A mathematical model and inversion procedure for Magneto-Acousto-Electric Tomography (MAET)

Leonid Kunyansky

University of Arizona, Tucson, AZ

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Hybrid methods: motivation

Conductivity carries important medical information.
Conductivity of tumors is much higher than that of healthy tissues. $\Longrightarrow$
EM measurements yield high contract.

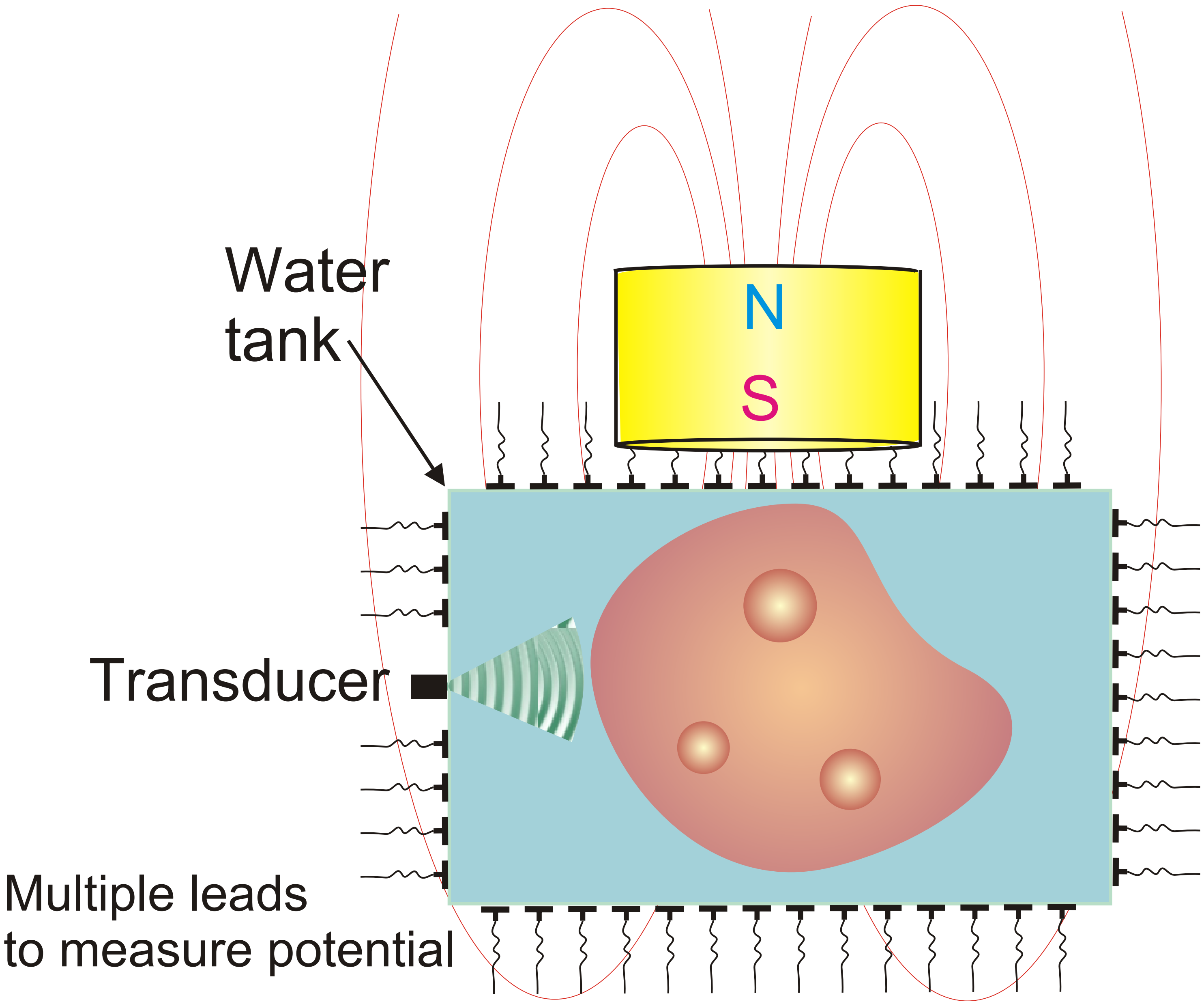
However, electrical impedance, optical and microwave tomographies lead to reconstruction problems that are strongly non-linear and severely ill-posed.

Acoustic waves yield high resolution but the contrast is low.

Use **hybrid** techniques; couple ultrasound with EM field:

- Thermo-Acoustic and Photo-Acoustic Tomography (TAT/PAT)
- Ultrasound Modulated Optical Tomography (UMOT)
- Acousto-Electric Tomography (AET)
- Magneto-Acousto-Electric Tomography (MAET)
- Magneto-Acoustic Tomography with Magnetic Induction (MAT-MI)

Magneto-Acousto-Electric Tomography (MAET)



Physics of MAET

Tissue moving with velocity $V(x,t)$ produces Lorentz currents $J_L(x,t)$:

$$J_L(x,t) = \sigma(x) B \times V(x,t)$$

There will also be Ohmic currents satisfying Ohm's law

$$J_O(x,t) = \sigma(x) \nabla u(x,t).$$

There are no sinks or sources, the total current is divergence-free

$$\nabla \cdot (J_L + J_O) = 0.$$

Thus

$$\nabla \cdot \sigma \nabla u = -\nabla \cdot (\sigma B \times V).$$

BC: the normal component of the total current $J_L(x,t) + J_O(x,t)$ vanishes:

$$\left. \frac{\partial}{\partial n} u(z) \right|_{\partial \Omega} = -(B \times V(z)) \cdot n(z)$$

Measuring functionals

At any given time t we measure potential $u(z,t)$ at all $z \in \partial\Omega$.

Integrate boundary values of u with weight $I(z)$ and get a functional $M(t)$:

$$M(t) = \int_{\partial\Omega} I(z) u(z,t) dA(z),$$

Consider lead potential $w_I(x)$ and lead current $J_I(x) = \sigma(x) \nabla w_I(x)$:

$$\nabla \cdot \sigma \nabla w_I(x) = 0,$$

$$\left. \frac{\partial}{\partial n} w_I(z) \right|_{\partial\Omega} = I\left(z\right).$$

Then (by the second Green's identity):

$$M(t) = \int_{\Omega} B \cdot J_I(x) \times V(x,t) dx$$

Previous models

(1) S. Haider, A. Hrbek, and Y. Xu, Magneto-acousto-electrical tomography: a potential method for imaging current density and electrical impedance, *Physiol. Meas.* **29** (2008) S41-S50.

Focused acoustic pulse, two-electrod acquisition

(2) B. J. Roth and K. Schalte, Ultrasonically-induced Lorentz force tomography, *Med. Biol. Eng. Comput.* **47** (2009) 573–7

Time-harmonic plane waves, two-electrod acquisition, first term only.

(3) H. Ammari, Y. Capdeboscq, H. Kang, and A. Kozhemyak, Mathematical models and reconstruction methods in magneto-acoustic imaging, *Euro. Jnl. of Appl. Math.*, **20** (2009) 303–17.

The present model generalizes (1) and (2).

Model (3) does not agree with all the others.

Analyzing the velocity field

Assume that speed of sound c and density ρ are constant.

Then velocity is the gradient of velocity potential $\varphi(x,t)$:

$$V(x,t) = \frac{1}{\rho}\nabla\varphi(x,t)$$

Velocity potential and pressure $p(x,t)$ satisfy the wave equation

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi(x,t) &= \Delta\varphi(x,t), \\ p(x,t) &= \frac{\partial}{\partial t}\varphi(x,t). \end{aligned}$$

Substitute into equation for $M(t)$ and integrate by parts:

$$M(t) = \frac{1}{\rho}B\cdot\left[\int\limits_{\partial\Omega}\varphi(z,t)J_I(z)\times n(z)dA(z) + \int\limits_{\Omega}\varphi(x,t)\nabla\times J_I(x)dx\right]$$

Volumetric part shows that we measure components of $\mathbf{curl} J_I(x)$!

Synthetic focusing

If $\varphi(x,t)$ could be focused into a point, i.e. $\varphi(x,0)=\delta(x-x_0)$, then

$$M_{x_0}(0)=\frac{1}{\rho}B\cdot\left[\int\limits_{\Omega}\delta(x-x_0)\mathrm{curl}J_I(x)dx\right]=\frac{1}{\rho}B\cdot\mathrm{curl}J_I(x_0).$$

If three differenent directions of B are used, we have $\mathrm{curl}J_I(x_0)$!

Perfect focusing is not possible! Let's use spherical fronts centered at y:

$$\varphi(x,y,t)=\frac{\delta(|x-y|-ct)}{4\pi|x-y|}.$$

Then measuring functional $M_{I,B}(y,t)$ equals

$$M_{I,B}(y,t)=\frac{1}{\rho}\int\limits_{\Omega}\frac{\delta(|x-y|-ct)}{4\pi|x-y|}B\cdot\nabla\times J_I(x)dx+\text{surface term}$$

It solves the wave equation

$$\frac{\partial^2}{\partial t^2}M_{I,B}(y,t)=\Delta_y M_{I,B}(y,t),\qquad \frac{\partial}{\partial t}M_{I,B}(y,0)=\frac{1}{\rho}B\cdot\mathrm{curl}J_I(y).$$

Thus, $\mathrm{curl}J_I(y)$ **can be reconstructed by the methods of TAT!**

From curls to currents

Denote curl$J(x)$ by $C(x)$. Since $J(x)$ is a purely solenoidal field:

$$J(x) = \nabla \times \int\limits_{\Omega} \frac{C(y)}{4\pi(x-y)} dy + \nabla\psi(x),$$

where $\psi(x)$ is a harmonic function.

Find $\psi(x)$ by solving the Laplace eq-n with Neumann BC's:

$$\begin{cases} \Delta\psi(x) = 0, \quad x \in \Omega \\ \frac{\partial}{\partial n}\psi(z) = I(z) - n \cdot \left(\nabla \times \int\limits_{\Omega} \frac{C(y)}{4\pi|z-y|} dy \right), \quad z \in \partial\Omega. \end{cases}$$

Got the current(s)!

From currents to conductivity

Is finding conductivity from known currents a linear problem?

$$0 = \nabla \times \frac{J}{\sigma} = \left(\nabla \frac{1}{\sigma}\right) \times J + \frac{1}{\sigma} C = -\frac{1}{\sigma^2} (\nabla \sigma) \times J + \frac{1}{\sigma} C$$

or

$$\nabla \ln \sigma \times J = C.$$

Yes, it is!

If we have two lead currents $J^{(j)}(x)$, $j = 1, 2$, then:

$$\begin{cases} \nabla \ln \sigma(x) \times J^{(1)}(x) = C^{(1)}(x) \\ \nabla \ln \sigma(x) \times J^{(2)}(x) = C^{(2)}(x) \end{cases}.$$

This system w. r. to $\nabla \ln \sigma$ is overdetermined, easily solved at each x

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At no cost (?) we can have three lead currents $J^{(j)}(x)$, $j = 1, 2, 3$, then:

$$\begin{cases} \nabla \ln \sigma \times J^{(1)} = C^{(1)} \\ \nabla \ln \sigma \times J^{(2)} = C^{(2)} \\ \nabla \ln \sigma \times J^{(3)} = C^{(3)} \end{cases}.$$

Explicit formula with three lead currents

If M is the following matrix

$$M = \left(J^{(1)} \middle| J^{(2)} \middle| J^{(3)} \right),$$

then

$$\Delta \ln \sigma = \frac{1}{2} \nabla \cdot \left[\frac{1}{J^{(1)} \cdot (J^{(2)} \times J^{(3)})} M \begin{pmatrix} C^{(2)} \cdot J^{(3)} - C^{(3)} \cdot J^{(2)} \\ -C^{(1)} \cdot J^{(3)} + C^{(3)} \cdot J^{(1)} \\ C^{(1)} \cdot J^{(2)} - C^{(2)} \cdot J^{(1)} \end{pmatrix} \right],$$

subject to the Dirichlet boundary conditions

$$\ln \sigma \big|_{\partial\Omega} = 0.$$

Solve the above Poisson equation, find $\ln \sigma$!

Fast algorithm for a rectangular domain

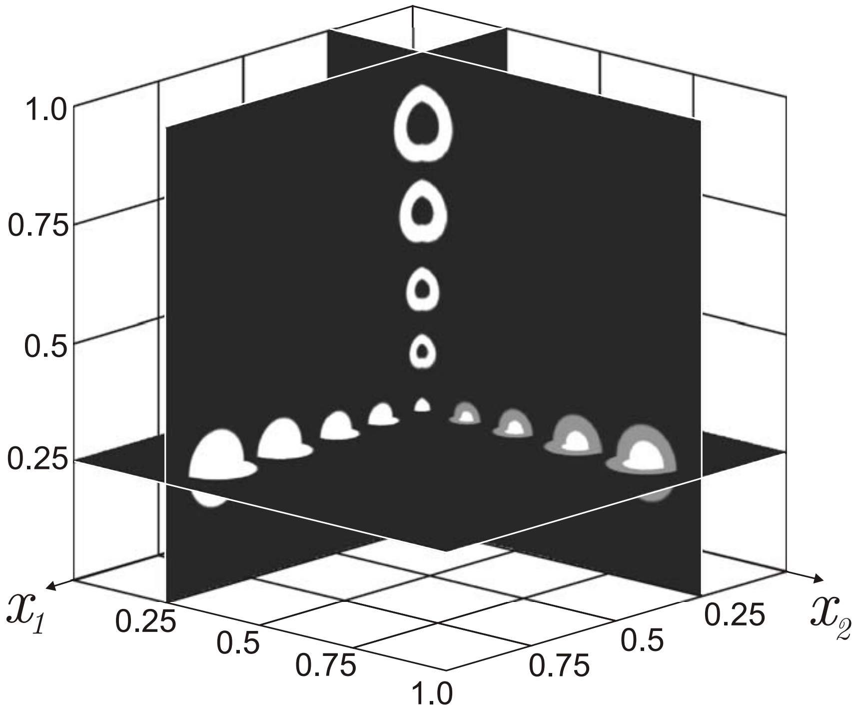
A three step procedure:

(1) Synthetic focusing: fast algorithm for a cube, Kunyansky [2007]

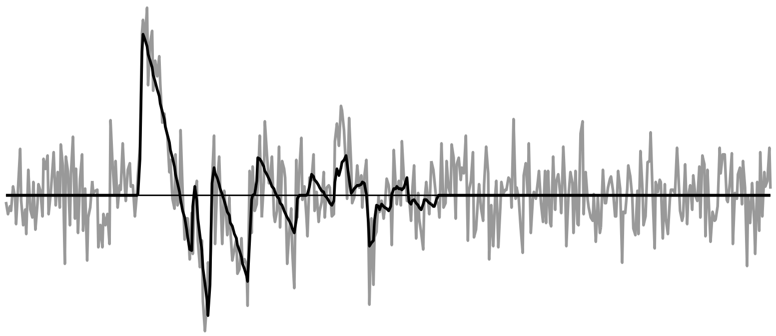
(2) Finding currents from curls:
Fast Cosine Fourier Transform yields correct BC!

(3) Solving Poisson equation in a cube: use Fast Sine Fourier Transform

Simulations: phantom and noisy data



One of the simulated measurement functionals, with added 100% noise



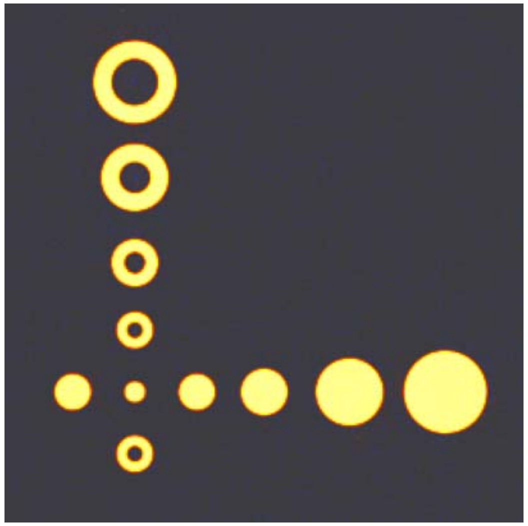
Simulations: reconstruction

Phantom:

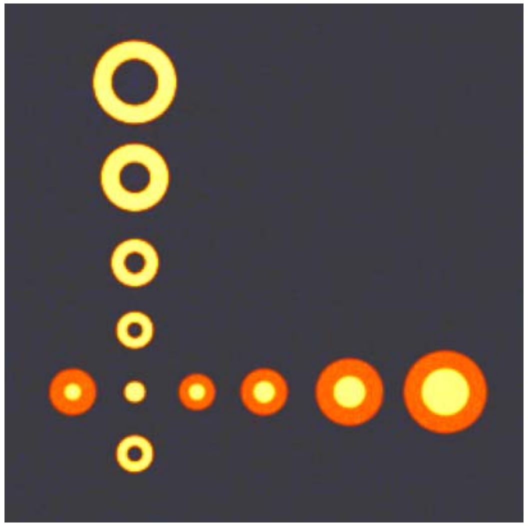
$x_2 = 0.25$

$x_3 = 0.25$

Reconstruction:

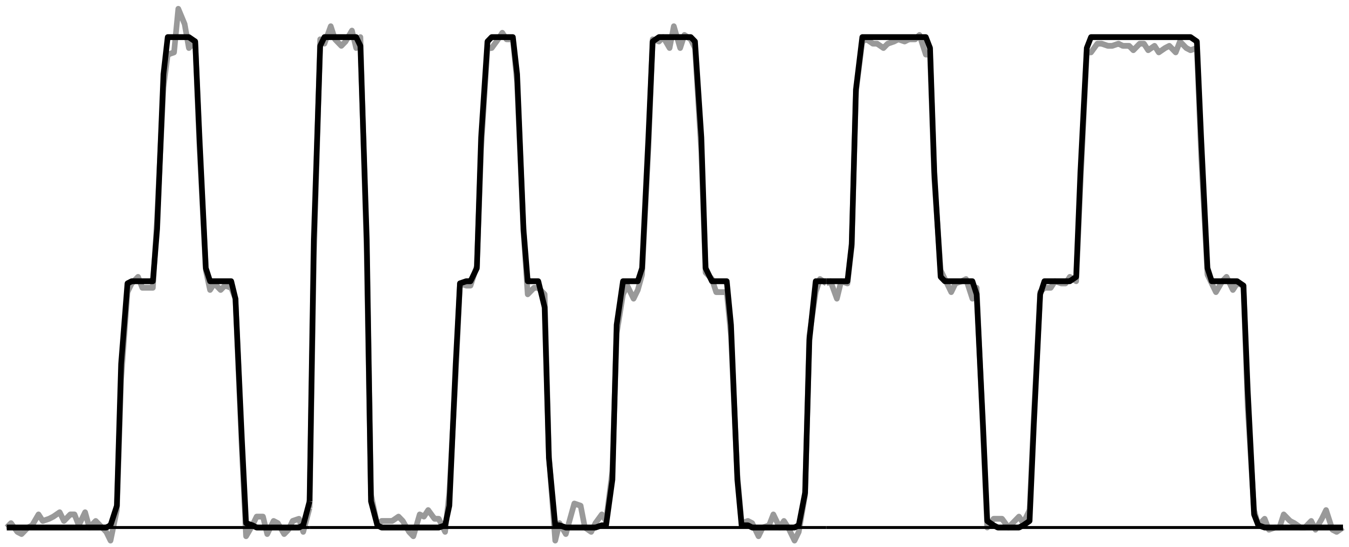


$x_2 = 0.25$



$x_3 = 0.25$

Reconstruction: profile



Cross section of the reconstruction by the line $x_1 = 0.25$, $x_3 = 0.25$.

Remarks and open questions

(1) Reconstruction with only two directions of magnetic field

If only $B^{(1)}$ and $B^{(2)}$ are used, then only C_1 and C_2 can be found.
But $\operatorname{div}\operatorname{curl} J = 0$. To find C_3 solve

$$\frac{\partial}{\partial x_3} C_3 = -\frac{\partial}{\partial x_1} C_1 - \frac{\partial}{\partial x_2} C_2.$$

(2) Cannot guarantee three linearly independent currents. Counterexample.

(3) Can one always have two non-parallel currents?