Asymptotics of radiation fields in asymptotically Minkowski spacetimes

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Radiation fields in Minkowski space

- Suppose u solves $\Box u = 0$ with smooth, compactly supported initial data in $\mathbb{R} \times \mathbb{R}^n$. ($\Box u = f \in C_c^\infty(\mathbb{R}^{n+1})$ with $u = 0$ for $t \ll 0$ works as well.)
- In polar coordinates (t, r, ω), introduce $s = t - r$ $\rho = \frac{1}{r}$, and introduce

$$v(\rho, s, \omega) = \rho^{-\frac{n-1}{2}} u\left(s + \frac{1}{\rho}, \frac{1}{\rho}\omega\right)$$

Fact

v is smooth down to $\rho = 0$, i.e., to null infinity.

Definition

The forward radiation field is the function given by

$$\mathcal{R}_+[u](s, \omega) = \partial_s v(0, s, \omega)$$

- In 1-d, these are the waves moving to the left and right.

Radiation fields in Minkowski space

The radiation field is of independent interest: $\mathcal{R}_+$ is

- an FIO
- a unitary isomorphism $\dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R} \times \mathbb{S}^{n-1})$
- a translation representation
- related to the Radon transform
- a concrete realization of the wave operators in Lax-Phillips scattering theory

The radiation field is understood in a variety of geometric contexts. See Friedlander, Sá Barreto, Wang, Melrose–Wang, Sá Barreto–Wunsch, ...

Motivating question

How does $\mathcal{R}_+$ behave as $s \to \infty$?

- On Minkowski space $\mathbb{R} \times \mathbb{R}^n$,

$$|\mathcal{R}_+[u](s,\omega)| \lesssim \begin{cases} (1+s)^{-\infty} & n \text{ odd} \\ (1+s)^{-\frac{n+1}{2}} & n \text{ even} \end{cases}$$

- Klainerman–Sobolev inequalities yield

$$|\mathcal{R}_+[u](s,\omega)| \lesssim (1+s)^{-1/2}$$

on perturbations of MInkowski space

Where does the radiation field live?

- Take the radial compactification of Minkowski space ($\rho = (t^2 + r^2)^{-1/2}$, $\theta = (t, r)/\rho \in \mathbb{S}^1$):

$$dt^2 - \sum dz_j^2 = \cos 2\theta \frac{d\rho^2}{\rho^4} - \cos 2\theta \frac{d\theta}{\rho^2} + 2\sin 2\theta \frac{d\rho}{\rho^2}\frac{d\theta}{\rho} - \sin^2\theta \frac{d\omega^2}{\rho^2}.$$

- Introduce $v = \cos 2\theta$ and metric becomes

$$v\frac{d\rho^2}{\rho^4} - \frac{v}{4(1-v^2)}\frac{dv^2}{\rho^2} - \frac{d\rho}{\rho^2}\frac{dv}{\rho} - \frac{1-v}{2}\frac{d\omega^2}{\rho^2}$$

- The radiation field is the (rescaled) restriction of the solution u to the front face of the blow up of $\{v = \rho = 0\}$.

Asymptotically Minkowski spaces

- Suppose (M, g) is an $(n+1)$-dimensional compact manifold with connected boundary, g a time-oriented Lorentzian metric on M that extends to a nondegenerate quadratic form on ${}^{sc}TM$.

Definition

g is a Lorentzian scattering metric if there is a boundary defining function ρ and a Morse-Bott function $v \in C^\infty(M)$ so that 0 is a regular value for v and, in a neighborhood of ∂M,

$$g = v\frac{d\rho^2}{\rho^4} - 2f\frac{d\rho}{\rho^2}\frac{dv}{\rho} - \frac{h}{\rho^2},$$

where $f = \frac{1}{2} + O(v) + O(\rho)$ near $v = \rho = 0$, and $h|_{\mathrm{Ann}(d\rho, dv)}$ is positive definite near ∂M.

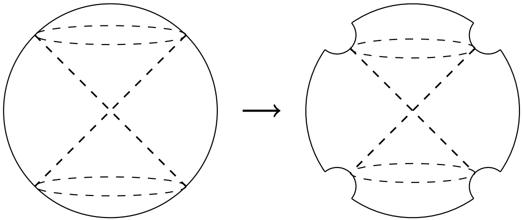
- Also impose a non-trapping assumption on the light rays.

Radiation fields

Proposition

The radiation field exists for metrics of this form.

The radiation field blow-up:



Asymptotics of radiation fields

Theorem

Suppose (M, g) is as above (non-trapping Lorentzian scattering), u is a tempered solution of $\Box_g u = f \in C_c^\infty(M^\circ)$. Then $\mathcal{R}_+[u]$ has an asymptotic expansion of the form

$$\mathcal{R}_+[u] \sim \sum_j \sum_{\kappa \leq m_j} s^{-i\sigma_j} \left|\log s\right|^\kappa a_{j\kappa}$$

Note

This is really a full asymptotic expansion for u in terms of ρ and s.

Note

This is not an existence theorem!

Some remarks

- The σ_j and m_j in the expansion are related to the resonances of an asymptotically hyperbolic problem in the region of ∂M where $\{v > 0\}$ (and in particular are independent of u).
  - This region inherits an AH metric: $k(X, Y) = \frac{-1}{v} g(\rho \tilde{X}, \rho \tilde{Y})$, where $\tilde{X}$, $\tilde{Y} \perp \rho^2 \partial_\rho$. The σ_j are the locations of the poles of an operator related to $(\Delta_k - \sigma^2)^{-1}$.
- Resonance gap (known) for k yields rate of decay for $\mathcal{R}_+[u]$.
- In Minkowski space, k is the hyperbolic metric, and the expansion for u is of the form

$$u \sim \begin{cases} O(\rho^{\frac{n-1}{2}} s^{-\infty}) & n \text{ odd} \\ \sum_j \rho^{\frac{n-1}{2}} s^{-\frac{n-1}{2} - j} a_j & n \text{ even} \end{cases}$$

Ideas in the proof

- Much heavy lifting done in recent paper of Vasy.
- Mellin transform reduces to problem on ∂M.
- P_σ fits into framework of Vasy paper, yielding a preliminary asymptotic expansion.
- Propagation of singularities estimate implies remainder term is lower order.
- Work of Haber-Vasy implies the coefficients are L^2-based conormal distributions.
- Coefficients are classical conormal, so have expansions in v.
- Blow-up turns v expansion into s expansion (since $v = s\rho$).

Thank you