Linearized internal functionals for anisotropic conductivities

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June 20th, 2012

UC Irvine Conference in honor of Gunther Uhlmann
1. Background

\[ -\nabla \cdot (\gamma \nabla u) \equiv - \sum_{i,j=1}^{n} \partial_i (\gamma^{ij} \partial_j u) = 0 \quad (\Omega) \quad u|_{\partial \Omega} = g \]

\( \gamma \) is a real-valued symmetric positive definite tensor with bounded coefficients, satisfying a uniform ellipticity condition for some \( \kappa \geq 1 \):

\[ \kappa^{-1} |\xi|^2 \leq \xi \cdot \gamma(x) \xi \leq \kappa |\xi|^2 \quad \xi \in \mathbb{R}^n \quad x \in \Omega \]

- **Conductivity** equation: rules the equilibrium distribution of the electrostatic potential \( u \) inside the domain \( \Omega \) in response to a prescribed boundary voltage \( g \). Electrical Impedance Tomography (EIT).

\[ \gamma \nabla u \cdot \nu|_{\partial \Omega} \quad \rightarrow \quad \text{Calderon’s problem} \]

- Internal measurement

  power density of a solution \( u \)

  \[ H_{\gamma}[g](x) := \nabla u(x) \cdot \gamma(x) \nabla u(x) \]

Application: Hybrid imaging

How to construct power densities: Ammari et al. (2008), Kuchment-Kunyansky (2010)
\[- \nabla \cdot (\gamma \nabla u) = 0 \quad (\Omega) \quad \left. u \right|_{\partial \Omega} = g \quad H_{\gamma}[g](x) := \nabla u(x) \cdot \gamma(x) \nabla u(x) \]

History (non-linear case):

- Isotropic case $\gamma = \sigma I_n$ from only one measurement $H = \sigma |\nabla u|^2$

\[\nabla \cdot (\frac{H}{|\nabla u|^2} \nabla u) = 0 \quad (\Omega) \quad \left. u \right|_{\partial \Omega} = g\]

- Newton-based method see (Gebauer and Scherzer (2009))
The conductivity equation

\[
\nabla \cdot \left( \gamma' \nabla u \right) = 0 \quad (\Omega), \quad u|_{\partial \Omega} = g_{\text{known}}
\]

(1)

Fixing a few boundary conditions \( g_1, \cdots, g_m \) with \( u_i \) solving (1) with \( g \equiv g_i \)

Measurement operator : \( H : \gamma' \rightarrow H(\gamma') = H_0(\gamma') = \gamma' \nabla u'_i \cdot \nabla u'_j \quad 1 \leq i, j \leq m \)

Problem: recover \( \gamma' \) from \( H \)

• Isotropic, 2 dimensional setting, see (Capdebsocq et al. (2009))

• Explicit reconstruction using a large number of functionals in an isotropic case in dimension 3. see (Bal, Monard, Bonnetier and Triki)

• Generalized to dimension n, isotropic tensor with more general type of measurements \( \sigma^{2\alpha} \left| \nabla u \right|^2 \)

where \( \alpha \) not necessary \( \frac{1}{2} \)

see Monard and Bal (2012).

• reconstruction formulas for the anisotropic two-dimensional problem see Monard and Bal (2012).
Linearization of the problem

The partial differential equation

\[ \nabla \cdot (\gamma' \nabla u) = 0 \quad (\Omega), \quad u|_{\partial \Omega} = g_{\text{known}} \quad (1) \]

Fix a few boundary conditions \( g_i, \ldots, g_m \) with \( u_i \) solving (1) with \( g = g_i \)

Measurement operator: \( H : \gamma' \rightarrow H(\gamma') = H_{ij}(\gamma') = \gamma' \nabla u_i \cdot \nabla u_j \quad 1 \leq i, j \leq m \)

Non-linear problem: recover \( \gamma' \) from \( H \)

Fréchet derivative:

\[ \gamma' = \gamma_0 + \varepsilon \gamma + o(\varepsilon^2) \]
\[ u_i' = u_i + \varepsilon v_i + o(\varepsilon^2) \]

PDE (1) of order \( O(1) \) and \( O(\varepsilon) \)

\[ -\nabla \cdot (\gamma_0 \nabla u_i) = 0 \quad (\Omega), \quad u_i|_{\partial \Omega} = g_i_{\text{known}} \]

\[ -\nabla \cdot (\gamma_0 \nabla v_i) = \nabla \cdot (\gamma \nabla u_i) \quad (\Omega), \quad v_i|_{\partial \Omega} = 0 \]
The measurements look like:

\[ H_{ij} = \gamma_0 \nabla u_i \cdot \nabla u_j + \varepsilon (\gamma \nabla u_i \cdot \nabla u_j + \gamma_0 \nabla u_i \cdot \nabla v_j + \gamma_0 \nabla u_j \cdot \nabla v_i) + o(\varepsilon^2) \]

Linearized measurements:

\[ dH_{ij} = \gamma \nabla u_i \cdot \nabla u_j + \gamma_0 \nabla u_i \cdot \nabla v_j + \gamma_0 \nabla u_j \cdot \nabla v_i \]

Linearized problem: recover \( \gamma \) from \( dH_{ij} \)

References on the linearized problem

- isotropic case in dimension 2 and 3 with numerical implementation see (Kuchment and Kunyansky (2011))

- isotropic case, studied using pseudo-differential calculus, inversion modulo a compact operator. see (Kuchment and Steinhauer (2011))
2. Microlocal inversion

Study of the principal symbol \( M_{ij}(x, \xi) \) of \( dH_{ij} \)

Recall the equations and measurements:

\[
\begin{align*}
- \nabla \cdot (\gamma_0 \nabla u_i) &= 0 \quad (\Omega), \quad u_i \big|_{\partial \Omega} = g_i \\
- \nabla \cdot (\gamma_0 \nabla v_i) &= \nabla \cdot (\gamma \nabla u_i) \quad (\Omega), \quad v_i \big|_{\partial \Omega} = 0 \\
dH_{ij} &= \gamma \nabla u_i \cdot \nabla u_j + \gamma_0 \nabla u_i \cdot \nabla v_j + \gamma_0 \nabla u_j \cdot \nabla v_i
\end{align*}
\]

Denote \( L_0 := -\nabla \cdot (\gamma_0 \nabla) \), \( \nu_i = L_0^{-1}(\nabla \cdot (\gamma \nabla u_i)) \)

Suppose \( \gamma \) compactly supported inside \( \Omega \)

Insert \( \nu_i = L_0^{-1}(\nabla \cdot (\gamma \nabla u_i)) \) into \( dH_{ij} \) and express as a pseudo-DO

\[
dH_{ij}(\gamma, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\xi \cdot (x-y)} (M_{ij}(x, \xi) + M_{ij} \big|_{-1}(x, \xi)) : \gamma(y) d\xi dy
\]

\[
M_{ij}(x, \xi) = o(|\xi|^0) \quad M_{ij} \big|_{-1}(x, \xi) = o(|\xi|^{-1})
\]

principal symbol \( \downarrow \) \( \text{the symbol of order -1} \downarrow \)
\[ dH_y(\gamma, x) = (2\pi)^{-n} \iint_{R^n \times R^n} e^{\xi \cdot (x-y)} (M_y(x, \xi) + M_y \mid_{-1} (x, \xi)) : \gamma(y) d\xi dy \]

\[ M_y(x, \xi) = o(|\xi|^0) \quad M_y \mid_{-1} (x, \xi) = o(|\xi|^{-1}) \]

principal symbol \quad the symbol of order -1

**Goal:** determine under which conditions the operator \( dH = \{dH_y \}_{1 \leq i, j \leq m} \)
is an elliptic pseudo-differential operator

**Conclusion:** with only principal symbols \( M_y(x, \xi) \), \( dH = \{dH_y \}_{1 \leq i, j \leq m} \) will never
be elliptic, no matter how large \( m \)

Define \( A_0 = (\gamma_0)^{\frac{1}{n}} \xi_0 := \hat{A}_0 \xi \) and \( V_i := A_0 \nabla u \); rewrite the principal symbol \( M_y(x, \xi) \)

\[ \tilde{M}_{ij}(x, \xi) = A_0 M_{ij}(x, \xi) A_0 \]

\[ = V_i \odot V_j - (\xi_0 \cdot V_i) \xi_0 \odot V_j - (\xi_0 \cdot V_j) \xi_0 \odot V_i \]

**Notation:** \( U \odot V = \frac{1}{2} (U \otimes V + V \otimes U) \)
Lemma:

For any \( i, j \) and vector fields \( V_i, V_j \) defined as above \( V_i := A_0 \nabla u_i \) the symbol \( \tilde{M}_g(x, \xi) \) satisfies

\[
\tilde{M}_g(x, \xi) : \xi_0 \otimes \eta = 0 \quad \text{for all} \quad \eta \in S^{n-1} \quad \text{and} \quad \eta \perp \xi_0
\]

Conclusion: the \( M_g \) can never control a subspace of \( S_n(R) \) of dimension \( n-1 \)

**Basic Hypothesis:** the gradients \( \{\nabla u_i\}_{i=1}^n \) form a frame in \( R^n \)

Lemma:

Suppose that the vector fields \( \{V_i\}_{i=1}^n \) form a basis of \( R^n \). If a matrix \( P \in S_n(R) \)

\[
\tilde{M}_g(x, \xi) : P = 0 \quad 1 \leq i \leq j \leq n
\]

Then \( P \) is of the form \( P = \xi_0 \otimes \eta \) for some vector \( \eta \) satisfying \( \eta \perp \xi_0 \)

**Conclusion:** the only directions that are not controlled by the principal symbols are \( P = \xi_0 \otimes \eta \quad \eta \perp \xi_0 \)
The second term of \( dH_{ij} \)

\[
dH_{ij}(\gamma, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x - y \rangle} (M_{ij}(x, \xi) + M_{ij}|_{-1}(x, \xi)) : \gamma(y) d\xi dy
\]

**Goal:** invert \( dH = \{dH_{ij}\}_{1 \leq i, j \leq m} \) microlocally from \( M_{ij}(x, \xi) + M_{ij}|_{-1}(x, \xi) \)

- case \( \gamma_0 \) is constant

\[
\tilde{M}_{ij}(x, \xi) = A_0 M_{ij} |_{-1}(x, \xi) A_0
\]

\[
= \|A_0 \xi\|^{-1} \sqrt{-1}[H_i((\xi_0 \cdot V_j)(I_n - 2\xi_0 \otimes \xi_0) + V_j \otimes \xi_0) + H_j((\xi_0 \cdot V_i)(I_n - 2\xi_0 \otimes \xi_0) + V_i \otimes \xi_0)^{ym}]
\]

where \( H_i = A_0 \nabla^2 u_i A_0 \)

**Lemma**

Suppose \( \gamma_0 \) is constant, pick \( u_i = x_i, 1 \leq i \leq n \) and add an additional solution denoted by \( u_{n+1} \) with full-rank Hessian \( \nabla^2 u_{n+1} \). If there exist \( P, Q \in S_u(\mathbb{R}) \) such that for all \( 1 \leq i, j \leq n \)

\[
(\tilde{M}_{ij} + M_{ij}|_{-1})(x, \xi) : (P + \sqrt{-1}Q) = 0
\]

\[
(\tilde{M}_{i,n+1} + \tilde{M}_{i,n+1}|_{-1})(x, \xi) : (P + \sqrt{-1}Q) = 0
\]

then \( P = Q = 0 \)
• case $\gamma_0$ is not constant

**Basic Hypotheses**

- $(\nabla u, \cdots \nabla u_n)$ form a frame $-\nabla \cdot (\gamma_0 \nabla u_j) = 0 \quad u_i|_{\partial \Omega} = g_i$

- Add one addition $u_{n+1}$ such that $\sum_j \mu_j \nabla u_j + \nabla u_{n+1} = 0$

we construct the matrix $Z=[\nabla \mu_1, \cdots, \nabla \mu_n]$ assume $Z$ to be invertible

**Remark:** in the case $\gamma_0$ constant, the above hypothesis is automatically satisfied, by choosing

$u_i = x_i \quad 1 \leq i \leq n$

$u_{n+1} = \frac{1}{2} x^T Qx \quad Q \text{ invertible} \quad tr(Q) = 0$

Recall $dH_{ij}(\gamma, x) = (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i\xi(x - y)} (M_{ij}(x, \xi) + M_{ij} |_{-1} (x, \xi)) : \gamma(y) d\xi d\eta$

**Goal:** under the above hypotheses, $\tilde{M}_{ij}$ can control all bad directions $\xi_0 \otimes \eta$
Recall:
\[
dH_{ij} = \gamma \nabla u_i \cdot \nabla u_j + \gamma_0 \nabla u_i \cdot \nabla v_j + \gamma_0 \nabla u_j \cdot \nabla v_i
\]
known
\[
v_i = L^{-1}_0 (\nabla \cdot (\gamma \nabla u_i))
\]

\[
dH_{ij}(\gamma, x) = (2\pi)^{-n} \int_{R^n} e^{\xi \cdot (x-y)} (M_{ij}(x, \xi) + M_{ij} \mid_{-1} (x, \xi)) : \gamma(y) d\xi dy
\]

**Lemma**
Assume that \( Z = [\nabla \mu_1, \cdots, \nabla \mu_n] \) is invertible, if \( \tilde{M} \mid_{-1} \xi_0 \odot \eta = 0 \) and \( \tilde{M} \mid_{n+1} \xi_0 \odot \eta = 0 \) for any \( 1 \leq i, j \leq n \) and \( \xi_0 \perp \eta \) then \( \eta = 0 \)

**Theorem**
Suppose that we have measurements \( \{dH_{ij}\}_{1 \leq i,j \leq n} \) and \( \{dH_{i,n+1}\}_{1 \leq i \leq n} \) such that \( Z = [\nabla \mu_1, \cdots, \nabla \mu_n] \) is invertible and \( (\nabla u_1, \cdots, \nabla u_n) \) form a frame.

we denote the operator \( dH \) as follows
\[
dH := (dH_{ij}, \cdots, dH_{i,n+1})_{1 \leq i,j \leq n}
\]

Then the operator \( G \circ dH : \{\tilde{L}(\Omega')\}^2 \rightarrow \{\tilde{L}(\Omega')\}^2 \)

is semi-Fredholm, where \( G \) is the restriction operator to \( \tilde{L}(\Omega') \)

**Proof:** construct a parametrix \( Q \) such that \( Q \circ dH - Id \) is of order -1
3. Explicit reconstruction

\[-\nabla \cdot (\gamma_0 \nabla u_i) = 0 \quad (\Omega), \quad u_i|_{\partial\Omega} = g_i \quad (\star)\]

\[
\begin{cases}
-\nabla \cdot (\gamma_0 \nabla \nu_j) = \nabla \cdot (\gamma \nabla u_i) \quad (\Omega), \quad \nu_j|_{\partial\Omega} = 0 \\
\frac{dH_{ij}}{\partial \gamma} = \gamma \nabla u_i \cdot \nabla u_j + \gamma_0 \nabla u_i \cdot \nabla \nu_j + \gamma_0 \nabla u_j \cdot \nabla \nu_i
\end{cases}
\]

Let us now add another solution of (\star) \(U\) with corresponding \(V\) at order \(\varepsilon\)

\[u' = u + \varepsilon v + o(\varepsilon^2)\]

\(\nabla u\) may be expressed in the basis \(\nabla u_1, \ldots, \nabla u_n\)

\[\sum_{j} \mu_j \nabla u_j + \nabla u = 0\]

Suppose \([\nabla U] = [\nabla u_1 | \cdots | \nabla u_n]\) and \(Z := [Z_1 | \cdots | Z_n]\) be invertible, where \(Z_j = \nabla \mu_j\)
In the end, we obtain the strongly **coupled** elliptic system

\[-\nabla \cdot (\gamma_0 \nabla v_i) + \sum_{j=1}^{n} W_{ij} \cdot \nabla v_j = f_i, \quad v_i|_{\partial \Omega} = 0 \quad 1 \leq i \leq n\]

where \( W_{ij} \) depends on \( \gamma_0 \) and \( (u_1, \ldots, u_n) \)

\( f_i \) depends on \( dH_{pq} \) and their derivatives

With \( L_0 := -\nabla \cdot (\gamma_0 \nabla) \) define \( L_0^1 : f \in \vec{E} \mapsto u \in H_0^1(\Omega) \) where \( u \) solves \( -\nabla \cdot (\gamma_0 \nabla u) = f' \)

\( u|_{\partial \Omega} = 0 \)

By Lax-Milgram theorem, and by Rellich imbedding, \( L_0^1 : \vec{E}(\Omega) \mapsto H_0^1(\Omega) \) is compact.

Applying \( L_0^1 \) to the elliptic system yields \( v_i + \sum_{j=1}^{n} L_0^{-1}(W_{ij} \cdot \nabla v_j) = h_i := L_0^{-1} f_i \)

\[ P_{ij} : H_0^1(\Omega) \ni v \mapsto P_{ij}v := L_0^{-1}(W_{ij} \cdot \nabla v) \in H_0^1(\Omega) \] compact if \( W_{ij} \) bounded.

we obtain \( (I + P)v = h \quad (\star) \)

where \( v = (v_1, \ldots, v_n), \quad h = (h_1, \ldots, h_n), \) and \( P \) compact

(\( \star \)) satisfies a Fredholm alternative, if -1 is not an eigenvalue of \( P \)

and we obtain the stability: \( \| v_i \|_{H_0^1(\Omega)} \leq C \| dH \|_{H_0^1(\Omega)} \)

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once \((v_1, \cdots v_n)\) are reconstructed

the relations

\[
\underbrace{dH_{ij}}_{\text{known}} = \gamma \nabla u_i \cdot \nabla u_j + \gamma_0 \nabla u_i \cdot \nabla v_j + \gamma_0 \nabla v_i \cdot \nabla v_j
\]

can be inverted for \(\gamma\)

\[
\gamma = \gamma_0 (dH_{ip} (H_{pq} \nabla u_q \otimes H^{ij} \nabla u_j) - 2 H^{ij} \nabla v_i \otimes \nabla u_j) \gamma_0
\]

and we get the stability

\[
\|\gamma\|_{L^2(\Omega)} \leq C \|dH\|_{H^1(\Omega)}
\]
• **case** $\gamma_0$ is constant

\[
- \nabla \cdot (\gamma_0 \nabla u_i) = 0 \quad (\Omega), \quad u_i|_{\partial \Omega} = g_i^{\text{known}} \quad (\ast)
\]

\[
- \nabla \cdot (\gamma_0 \nabla v_j) = \nabla \cdot (\gamma \nabla u_i) \quad (\Omega), \quad v_i|_{\partial \Omega} = 0
\]

\[
dH_{ij} = \gamma \nabla u_i \cdot \nabla u_j + \gamma_0 \nabla u_i \cdot \nabla v_j + \gamma_0 \nabla u_j \cdot \nabla v_i
\]

Taking $\gamma_0 = Id$ using harmonic polynomials as solutions of $(\ast)$

\[
\begin{align*}
  u_i &= x_i & 1 \leq i \leq n \\
  u &= \frac{1}{2} \sum_{p=1}^{n} a_p x_p^2 & a_p \neq 0 & \sum_p a_p = 0
\end{align*}
\]

Then we get n additional measurements

\[
dH_i = \gamma : \nabla u_i \odot \nabla u_i + \nabla u_i \odot \nabla v_i + \nabla v_i \cdot \nabla v_i \quad 1 \leq i \leq n
\]

Together with

\[
dH_{ij} = \gamma \nabla u_i \cdot \nabla u_j + \nabla u_i \cdot \nabla v_j + \nabla u_j \cdot \nabla v_i \quad 1 \leq i, j \leq n
\]
\[
\frac{1}{2} n(n+1) + n
\]

\[
\begin{align*}
    dH_i &= \gamma : \nabla u_i \otimes \nabla u_i' + \nabla u_i \otimes \nabla v_i + \nabla v_i \cdot \nabla u_i' & 1 \leq i \leq n \\
    dH_{ij} &= \gamma \nabla u_i \cdot \nabla u_j + \nabla u_i \cdot \nabla v_j + \nabla u_j \cdot \nabla v_i & 1 \leq i, j \leq n
\end{align*}
\]

For this choice of \((u_1, \cdots, u_n)\), the elliptic system is decoupled, hence solvable.

\[
\Delta v_i = \frac{1}{a_i} \sum_{j=1}^{n} \partial_j (\sum_{p=1}^{n} a_{ip} \partial_p dH_p + \partial_j dH_l) + \frac{1}{a_i} \partial_i \sum_{p=1}^{n} a_{ip} dH_{pp} \\
\gamma_{ij} = dH_{ij} - \partial_j v_i - \partial_i v_j
\]

we obtain a standard estimate for \(v_i\), notice that the RHS is a linear combination of \(dH_y, dH_x\) and their first derivatives

\textbf{stability:} \quad \|v_i\|_{H^r(\Omega)} \leq C \|dH\|_{H^r(\Omega)} \quad \text{then} \quad \|\gamma_{ij}\|_{\mathcal{L}(\Omega)} \leq C \|dH\|_{H^r(\Omega)}

\Rightarrow \quad \text{we lose one derivative}