Transmission Eigenvalues in Inverse Scattering Theory

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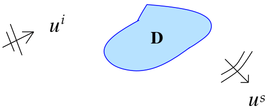
Scattering by an Inhomogeneous Media



$$\Delta u + k^2 n(x) u = 0 \quad \text{in } \mathbb{R}^d, \quad d = 2, 3$$

$$u = u^s + u^i$$

$$\lim_{r \to \infty} r^{\frac{d-1}{2}} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0$$

We assume that $n - 1$ has compact support $\overline{D}$ and $n \in L^\infty(D)$ is such that $\Re(n) \geq \gamma > 0$ and $\Im(n) \geq 0$ in $\overline{D}$. Here $k > 0$ is the wave number proportional to the frequency ω.

Question: Is there an incident wave u^i that does not scatter?

The answer to this question leads to the transmission eigenvalue problem.

Transmission Eigenvalues

If there exists a nontrivial solution to the homogeneous interior transmission problem

$$\begin{aligned} \Delta w + k^2 n(x) w = 0 & \quad \text{in} \quad D \\ \Delta v + k^2 v = 0 & \quad \text{in} \quad D \\ w = v & \quad \text{on} \quad \partial D \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} & \quad \text{on} \quad \partial D \end{aligned}$$

such that v can be extended outside D as a solution to the Helmholtz equation $\tilde{v}$, then the scattered field due to $\tilde{v}$ as incident wave is identically zero.

Values of k for which this problem has non trivial solution are referred to as transmission eigenvalues and the corresponding nontrivial solution w, v as eigen-pairs.

Transmission Eigenvalues

In general such an extension of v does not exist!

Since Herglotz wave functions

$$v_g(x) := \int_\Omega e^{ikx\cdot d} g(d)ds(d), \qquad \Omega := \{x : |x| = 1\},$$

are dense in the space

$$\left\{v \in L^2(D) : \ \Delta v + k^2 v = 0 \quad \text{in } D\right\}$$

at a transmission eigenvalue there is an incident field that produces arbitrarily small scattered field.

Motivation

Two important issues:

- Real transmission eigenvalues can be determined from the scattered data.
- Transmission eigenvalues carry information about material properties.

Therefore, transmission eigenvalues can be used

- to quantify the presence of abnormalities inside homogeneous media and use this information to test the integrity of materials.

How are real transmission eigenvalues seen in the scattering data?

Measurements

We assume that $u^i(x) = e^{ikx\cdot d}$ and the far field pattern $u_\infty(\hat{x}, d, k)$ of the scattered field $u^s(x, d, k)$ is available for $\hat{x}, d \in \Omega$, and $k \in [k_0, k_1]$

$$\text{where} \qquad u^s(x, d, k) = \frac{e^{ikr}}{r^{\frac{d-1}{2}}} u_\infty(\hat{x}, d, k) + O\left(\frac{1}{r^{(d+1)/2}}\right)$$

as $r \to \infty$, $\hat{x} = x/|x|$, $r = |x|$.

Define the far field operator $F : L^2(\Omega) \to L^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d, k) g(d) ds(d).$$

The Far Field Operator

Theorem

The far field operator $F : L^2(\Omega) \to L^2(\Omega)$ is injective and has dense range if and only if k is not a transmission eigenvalue such that for a corresponding eigensolution (w, v), v takes the form of a Herglotz wave function.

For $z \in D$ the far field equation is

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z, k), \quad g \in L^2(\Omega)$$

where $\Phi_\infty(\hat{x}, z, k)$ is the far field pattern of the fundamental solution $\Phi(x, z, k)$ of the Helmholtz equation $\Delta v + k^2 v = 0$.

Computation of Real TE

Theorem (Cakoni-Colton-Haddar, *Comp. Rend. Math. 2010*)

Assume that either $n > 1$ or $n < 1$ and $z \in D$.

- *If k is not a transmission eigenvalue then for every $\epsilon > 0$ there exists $g_{z,\epsilon,k} \in L^2(\Omega)$ satisfying $\|Fg_{z,\epsilon,k} - \Phi_\infty\|_{L^2(\Omega)} < \epsilon$ and*

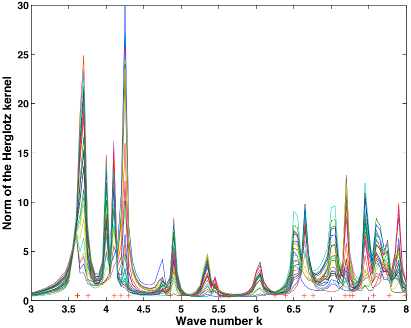
$$\lim_{\epsilon \to 0} \|v_{g_{z,\epsilon,k}}\|_{L^2(D)} \qquad \text{exists.}$$

- *If k is a transmission eigenvalue for any $g_{z,\epsilon,k} \in L^2(\Omega)$ satisfying $\|Fg_{z,\epsilon,k} - \Phi_\infty\|_{L^2(\Omega)} < \epsilon$ and for almost every $z \in D$*

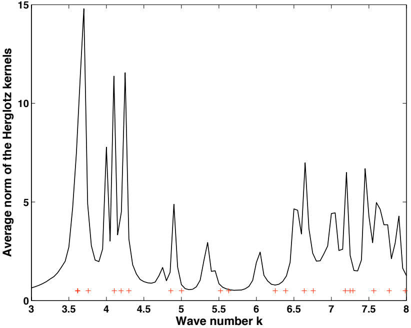
$$\lim_{\epsilon \to 0} \|v_{g_{z,\epsilon,k}}\|_{L^2(D)} = \infty.$$

Note: $g_{z,\epsilon,k}$ is computed using Tikhonov regularization, see *Arens, Inverse Problems (2004)*.

Computation of Real TE



A composite plot of $\|g_{z_i}\|_{L^2(\Omega)}$ against k for 25 random points $z_i \in D$



The average of $\|g_{z_i}\|_{L^2(\Omega)}$ over all choices of $z_i \in D$.

Computation of the transmission eigenvalues from the far field equation for the unit square D.

Transmission Eigenvalue Problem

Recall the transmission eigenvalue problem

$$\begin{aligned} \Delta w + k^2 n(x) w = 0 & \quad \text{in} \quad D \\ \Delta v + k^2 v = 0 & \quad \text{in} \quad D \\ w = v & \quad \text{on} \quad \partial D \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} & \quad \text{on} \quad \partial D \end{aligned}$$

It is a nonstandard eigenvalue problem:

$$\int_D \left(\nabla w \cdot \nabla \overline{\psi} - k^2 n(x) w \overline{\psi} \right) \, dx = \int_D \left(\nabla v \cdot \nabla \overline{\phi} - k^2 v \, \overline{\phi} \right) \, dx$$

- If $n = 1$ the interior transmission problem is degenerate.
- If $\Im(n) > 0$ in $\overline{D}$, there are no real transmission eigenvalues.

Historical Overview

- The transmission eigenvalue problem in scattering theory was introduced by *Kirsch (1986)* and *Colton-Monk (1988)*
- Research was focused on the discreteness of transmission eigenvalues for variety of scattering problems: *Colton-Kirsch-Päivärinta (1989), Rynne-Sleeman (1991), Cakoni-Haddar (2007), Colton-Päivärinta-Sylvester (2007), Kirsch (2009), Cakoni-Haddar (2009).*

In the above work, it is always assumed that either $n - 1 > 0$ or $1 - n > 0$.

Historical Overview, cont.

- The first proof of existence of at least one transmission eigenvalues for large enough contrast is due to *Päivärinta-Sylvester (2009)*.
- The existence of an infinite set of transmission eigenvalues is proven by *Cakoni-Gintides-Haddar (2010)* under only the assumption that either $n-1>0$ or $1-n>0$. The existence has been extended to other scattering problems by *Kirsch (2009)*, *Cakoni-Haddar (2010) Cakoni-Kirsch (2010)*, *Cossonniere (Ph.D. thesis, 2011)* etc.
- *Hitrik-Krupchyk-Ola-Päivärinta (2010)*, in a series of papers have extended the transmission eigenvalue problem to a more general class of differential operators with constant coefficients.

Historical Overview, cont.

- *Cakoni-Colton-Haddar (2010)* and then *Cossonniere-Haddar (2011)* have investigated the case when $n = 1$ in $D_0 \subset D$ and $n - 1 > \alpha > 0$ in $D \setminus \overline{D}_0$.
- Recently *Sylvester (2012)* has shown that the set of transmission eigenvalues is at most discrete if $n - 1$ is positive (or negative) only in a neighborhood of ∂D but otherwise could changes sign inside D. A similar result is obtained by *Bonnet Ben Dhia - Chesnel - Haddar (2011)* using T-coercivity for the case when there is contrast in both the main differential operator and the lower term.

Scattering by a Spherically Stratified Medium

We consider the interior eigenvalue problem for a ball of radius a with index of refraction $n(r)$ being a function of $r := |x|$

$$
\begin{aligned}
\Delta w + k^2 n(r) w = 0 \qquad & \text{in } B \\
\Delta v + k^2 v = 0 \qquad & \text{in } B \\
w = v \qquad & \text{on } \partial B \\
\frac{\partial w}{\partial r} = \frac{\partial v}{\partial r} \qquad & \text{on } \partial B
\end{aligned}
$$

where $B := \left\{x \in \mathbb{R}^3 : |x| < a\right\}$.

Scattering by a Spherically Stratified Medium

Look for solutions in polar coordinates (r, θ, φ)

$$v(r,\theta) = a_\ell j_\ell(kr) P_\ell(\cos\theta) \qquad \text{and} \qquad w(r,\theta) = a_\ell Y_\ell(kr) P_\ell(\cos\theta)$$

where j_ℓ is a spherical Bessel function and Y_ℓ is the solution of

$$Y_\ell'' + \frac{2}{r} Y_\ell' + \left(k^2 n(r) - \frac{\ell(\ell+1)}{r^2} \right) Y_\ell = 0$$

such that $\lim_{r \to 0} \left(Y_\ell(r) - j_\ell(kr) \right) = 0$. There exists a nontrivial solution of the interior transmission problem provided that

$$d_\ell(k) := det \begin{pmatrix} Y_\ell(a) & -j_\ell(ka) \\ Y_\ell'(a) & -kj_\ell'(ka) \end{pmatrix} = 0.$$

Values of k such that $d_\ell(k) = 0$ are the transmission eigenvalues. $d_\ell(k)$ are entire functions of k of finite type and bounded for k real.

Scattering by a Spherically Stratified Medium

Assume that $\Im(n) = 0$ and $n \in C^2[0, a]$.

- If either $n(a) \neq 1$ or $n(a) = 1$ and $\int_0^a \sqrt{n(\rho)}d\rho \neq a$.
  - The set of all transmission eigenvalue is discrete.
  - There exists an infinite number of real transmission eigenvalues accumulating only at $+\infty$.
- For a subclass of $n(r)$ there exist infinitely many complex transmission eigenvalues, *Leung-Colton, (to appear)*.

Inverse spectral problem

- All transmission eigenvalues uniquely determine $n(r)$ provide $n(0)$ is given and either $n(r) > 1$ or $n(r) < 1$. *Cakoni-Colton-Gintides, SIAM Journal Math Analysis, (2010)*.
- If $n(r) < 1$ then transmission eigenvalues corresponding to spherically symmetric eigenfunctions uniquely determine $n(r)$ *Aktosun-Gintides-Papanicolaou, Inverse Problems, (2011)*.

Transmission Eigenvalue Problem

Recall the transmission eigenvalue problem

$$\begin{aligned} \Delta w + k^2 n(x) w = 0 \qquad & \text{in} \quad D \\ \Delta v + k^2 v = 0 \qquad & \text{in} \quad D \\ w = v \qquad & \text{on} \quad \partial D \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad & \text{on} \quad \partial D \end{aligned}$$

Let $u = w - v$, we have that

$$\Delta u + k^2 n u = k^2 (n - 1) v.$$

Then eliminate v to get an equation only in terms of u by applying $(\Delta + k^2)$

Transmission Eigenvalues

Let $n \in L^\infty(D)$, and denote $n^* = \sup_{x \in D} n(x)$ and $0 < n_* = \inf_{x \in D} n(x)$. To fix our ideas assume $n_* > 1$ (similar analysis if $n^* < 1$).

Let $u := w - v \in H_0^2(D)$. The transmission eigenvalue problem can be written for u as an eigenvalue problem for the fourth order equation:

$$(\Delta + k^2)\frac{1}{n-1}(\Delta + k^2 n)u = 0$$

i.e. in the variational form

$$\int_D \frac{1}{n-1}(\Delta u + k^2 n u)(\Delta \varphi + k^2 \varphi)\, dx = 0 \qquad \text{for all } \varphi \in H_0^2(D)$$

Definition: $k \in \mathbb{C}$ is a transmission eigenvalue if there exists a nontrivial solution $v \in L^2(D)$, $w \in L^2(D)$, $w - v \in H_0^2(D)$ of the homogeneous interior transmission problem.

Transmission Eigenvalues

Obviously we have

$$0 = \int_D \frac{1}{n-1} \left|(\Delta u + k^2 n u)\right|^2 \, dx + k^2 \int_D \left(|\nabla u|^2 - k^2 n |u|^2\right) dx.$$

The Poincare inequality yields the Faber-Krahn type inequality for the first transmission eigenvalue (not isoperimetric)

$$k^2_{1,D,n} > \frac{\lambda_1(D)}{n^*}.$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D.

In particular there are no real transmission eigenvalues in the interval $(0, \lambda_1(D)/n^*)$.

Transmission Eigenvalues

Theorem (Cakoni-Gintides-Haddar, SIMA (2010))

Assume that $1 < n_*$. Then, there exists an infinite discrete set of real transmission eigenvalues accumulating at infinity $+\infty$. Furthermore

$$k_{1,n^*,B_1} \leq k_{1,n^*,D} \leq k_{1,n(x),D} \leq k_{1,n_*,D} \leq k_{1,n_*,B_2}.$$

where $B_2 \subset D \subset B_1$.

One can prove that, for n constant, the first transmission eigenvalue $k_{1,n}$ is continuous and strictly monotonically decreasing with respect to n. In particular, this shows that the first transmission eigenvalue uniquely determines the constant index of refraction, provided that it is known a priori that $n > 1$.

Similar results can be obtained for the case when $0 < n^* < 1$.

Detection of Anomalies in an Isotropic Medium

What does the first transmission eigenvalue say about the inhomogeneous media $n(x)$?

We find the constant n_0 such that the first transmission eigenvalue of

$$\begin{aligned} \Delta w + k^2 n_0 w = 0 & \quad \text{in} \quad D \\ \Delta v + k^2 v = 0 & \quad \text{in} \quad D \\ w = v & \quad \text{on} \quad \partial D \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} & \quad \text{on} \quad \partial D \end{aligned}$$

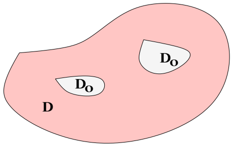
is $k_{1,n(x)}$ (which can be determined from the measured data).

Then from the previous discussion we have that $n_* \le n_0 \le n^*$.

Open Question: Find an exact formula that connects n_0 to $n(x)$ and D.

The Case with Cavities

Can the assumption $n > 1$ or $0 < n < 1$ in D be relaxed?



$n = 1$ in D_0

$n - 1 \geq \delta > 0$ in $D \setminus D_0$

The case when there are regions D_0 in D where $n = 1$ (i.e. cavities) is more delicate. The same type of analysis can be carried through by looking for solutions of the transmission eigenvalue problem

$v \in L^2(D)$ and $w \in L^2(D)$ such that $w - v$ is in

$$V_0(D, D_0, k) := \{u \in H_0^2(D) \text{ such that } \Delta u + k^2 u = 0 \text{ in } D_0\}.$$

Cakoni-Colton-Haddar, SIMA (2010)

The Case with Cavities

In particular if $n > 1$ and $k(D_0, n(x))$ is the first eigenvalue for a fixed D, one has the following properties:

- The Faber Krahn inequality

$$0 < \frac{\lambda_1(D)}{n^*} \leq k(D_0, n(x)).$$

- Monotonicity with respect to the index of refraction

$$k(D_0, n(x)) \leq k(D_0, \tilde{n}(x)), \qquad \tilde{n}(x) \leq n(x).$$

- Monotonicity with respect to voids

$$k(D_0, n(x)) \leq k(\tilde{D}_0, n(x)), \qquad D_0 \subset \tilde{D}_0.$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D.

The Case of $n-1$ Changing Sign

Recently, progress has been made in the case of the contrast $n-1$ changing sign inside D with state of the art a result by *Sylvester (2012)*. Roughly speaking he shows that transmission eigenvalues form a discrete (possibly empty) set provided $n-1$ has fixed sign **only in a neighborhood** of ∂D. There are two aspects in the proof:

- Fredholm property. Sylvester considers the problem in the form

  $$\Delta u + k^2 n u = k^2(n-1)v, \ \Delta v + k^2 v = 0, \quad u \in H_0^2(D), \ v \in H^1(D)$$

  and uses the concept of upper-triangular compact operators.
- Find a k that is not a transmission eigenvalue. This requires careful estimates for the solution inside D in terms of its values in a neighborhood of ∂D.

The existence of transmission eigenvalues under such weaker assumptions is still open.

Complex Eigenvalues

Current results on complex transmission eigenvalues for media of general shape are limited to identifying eigenvalue free zones in the complex plane.

- The first result for homogeneous media is given in *Cakoni-Colton-Gintides SIMA (2010)*.
- The best result to date is due *Hitrik-Krupchyk-Ola-Päivärinta, Math. Research Letters (2011)*, where they show that almost all transmission eigenvalues are confined to a parabolic neighborhood of the positive real axis. More specifically they show

Theorem (Hitrik-Krupchyk-Ola-Päivärinta)

For $n \in C^\infty(\overline{D}, \mathbb{R})$ *and* $1 < \alpha \leq n \leq \beta$, *there exists a* $0 < \delta < 1$ *and* $C > 1$ *both independent of* α, β *such that all transmission eigenvalues* $\tau := k^2 \in \mathbb{C}$ *with* $|\tau| > C$ *satisfies* $\Re(\tau) > 0$ *and* $\Im(\tau) \leq C|\tau|^{1-\delta}$.

Absorbing-Dispersive Media

The interior transmission eigenvalue problem for this case reads:

$$\begin{aligned} \Delta w + k^2 \left(\epsilon_1 + i\frac{\gamma_1}{k} \right) w = 0 \qquad & \text{in} \quad D \\ \Delta v + k^2 \left(\epsilon_0 + i\frac{\gamma_0}{k} \right) v = 0 \qquad & \text{in} \quad D \\ w = v \qquad & \text{on} \quad \partial D \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad & \text{on} \quad \partial D \end{aligned}$$

where $\epsilon_0 \geq \alpha_0 > 0$, $\epsilon_1 \geq \alpha_1 > 0$, $\gamma_0 \geq 0$, $\gamma_1 \geq 0$ are bounded functions.

For the corresponding spherically stratified case it can be shown using Hadamard's factorization theorem there exists an infinite discrete set of (complex) transmission eigenvalues.

Absorbing-Dispersive Media

In the general case we have proven *Cakoni-Colton-Haddar (2012)*:

- The set of transmission eigenvalues $k \in \mathbb{C}$ in the right half plane is discrete, provided $\epsilon_1(x) - \epsilon_0(x) > 0$.
- Using Kato's perturbation theory of linear operators it can be shown that if $\sup_D(\gamma_0 + \gamma_1)$ is small enough there exist at least $\ell > 0$ transmission eigenvalues each in a small neighborhood of the first ℓ real transmission eigenvalues corresponding to $\gamma_0 = \gamma_1 = 0$.
- For the case of $\epsilon_0, \epsilon_1, \gamma_0, \gamma_1$ constant, we have identified eigenvalue free zones in the complex plane

The existence of transmission eigenvalues for general media if absorption is present is still open.

Open Problems

- Can the existence of real transmission eigenvalues for non-absorbing media be established if the assumptions on the sign of the contrast are weakened?
- Do complex transmission eigenvalues exists for general non-absorbing media?
- Do real transmission eigenvalues exist for absorbing media?
- What would the necessary conditions be on the contrast that guaranty the discreteness of transmission eigenvalues?
- Can an inverse spectral problem be developed for the general transmission eigenvalue problem? (Completeness of eigen-solutions?)