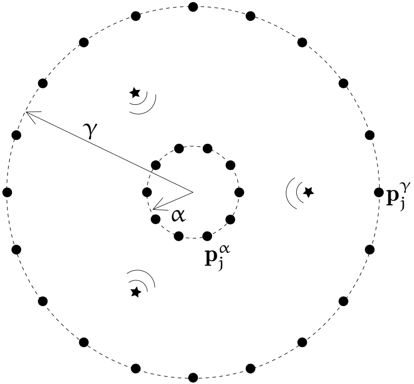
(a) $\Rightarrow u_{\mathsf{tot}}(\mathbf{x}) \approx 0$ for $|\mathbf{x}| \leqslant \alpha$

(b) $\Rightarrow u_{\mathsf{tot}}(\mathbf{x}) \approx u_{\mathsf{inc}}(\mathbf{x})$ for $|\mathbf{x}| \geqslant \gamma$

Caveats

- We need to know the incident field in advance, from e.g. sensors.
- Information from sensors needs to travel faster than incident field (OK for acoustics. For electromagnetics: periodicity?)
- Need very accurate reproduction of incident field (OK in controlled environments like MRI?)

Finding the coefficients numerically



(a') $u_{\text{dev}}(\mathbf{x}) \approx -u_{\text{inc}}(\mathbf{x})$ for $|\mathbf{x}| = \alpha$

(b') $u_{\text{dev}}(\mathbf{x}) \approx 0$ for $|\mathbf{x}| = \gamma$

Construct matrices $\mathbf{A}$, $\mathbf{B}$ s.t.

$$\mathbf{A}\mathbf{b} = [u_{\text{dev}}(\mathbf{p}_1^\alpha), \ldots, u_{\text{dev}}(\mathbf{p}_{N^\alpha}^\alpha)]^T,$$

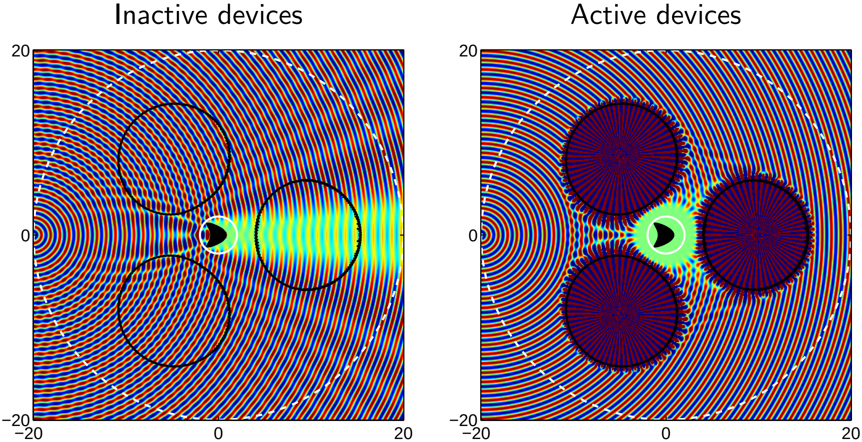
$$\mathbf{B}\mathbf{b} = [u_{\text{dev}}(\mathbf{p}_1^\gamma), \ldots, u_{\text{dev}}(\mathbf{p}_{N^\gamma}^\gamma)]^T,$$

where $\mathbf{b} \in \mathbb{C}^{(2M+1)D}$

$\equiv$ device coefficients.

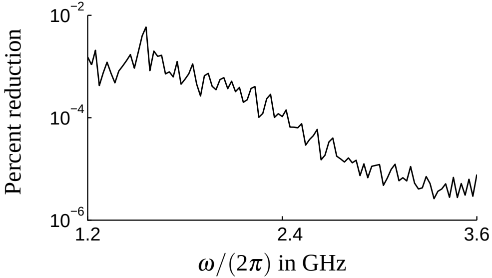
1. Find $\mathbf{b}_0 = \operatorname{argmin} \|\mathbf{A}\mathbf{b} + u_{\text{inc}}(|\mathbf{x}| = |\alpha|)\|_2^2$ (enforce (a'))
2. Find $\mathbf{b}_* = \underset{\mathbf{A}\mathbf{b}=\mathbf{A}\mathbf{b}_0}{\operatorname{argmin}} \|\mathbf{B}\mathbf{b}\|_2^2$ (enforce (b'))

Cloaking for one single frequency



$c = 3 \times 10^8$m/s, $\lambda = 12.5$cm, $\omega/(2\pi) = 2.4$GHz
$\alpha = 2\lambda$, $\delta = 5\lambda$, $\gamma = 10\lambda$.

Scattering reduction

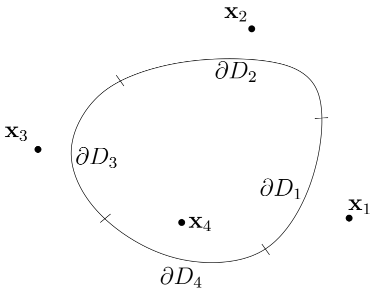


$c = 3 \times 10^8$m/s, $\lambda_0 = 12.5$cm, $\omega/(2\pi) \in [1.2, 3.6]$GHz

Devices for many frequencies (pulse)

By superposition principle: sum device fields for many ω to get cloaking in a bandwidth (i.e. in the time domain).

Green cloak devices idea



Idea The contribution of portion ∂D_j to the single and double layer potentials in Green's formula is replaced by a multipolar source located at $\mathbf{x}_j \notin \partial D$.

Graf's addition formula

The Green's function $G(\mathbf{x}, \mathbf{y})$ can be written as a superposition of sources located at $\mathbf{x}_j$:

$$G(\mathbf{x}, \mathbf{y}) = \frac{i}{4} H_0^{(1)}(k \left|\mathbf{x} - \mathbf{x}_j - (\mathbf{y} - \mathbf{x}_j)\right|)$$
$$= \frac{i}{4} \sum_{m=-\infty}^{\infty} V_m(\mathbf{x} - \mathbf{x}_j) \overline{U_m(\mathbf{y} - \mathbf{x}_j)},$$

where the entire cylindrical waves are

$$U_m(\mathbf{x}) \equiv J_m(k|\mathbf{x}|) \exp[i m \arg(\mathbf{x})]$$

and the sum converges uniformly in compact subsets of $\left|\mathbf{x} - \mathbf{x}_j\right| > \left|\mathbf{y} - \mathbf{x}_j\right|$.

Use summation formula to "move" monopoles and dipoles from a portion of the boundary to the corresponding $\mathbf{x}_j$.

Green cloak devices

The device field

$$u_d(\mathbf{x}) = \sum_{j=1}^{n_{dev}} \sum_{m=-\infty}^{\infty} b_{j,m} V_m(\mathbf{x} - \mathbf{x}_j),$$

with

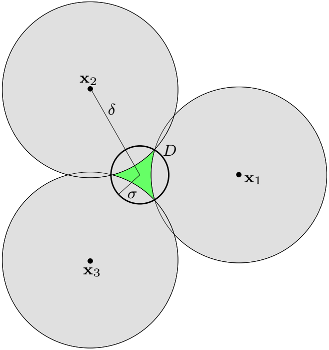
$$b_{j,m} = \int_{\partial D_j} dS_{\mathbf{y}} \{ (-\mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} u_i(\mathbf{y})) \, \overline{U_m(\mathbf{y} - \mathbf{x}_j)} \\ + u_i(\mathbf{y}) \mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \overline{U_m(\mathbf{y} - \mathbf{x}_j)} \}$$

converges (uniformly in compact subsets) outside of the region

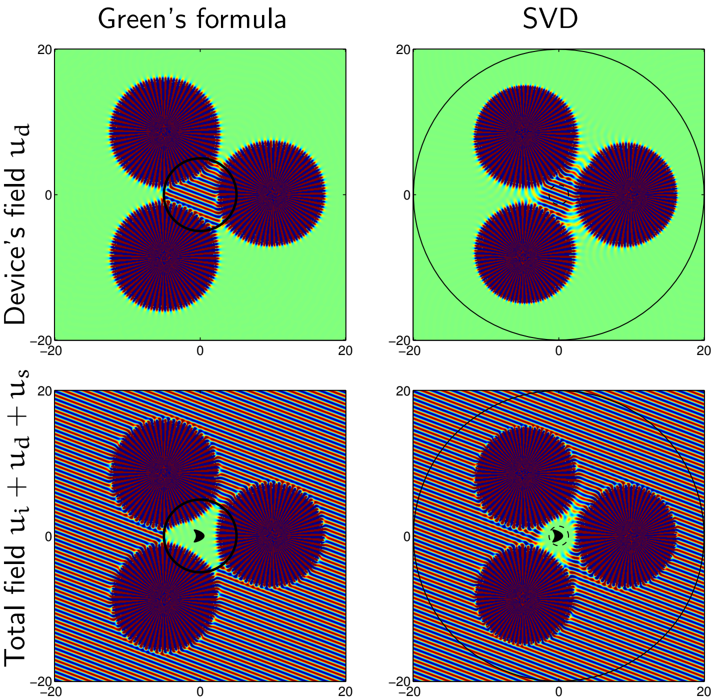
$$R = \bigcup_{l=1}^{n_{dev}} B\left(\mathbf{x}_l, \sup_{\mathbf{y} \in \partial D_l} |\mathbf{y} - \mathbf{x}_l| \right).$$

A specific configuration

With $D = B(0, \sigma)$ and devices $|\mathbf{x_j}| = \delta$:

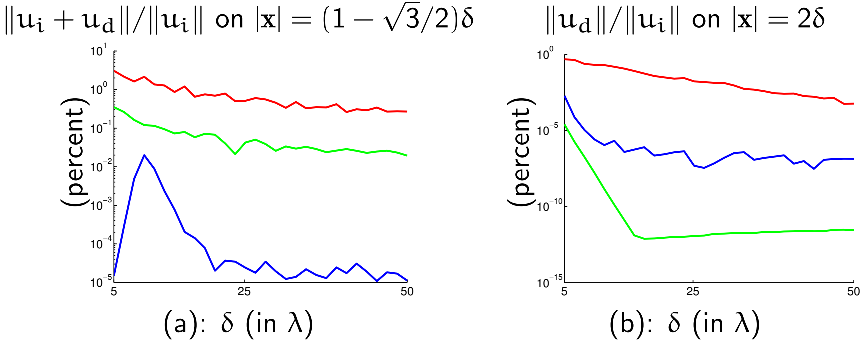


- Gray disks have radius: $r(\sigma, \delta) = ((\sigma - \delta/2)^2 + 3\delta^2/4)^{1/2}$.
- Largest disk in cloaked region radius: $r_{\text{eff}}(\sigma, \delta) = \delta - r(\sigma, \delta)$.
- Largest cloaked region ($\sigma^* = \delta/2$): $r^*_{\text{eff}}(\delta) = (1 - \sqrt{3}/2)\delta \approx 0.13\delta$.



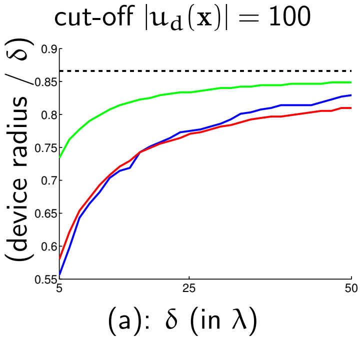
Green's formula
SVD
Device's field u_d
Total field $u_i + u_d + u_s$
20
0
−20
−20
0
20

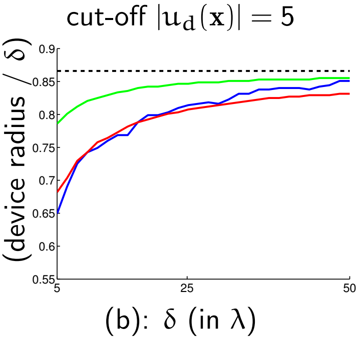
Cloak performance



- blue: SVD method with $M(\delta)$ terms
- red: Green's identity method with $M(\delta)$ terms
- green: Green's identity method with $2M(\delta)$ terms

Size of the "throats"

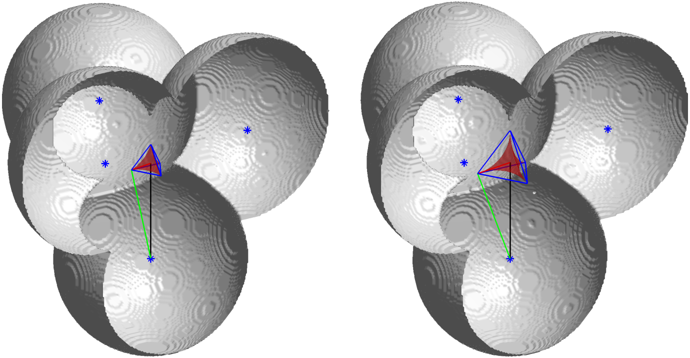




Estimated device radius relative to δ for different values of δ.

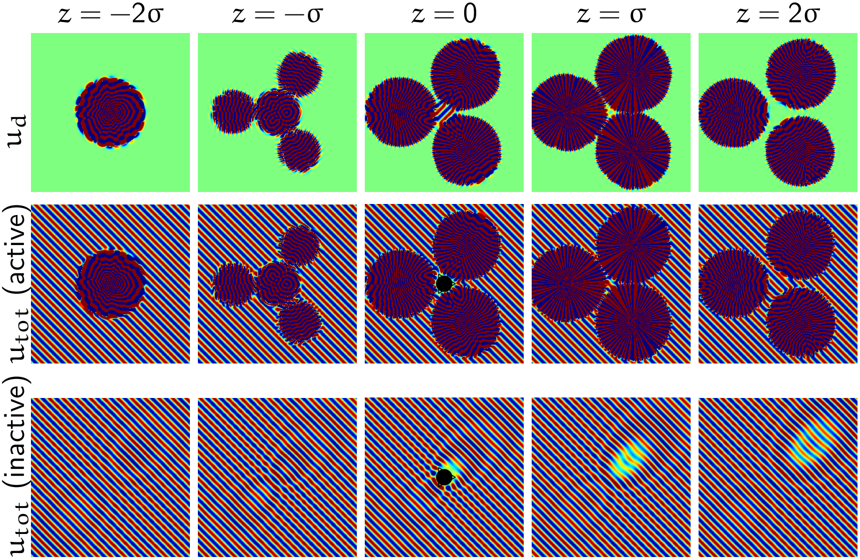
Cloaking for Helmholtz equation in 3D

With D =tetrahedron inscribed in $B(0,\sigma)$, devices $|\mathbf{x}_j| = \delta$:

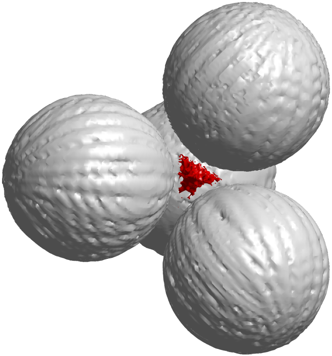


(a) suboptimal, $\sigma = \delta/5$ (b) optimal, $\sigma = \delta/3$

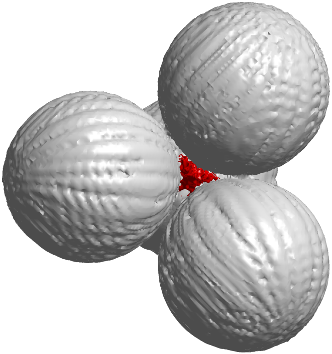
- Radius of gray balls: $r(\sigma,\delta) = \left(\left(\sigma - \frac{\delta}{3}\right)^2 + \frac{8}{9}\delta^2\right)^{\frac{1}{2}}$. (green)
- Largest ball in cloaked region: $r_{\text{eff}}(\sigma,\delta) = \delta - r(\sigma,\delta)$. (red)
- Largest cloaked region: $r^*_{\text{eff}} = \left(1 - \frac{2\sqrt{2}}{3}\right)\delta \approx 0.057\delta$.



$z = -2\sigma$
$z = -\sigma$
$z = 0$
$z = \sigma$
$z = 2\sigma$
u_d
u_{tot} (active)
u_{tot} (inactive)

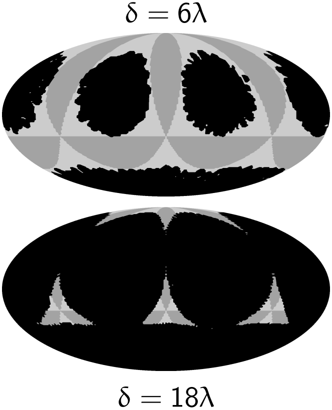


$|u_d| = 100$



$|u_d| = 5$

Contours of $|u_d|$ (gray) and $|u_d + u_i| = 10^{-2}$ (red).



Cross-section of level set $|u_d| \geqslant 10^2$ (black) and of the region R (shades of gray) on the sphere $|\mathbf{x}| = \sigma$ for the optimal $\sigma = \delta/3$.

Main ingredients for Helmholtz 3D active cloaking

- Green's identity: mono- and dipole density on ∂D reproduces incident field u_i in D.
- Device Ansatz:

$$u_d(\mathbf{x}) = \sum_{l=1}^{n_{dev}} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_{l,n,m} V_n^m(\mathbf{x} - \mathbf{x}_l).$$

- Movable source: (Graf's Identity)

$$G(\mathbf{x}, \mathbf{y}) = \text{ linear combination of } V_n^m(\mathbf{x} - \mathbf{x}_l).$$

Main ingredients for Maxwell active cloaking

- Stratton-Chu Formula: magnetic and electric dipole density on ∂D reproduces incident field E_i, H_i in D.
- Device Ansatz:

$$E_d(\mathbf{x}) = \sum_{l=1}^{n_{dev}} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} a_{l,n,m} \nabla\times((\mathbf{x}-\mathbf{x}_l)V_n^m(\mathbf{x}-\mathbf{x}_l)) + b_{l,n,m} \nabla\times\nabla\times((\mathbf{x}-\mathbf{x}_l)V_n^m(\mathbf{x}-\mathbf{x}_l))$$

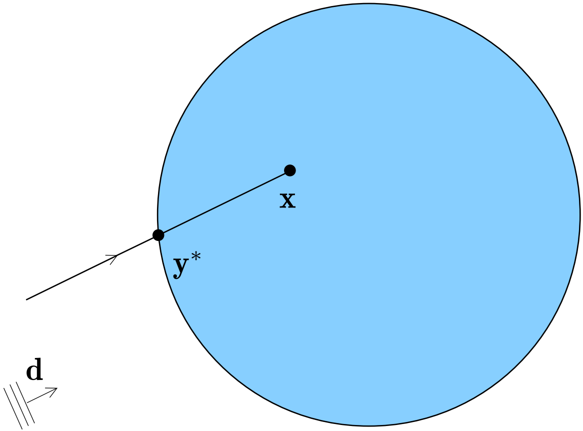
- Movable source: (vector addition theorem)

$$\begin{aligned} G(\mathbf{x},\mathbf{y})\mathbf{p} = \text{linear combination of } & \nabla\times((\mathbf{x}-\mathbf{x}_l)V_n^m(\mathbf{x}-\mathbf{x}_l)), \\ & \nabla\times\nabla\times((\mathbf{x}-\mathbf{x}_l)V_n^m(\mathbf{x}-\mathbf{x}_l)), \text{ and} \\ & \nabla V_n^m(\mathbf{x}-\mathbf{x}_l). \end{aligned}$$

(REU with Michael Bentley)

Directionality with stationary phase method

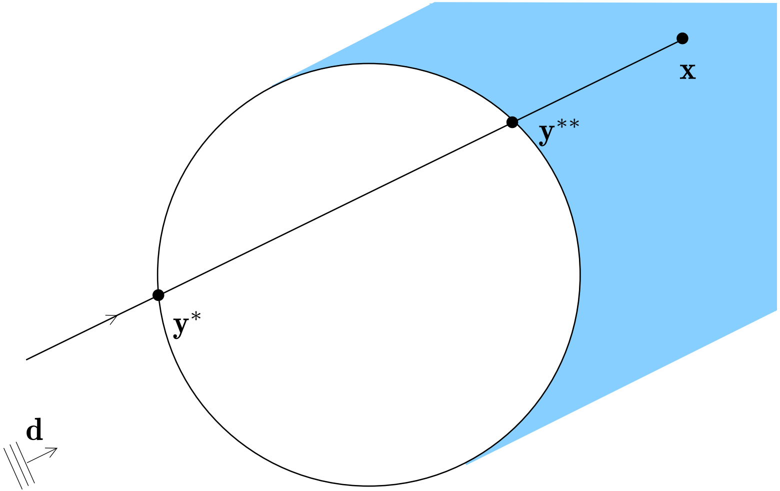
$u(\mathbf{x}) = \exp[ik\mathbf{d} \cdot \mathbf{x}]$



(with Leonid Kunyansky)

Directionality with stationary phase method

$u(\mathbf{x}) = 0$



(with Leonid Kunyansky)

Future work

- Time domain problems (active control of waves)
- Approximate Green's identities with a few devices while enforcing a constraint (e.g. penalize size of devices)

Thank you!