Linearized internal functionals for anisotropic conductivities

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1.Background

$$-\nabla\cdot(\gamma\nabla u)\equiv-\sum_{i,j=1}^{n}\partial_i(\gamma^{ij}\partial_j u)=0 \qquad (\Omega) \qquad u|_{\partial\Omega}=g$$

γ is a real-valued symmetric positive definite tensor with bounded coefficients, satisfying a uniform ellipticity condition for some $\kappa\geq 1$

$$\kappa^{-1}|\xi|^2\leq\xi\cdot\gamma(x)\xi\leq\kappa|\xi| \quad \xi\in R^n \qquad x\in\Omega$$

- *Conductivity* equation: rules the equilibrium distribution of the electrostatic potential u inside the domain Ω, in response to a prescribed boundary voltage g. Electrical Impedance Tomography(EIT).

$\gamma\nabla u\cdot\nu|_{\partial\Omega}$ $\rightarrow$ Calderon’s problem

- Internal measurement

power density of a solution u $H_\gamma[g](x):=\nabla u(x)\cdot\gamma(x)\nabla u(x)$

Application: Hybrid imaging
How to construct power densities: Ammari et al. (2008), Kuchment-Kunyansky (2010)

$$-\nabla \cdot (\gamma \nabla u) = 0 \quad (\Omega) \qquad u|_{\partial\Omega} = g \qquad H_\gamma[g](x) := \nabla u(x) \cdot \gamma(x) \nabla u(x)$$

History (non-linear case):

- Isotropic case $\gamma = \sigma I_n$ from only one measurement $H = \sigma |\nabla u|^2$

$$\nabla \cdot \left(\frac{H}{|\nabla u|^2} \nabla u\right) = 0 \quad (\Omega) \qquad u|_{\partial\Omega} = g$$

  - Newton-based method see (Gebauer and Scherzer (2009))
  - Theoretically, by Bal (2012).

The conductivity equation

$$\nabla \cdot (\underbrace{\gamma'}_{unknown} \nabla u) = 0 \qquad (\Omega), \qquad u|_{\partial\Omega} = \underbrace{g}_{known} \qquad (1)$$

Fixing a few boundary condtions $g_1, \cdots, g_m$ with u_i' solving (1) with $g \equiv g_i$

Measurement operator : $H: \gamma' \to H(\gamma') = H_{ij}(\gamma') = \gamma' \nabla u_i' \cdot \nabla u_j' \qquad 1 \le i, j \le m$

Problem: recover γ' from H

- Isotropic, 2 dimensional setting, see (Capdeboscq et al. (2009))
- Explicit reconstruction using a large number of functionals in an isotropic case in dimension 3. see (Bal,Monard, Bonnetier and Triki)
- Generalized to dimension n, isotropic tensor with more general type of measurements $\sigma^{2\alpha}|\nabla u|^2$ where α not necessary $\frac{1}{2}$

  see Monard and Bal (2012).
- reconstruction formulas for the anisotropic two-dimensional problem see Monard and Bal (2012).

Linearization of the problem

The partial differential equation

$$\nabla\cdot(\underbrace{\gamma'}_{unknown}\nabla u)=0 \qquad (\Omega), \qquad u|_{\partial\Omega}=\underbrace{g}_{known} \qquad (1)$$

Fix a few boundary conditions $g_1,\cdots,g_m$ with $u_i^{'}$ solving (1) with $g\equiv g_i$

Measurement operator : $H:\gamma'\to H(\gamma')=H_{ij}(\gamma')=\gamma'\nabla u_i^{'}\cdot\nabla u_j^{'} \quad 1\le i,j\le m$

Non-linear problem: recover γ' from H

Fréchet derivative:

$$\gamma'=\underbrace{\gamma_0}_{known}+\varepsilon\gamma+o(\varepsilon^2)$$

$$u_i^{'}=u_i+\varepsilon v_i+o(\varepsilon^2)$$

PDE (1) of order $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$

$$-\nabla\cdot(\gamma_0\nabla\underbrace{u_i}_{known})=0 \quad (\Omega), \qquad u_i|_{\partial\Omega}=\underbrace{g_i}_{known}$$

$$-\nabla\cdot(\gamma_0\nabla\underbrace{v_i}_{unknown})=\nabla\cdot(\gamma\nabla u_i) \quad (\Omega), \quad v_i|_{\partial\Omega}=0$$

The measurements look like $H_{ij} = \gamma_0 \nabla u_i \cdot \nabla u_j + \varepsilon(\gamma \nabla u_i \cdot \nabla u_j + \gamma_0 \nabla u_i \cdot \nabla v_j + \gamma_0 \nabla u_j \cdot \nabla v_i) + o(\varepsilon^2)$

$$\downarrow$$

Linearized measurements $\underbrace{dH_{ij}}_{known} = \gamma \nabla u_i \cdot \nabla u_j + \gamma_0 \nabla u_i \cdot \nabla v_j + \gamma_0 \nabla u_j \cdot \nabla v_i$

Linearized problem: recover γ from dH_{ij}

References on the linearized problem

- isotropic case in dimension 2 and 3 with numerical implementation
  see (Kuchment and Kunyansky (2011))

- isotropic case, studied using pseudo-differential calculus, inversion modulo a compact operator. see (Kuchment and Steinhauer (2011))

2.Microlocal inversion

Study of the principal symbol $M_{ij}(x,\xi)$ of dH_{ij}

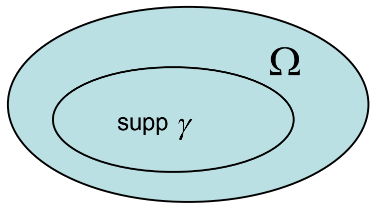
Recall the equations and measurements:

$$-\nabla\cdot(\gamma_0\nabla \underbrace{u_i}_{known})=0 \qquad (\Omega), \qquad u_i|_{\partial\Omega}=\underbrace{g_i}_{known}$$

$$-\nabla\cdot(\gamma_0\nabla \underbrace{v_i}_{unknown})=\nabla\cdot(\gamma\nabla u_i) \quad (\Omega), \quad v_i|_{\partial\Omega}=0$$

$$\underbrace{dH_{ij}}_{known}=\gamma\nabla u_i\cdot\nabla u_j+\gamma_0\nabla u_i\cdot\nabla v_j+\gamma_0\nabla u_j\cdot\nabla v_i$$

Denote $L_0:=-\nabla\cdot(\gamma_0\nabla)$ $\qquad v_i=L_0^{-1}(\nabla\cdot(\gamma\nabla u_i))$



suppose γ compactly supported inside Ω

Insert $v_i=L_0^{-1}(\nabla\cdot(\gamma\nabla u_i))$ into dH_{ij} and express as a pseudo-DO

$$dH_{ij}(\gamma,x)=(2\pi)^{-n}\iint_{R^n\times R^n} e^{i\xi\cdot(x-y)}(M_{ij}(x,\xi)+M_{ij}|_{-1}(x,\xi)):\gamma(y)d\xi dy$$

$$M_{ij}(x,\xi)=o(|\xi|^0) \qquad M_{ij}|_{-1}(x,\xi)=o(|\xi|^{-1})$$

$\downarrow$ pricipal symbol $\qquad$ $\downarrow$ the symbol of order -1

$$dH_{ij}(\gamma, x) = (2\pi)^{-n} \iint_{R^n \times R^n} e^{i\xi\cdot(x-y)} (M_{ij}(x,\xi) + M_{ij}|_{-1}(x,\xi)) : \gamma(y) d\xi dy$$

$$M_{ij}(x,\xi) = o(|\xi|^0) \qquad M_{ij}|_{-1}(x,\xi) = o(|\xi|^{-1})$$

$\downarrow$ pricipal symbol $\qquad$ $\downarrow$ the symbol of order -1

Goal: determine under which conditions the operator $dH = \{dH_{ij}\}_{1\le i,j\le m}$ is an elliptic pseudo-differential operator

Conclusion: with only principal symbols $M_{ij}(x,\xi)$, $dH = \{dH_{ij}\}_{1\le i,j\le m}$ will never be elliptic, no matter how large m

Define $A_0 = (\gamma_0)^{\frac{1}{2}}$ $\xi_0 := \hat{A_0 \xi}$ and $V_i := A_0 \nabla u_i$ rewrite the principal symbol $M_{ij}(x,\xi)$

$$\tilde{M}_{ij}(x,\xi) = A_0 M_{ij}(x,\xi) A_0$$

$$= V_i \odot V_j - (\xi_0 \cdot V_i)\xi_0 \odot V_j - (\xi_0 \cdot V_j)\xi_0 \odot V_i$$

Notation: $U \odot V = \frac{1}{2}(U \otimes V + V \otimes U)$

Lemma:

For any i, j and vector fields V_i, V_j defined as above $V_i := A_0 \nabla u_i$ the symbol $\tilde{M}_{ij}(x,\xi)$ satisfies

$$\tilde{M}_{ij}(x,\xi) : \xi_0 \odot \eta = 0 \qquad \text{for all} \quad \eta \in S^{n-1} \quad \text{and} \quad \eta \perp \xi_0$$

Conclusion: the M_{ij} can never control a subspace of $S_n(R)$ of dimension $n-1$

Basic Hypothesis: the gradients $\{\nabla u_i\}_{i=1}^n$ form a frame in R^n

Lemma:

Suppose that the vector fields $\{V_i\}_{i=1}^n$ form a basis of R^n If a matrix $P \in S_n(R)$

$$\tilde{M}_{ij}(x,\xi) : P = 0 \qquad 1 \le i \le j \le n$$

Then P is of the form $P = \xi_0 \odot \eta$ for some vector η satisfying $\eta \perp \xi_0$

Conclusion: the **only** directions that are not controlled by the principal symbols are $P = \xi_0 \odot \eta \quad \eta \perp \xi_0$

The second term of dH_{ij}

$$dH_{ij}(\gamma, x) = (2\pi)^{-n} \iint_{R^n \times R^n} e^{i\xi\cdot(x-y)} (M_{ij}(x,\xi) + M_{ij}|_{-1}(x,\xi)) : \gamma(y) d\xi dy$$

Goal: invert $dH = \{dH_{ij}\}_{1\le i,j\le m}$ microlocally from $M_{ij}(x,\xi) + M_{ij}|_{-1}(x,\xi)$

- case γ_0 is constant

$$\tilde{M}_{ij}|_{-1}(x,\xi) = A_0 M_{ij}|_{-1}(x,\xi) A_0$$

$$= \|A_0\xi\|^{-1}\sqrt{-1}[H_i((\xi_0\cdot V_j)(I_n - 2\xi_0\otimes\xi_0) + V_j\otimes\xi_0)$$

$$+ H_j((\xi_0\cdot V_i)(I_n - 2\xi_0\otimes\xi_0) + V_i\otimes\xi_0]^{sym}$$

where $H_i = A_0\nabla^2 u_i A_0$

Lemma

Suppose γ_0 is constant, pick $u_i = x_i \quad 1\le i\le n$ and add an additional solution denoted by u_{n+1} with full-rank Hessian $\nabla^2 u_{n+1}$ If there exist $P, Q \in S_n(R)$ such that for all $1\le i, j\le n$

$$(\tilde{M}_{ij} + \tilde{M}_{ij}|_{-1})(x,\xi) : (P + \sqrt{-1}Q) = 0$$

$$(\tilde{M}_{i,n+1} + \tilde{M}_{i,n+1}|_{-1})(x,\xi) : (P + \sqrt{-1}Q) = 0$$

then $P=Q=0$

- case γ_0 is not constant

Basic Hypotheses

- $(\nabla u_1, \cdots \nabla u_n)$ form a frame $-\nabla \cdot (\gamma_0 \nabla \underbrace{u_i}_{known}) = 0$ $\quad u_i |_{\partial\Omega} = \underbrace{g_i}_{known}$

- Add one addition u_{n+1} such that $\sum_j^n \mu_j \nabla u_j + \nabla u_{n+1} = 0$

  we construct the matrix $Z = [\nabla \mu_1, \cdots, \nabla \mu_n]$ assume Z to be invertible

Remark: in the case γ_0 constant, the above hypothesis is automatically satisfied, by choosing

$$u_i = x_i \qquad 1 \le i \le n$$

$$u_{n+1} = \frac{1}{2} x^t Q x \quad Q \text{ invertible} \quad tr(Q) = 0$$

Recall $dH_{ij}(\gamma, x) = (2\pi)^{-n} \iint_{R^n \times R^n} e^{i\xi \cdot (x-y)} (M_{ij}(x,\xi) + M_{ij}|_{-1}(x,\xi)) : \gamma(y) d\xi dy$

Goal: under the above hypotheses, $\tilde{M}_{ij}|_{-1}$ can control all bad directions $\xi_0 \odot \eta$

Recall: $$\underbrace{dH_{ij}}_{known} = \gamma\nabla u_i \cdot \nabla u_j + \gamma_0 \nabla u_i \cdot \nabla v_j + \gamma_0 \nabla u_j \cdot \nabla v_i$$

$$v_i = L_0^{-1}(\nabla \cdot (\gamma \nabla u_i))$$

$$dH_{ij}(\gamma, x) = (2\pi)^{-n} \iint_{R^n \times R^n} e^{i\xi\cdot(x-y)}(M_{ij}(x,\xi) + M_{ij}|_{-1}(x,\xi)) : \gamma(y) d\xi dy$$

Lemma

Assume that $Z = [\nabla\mu_1, \cdots, \nabla\mu_n]$ is invertible, if $\tilde{M}_{ij}|_{-1}^{nc} : \xi_0 \odot \eta = 0$ and $\tilde{M}_{i,n+1}|_{-1}^{nc} : \xi_0 \odot \eta = 0$ for any $1 \le i, j \le n$ and $\xi_0 \perp \eta$ then $\eta = 0$

Theorem

Suppose that we have measurements $\{dH_{ij}\}_{1\le i,j\le n}$ and $\{dH_{i,n+1}\}_{1\le i\le n}$ such that $Z = [\nabla\mu_1, \cdots, \nabla\mu_n]$ is invertible and $(\nabla u_1, \cdots \nabla u_n)$ form a frame.

we denote the operator dH as follows $dH := (dH_{ij}, \cdots, dH_{i,n+1})_{1\le i,j\le n}$

Then the operator $G \circ dH : \{L^2(\Omega')\}^{\frac{n(n+1)}{2}} \Rightarrow \{L^2(\Omega')\}^{\frac{n(n+1)}{2}+n}$

is semi-Fredholm, where G is the restriction operator to $L^2(\Omega')$

Proof: construct a parametrix Q such that $Q \circ dH - Id$ is of order -1

3.Explicit reconstruction

$$-\nabla\cdot(\gamma_0\nabla \underbrace{u_i}_{known})=0 \quad (\Omega),\ \ u_i|_{\partial\Omega}=\underbrace{g_i}_{known} \qquad (*)$$

$$\begin{cases} -\nabla\cdot(\gamma_0\nabla \underbrace{v_i}_{unknown})=\nabla\cdot(\gamma\nabla u_i) \qquad (\Omega),\ \ v_i|_{\partial\Omega}=0 \\ \underbrace{dH_{ij}}_{known}=\gamma\nabla u_i\cdot\nabla u_j+\gamma_0\nabla u_i\cdot\nabla v_j+\gamma_0\nabla u_j\cdot\nabla v_i \end{cases}$$

Let us now add another solution of $(*)$ u with corresponding v at order ε

$$u' = u+\varepsilon v+o(\varepsilon^2)$$

∇u may be expressed in the basis $\nabla u_1,\cdots\nabla u_n$

$$\sum_j^n \mu_j\nabla u_j+\nabla u=0$$

Suppose $[\nabla U]=:[\nabla u_1|\cdots|\nabla u_n]$ and $Z:=[Z_1|\cdots|Z_n]$ be invertible, where $Z_i=\nabla\mu_i$

In the end, we obtain the strongly coupled elliptic system

$$-\nabla\cdot(\gamma_0\nabla v_i)+\sum_{j=1}^{n}W_{ij}\cdot\nabla v_j=f_i,\qquad v_i|_{\partial\Omega}=0\qquad 1\le i\le n$$

where W_{ij} depends on γ_0 and $(u_1,\cdots,u_n)$

f_i depends on dH_{pq} and their derivatives

With $L_0:=-\nabla\cdot(\gamma_0\nabla)$ define $L_0^{-1}:f\in L^2\mapsto u\in H_0^1(\Omega)$ where u solves $-\nabla\cdot(\gamma_0\nabla u)=f$, $u|_{\partial\Omega}=0$

By Lax-Milgram theorem, and by Rellich imbedding, $L_0^{-1}:L^2(\Omega)\mapsto H_0^1(\Omega)$ is compact.

Applying L_0^{-1} to the elliptic system yields $v_i+\sum_{j=1}^{n}L_0^{-1}(W_{ij}\cdot\nabla v_j)=h_i:=L_0^{-1}f_i$

$P_{ij}:H_0^1(\Omega)\ni v\to P_{ij}v:=L_0^{-1}(W_{ij}\cdot\nabla v)\in H_0^1(\Omega)$ compact if W_{ij} bounded.

we obtain $(I+P)v=h$ (＊)

where $v=(v_1,\cdots,v_n)$, $h=(h_1,\cdots,h_n)$, and P compact

(＊) satisfies a Fredholm alternative, if -1 is not an eigenvalue of P

and we obtain the stability: $\|v_i\|_{H^1(\Omega)}\le C\|dH\|_{H^1(\Omega)}$

once $(v_1, \cdots v_n)$ are reconstructed

the relations

$$\underbrace{dH_{ij}}_{known} = \gamma \nabla u_i \cdot \nabla u_j + \gamma_0 \nabla u_i \cdot \nabla v_j + \gamma_0 \nabla u_j \cdot \nabla v_i$$

can be inverted for γ

$$\gamma = \gamma_0 (dH_{ip} (H^{pq} \nabla u_q \otimes H^{ij} \nabla u_j) - 2 H^{ij} \nabla v_i \odot \nabla u_j) \gamma_0$$

and we get the stability

$$\|\gamma\|_{L^2(\Omega)} \leq C \|dH\|_{H^1(\Omega)}$$

• case γ_0 is constant

$$-\nabla\cdot(\gamma_0\nabla \underbrace{u_i}_{known})=0 \quad (\Omega), \quad u_i|_{\partial\Omega}=\underbrace{g_i}_{known} \quad (*)$$

$$-\nabla\cdot(\gamma_0\nabla \underbrace{v_i}_{unknown})=\nabla\cdot(\gamma\nabla u_i) \quad (\Omega), \quad v_i|_{\partial\Omega}=0$$

$$\underbrace{dH_{ij}}_{known}=\gamma\nabla u_i\cdot\nabla u_j+\gamma_0\nabla u_i\cdot\nabla v_j+\gamma_0\nabla u_j\cdot\nabla v_i$$

Taking $\gamma_0 = Id$ using harmonic polynomials as solutions of $(*)$

$$\begin{cases} u_i = x_i & 1\le i\le n \\ u' = \frac{1}{2}\sum_{p=1}^{n} a_p x_p^2 & a_p \neq 0 \quad \sum_p a_p = 0 \end{cases}$$

Then we get n additional measurements

$$dH_i' = \gamma : \nabla u_i \odot \nabla u' + \nabla u_i \odot \nabla v' + \nabla v_i \cdot \nabla u' \qquad 1\le i\le n$$

Together with $\quad dH_{ij}=\gamma\nabla u_i\cdot\nabla u_j+\nabla u_i\cdot\nabla v_j+\nabla u_j\cdot\nabla v_i \qquad 1\le i,j\le n$

$$\frac{1}{2}n(n+1)+n \qquad \begin{cases} dH_i' = \gamma : \nabla u_i \odot \nabla u' + \nabla u_i \odot \nabla v' + \nabla v_i \cdot \nabla u' & 1 \le i \le n \\ dH_{ij} = \gamma \nabla u_i \cdot \nabla u_j + \nabla u_i \cdot \nabla v_j + \nabla u_j \cdot \nabla v_i & 1 \le i, j \le n \end{cases}$$

For this choice of $(u_1, \cdots, u_n)$, the elliptic system is decoupled, hence solvable.

$$\Delta v_i = \frac{1}{a_i} \sum_{j=1}^{n} \partial_j \Big(-\partial_j \sum_{p=1}^{n} a_p x_p dH_{ip} + \partial_j dH_i'\Big) + \frac{1}{a_i} \partial_i \sum_{p=1}^{n} a_p dH_{pp}$$

$$\gamma' = \underbrace{\gamma_0}_{known} + \varepsilon\gamma + o(\varepsilon^2)$$

$$u_i' = u_i + \varepsilon v_i + o(\varepsilon^2)$$

$$\gamma_{ij} = dH_{ij} - \partial_j v_i - \partial_i v_j$$

we obtain a standard estimate for v_i notice that the RHS is a linear combination of dH_{ij}, dH_i' and their first derivatives

<u>stability</u>: $\|v_i\|_{H^1(\Omega)} \le C \|dH\|_{H^1(\Omega)}$ then $\|\gamma_{ij}\|_{L^2(\Omega)} \le C \|dH\|_{H^1(\Omega)}$

$\Rightarrow$ we lose one derivative