Local Inversions in Ultrasound Modulated Optical Tomography

Guillaume Bal
Shari Moskow
Ultrasound Modulated Optical Tomography (Acousto-Optics)

• Acoustic waves are emitted which perturb the optical properties of the medium
• Light propagating through the medium is used to recover the original optical parameters
Optical properties perturbed by acoustic waves

\[ \sigma_{\varepsilon} = \sigma + \varepsilon(2\beta + 1) \cos(k \cdot x + \phi) \]
\[ \gamma_{\varepsilon} = \gamma + \varepsilon(2\beta - 1) \cos(k \cdot x + \phi) \]

Linearization wrt epsilon and some manipulation yields boundary data

\[ \Sigma(k, \phi) = \int_{\Omega} \left[ (2\beta - 1)\gamma(\nabla \phi)^2 + (2\beta + 1)\sigma \phi^2 \right] \cos(k \cdot x + \phi) \]

Which is the Fourier transform of some internal data
Mathematical Problem

Given internal data of the form

\[ H_{ij}(x) = \gamma \nabla u_i \cdot \nabla u_j + \eta \sigma u_i u_j, \]

where \( \eta \) is a known fixed constant and

\[ -\nabla \cdot \gamma \nabla u_j + \sigma u_j = 0 \quad \text{in} \quad \Omega \]
\[ u_j = f_j \quad \text{on} \quad \partial \Omega \]

Find \( \gamma \) and \( \sigma \)
Previous work

• Recovery of $\gamma$ only, $\sigma = 0$

  Capdeboscq, Fehrenback, De Gournay, Kavian ($n=2$)
  Bal, Bonnetier, Monard, Triki ($n=3$)
  Bal, Monard ($n\geq 4$)
  Kuchment, Kunyansky
  Kuchment, Steinhauer- pseudo-differential calculus
  Ammari, Capdeboscq, Triki 2012- separation of terms
Assume we have some known background $\gamma_0$ and $\sigma_0$.

\[
\begin{align*}
\gamma &= \gamma_0 + \delta \gamma \\
\sigma &= \sigma_0 + \delta \sigma \\
u_j &= u_j^0 + \delta u_j
\end{align*}
\]

Where the background solutions satisfy

\[
-\nabla \cdot \gamma_0 \nabla u_j^0 + \sigma_0 u_j^0 = 0 \text{ in } \Omega \\
u_j^0 = f_j \text{ on } \partial \Omega
\]
\[ L_0 := -\nabla \cdot \gamma_0 \nabla + \sigma_0 \]

\[ \delta u_j = L_0^{-1} (\nabla \cdot \delta \gamma \nabla u_j^0 - \delta \sigma u_j^0) \]
linearized problem

\[ dH_{ij} = \delta\gamma \nabla u^0_i \cdot \nabla u^0_j + \gamma_0 \nabla \delta u_i \cdot \nabla u^0_j + \gamma_0 \nabla u^0_i \cdot \nabla \delta u^0_j \]

\[ + \eta \delta\sigma u^0_i u^0_j + \eta\sigma_0 \delta u_i u^0_j + \eta\sigma_0 u^0_i \delta u_j \]

Really have 3 unknowns here \( \delta u_j, \delta\gamma, \delta\sigma \)

\[ L_0 \delta u_j = \nabla \cdot \delta\gamma \nabla u^0_j - \delta\sigma u^0_j \]

But they are coupled

One approach: solve for \( \delta u_j \) and substitute back in
Take Laplacian of data

\[ \Delta dH_{ij}(\delta \gamma, \delta \sigma) = G_{ij}(\delta \gamma, \delta \sigma) + \Delta T_{ij}(\delta \gamma, \delta \sigma), \]

where \( T_{ij} \) is compact, and

\[ G_{ij}(\delta \gamma, \delta \sigma) = \nabla u_0^i \cdot \nabla u_0^j \Delta \delta \gamma - 2(\nabla u_0^i \otimes \nabla u_0^j)^s : D^2 \delta \gamma + \eta u_0^i u_0^j \Delta \delta \sigma \]
Simplest case

- Case where $n = 2$ and $\gamma_0 = 1, \sigma_0 = 0$
- Take $u_0 = 1$ to get
  $$dH_{00}(\delta\gamma, \delta\sigma) = \eta\delta\sigma$$
- Eliminate $\delta\sigma$
- Take
  $$u_i^0 = x_i$$
then

\[ \Delta \tilde{d}H_{11} = (\partial_{x_2}^2 - \partial_{x_1}^2) \]
\[ \Delta \tilde{d}H_{12} = -2\partial_{x_1}x_2 \]
\[ \Delta \tilde{d}H_{22} = (\partial_{x_1}^2 - \partial_{x_2}^2) \]

Separately get hyperbolic, not elliptic
Together elliptic as a redundant system
Hard to invert because redundant
but consider

\[
\Gamma^T \Gamma = \sum_{ij} (\Delta d \tilde{H}_{ij})^2
\]

\[
\Gamma^T \Gamma = 2 \partial^4_{x_1} + 2 \partial^4_{x_2}
\]

Which is elliptic.
And for \( n \geq 3 \)

\[
\sum_{i=1}^{n} \Delta d \tilde{H}_{ii} = (n - 2) \Delta
\]

Which we can invert
For general $\sigma_0$ this doesn’t work

So let us consider for general $\gamma_0, \sigma_0$

The highest order part

$$G_{ij}(\delta \gamma, \delta \sigma) = \nabla u^0_i \cdot \nabla u^0_j \Delta \delta \gamma - 2(\nabla u^0_i \otimes \nabla u^0_j)^s : D^2 \delta \gamma + \eta u^0_i u^0_j \Delta \delta \sigma$$

Define

$$\theta_i = \frac{\nabla u^0_i}{|\nabla u^0_i|}$$
Then we are interested in the system

\[ A_{ij} \delta \gamma + B_{ij} \delta \sigma = F_{ij}, \]

where

\[ A_{ij} = \theta_i \cdot \theta_j \Delta - 2(\theta_i \otimes \theta_j)^s : \nabla \otimes \nabla, \]

and

\[ B_{ij} = \eta d_i d_j \Delta, \quad d_i = \frac{u_i}{|\nabla u_i|}. \]
Define the operator

$$\Gamma = \begin{pmatrix} A_{ij} & B_{ij} \end{pmatrix}$$

With a row for each pair (i,j)

We want to show this operator is elliptic so that we can get a parametrix, or invertibility of the highest order part.

Construction of parametrices for similar problems in

Kuchment and Steinhauer for one coefficient.
Consider the 2x2 system

\[ \Gamma^T \Gamma \begin{pmatrix} \delta \gamma \\ \delta \sigma \end{pmatrix} = \Gamma^T F. \]

\[ \Gamma^T \Gamma = \begin{pmatrix} \sum_{ij} A_{ij}^T A_{ij} & \sum_{ij} A_{ij}^T B_{ij} \\ \sum_{ij} B_{ij}^T A_{ij} & \sum_{ij} B_{ij}^T B_{ij} \end{pmatrix}. \]
Using symbols, the system is invertible when we always have at least one of the sub-determinants not vanishing

$$\det \begin{pmatrix} A_{ij} & B_{ij} \\ A_{kl} & B_{kl} \end{pmatrix}$$
These determinants are zero when

$$(\theta_i \cdot \theta_j - 2\theta_i \cdot \hat{\xi}\theta_j \cdot \hat{\xi})d_p d_q = (\theta_p \cdot \theta_q - 2\theta_p \cdot \hat{\xi}\theta_q \cdot \hat{\xi})d_i d_j \quad \forall (i, j, p, q).$$

$$\theta_i = \frac{\nabla u_i^0}{|\nabla u_i^0|} \quad d_i = \frac{u_i^0}{|\nabla u_i^0|}$$
Thanks to Gunther Uhlmann and CGOs

\[ u_\rho = e^{\rho \cdot x} \]
\[ = e^{\rho_r \cdot x} (\cos \rho_I \cdot x + i \sin \rho_I \cdot x) \]
\[ \nabla u_\rho = e^{\rho_r \cdot x} [(\rho_r \cos \rho_I \cdot x - \rho_I \sin \rho_I \cdot x) + i(\rho_r \sin \rho_I \cdot x + \rho_I \cos \rho_I \cdot x)] \]

Which gives, by taking real and imaginary parts

\[ \theta_1 = \begin{pmatrix} \cos \rho_I \cdot x \\ -\sin \rho_I \cdot x \end{pmatrix} \quad \theta_2 = \begin{pmatrix} \sin \rho_I \cdot x \\ \cos \rho_I \cdot x \end{pmatrix} \]

\[ d_1 = \frac{\cos \rho_I \cdot x}{|\rho|} \quad d_2 = \frac{\sin \rho_I \cdot x}{|\rho|} \]
\[
(1 - 2(\theta_1 \cdot \xi)^2) d_2^2 = (1 - 2(\theta_2 \cdot \xi)^2) d_1^2 \\
-2\theta_1 \cdot \xi \theta_2 \cdot \xi d_1^2 = (1 - 2(\theta_1 \cdot \xi)^2) d_1 d_2 \\
-2\theta_1 \cdot \xi \theta_2 \cdot \xi d_2^2 = (1 - 2(\theta_2 \cdot \xi)^2) d_1 d_2
\]

Which is
\[
(s^2 - c^2)d_2^2 = (c^2 - s^2)d_1^2 \\
-2csd_1^2 = (s^2 - c^2)d_1 d_2 \\
-2csd_2^2 = (c^2 - s^2)d_1 d_2
\]

\( (1) \Rightarrow s^2 - c^2 = 0 \)

\((2) \text{ or } (3) \Rightarrow sc = 0 \)
• But ellipticity doesn’t guarantee injectivity

• Need injectivity for extensions to nonlinear problem
One approach: view as a differential operator with its natural square bilinear form

\[ B \left( \left( \begin{array}{c} v \\ w \end{array} \right), \left( \begin{array}{c} v \\ w \end{array} \right) \right) := \int_{\Omega} \Gamma \left( \begin{array}{c} v \\ w \end{array} \right) \cdot \Gamma \left( \begin{array}{c} v \\ w \end{array} \right) \]

on

\[ H_0^2(\Omega) \times H_0^2(\Omega) \]
Variational formulation: find

\[(\delta \gamma, \delta \sigma) \in H^2_0(\Omega) \times H^2_0(\Omega)\]

Such that

\[B\left(\left( \begin{array}{c} \delta \gamma \\ \delta \sigma \end{array} \right), \left( \begin{array}{c} v \\ w \end{array} \right)\right) + L\left(\left( \begin{array}{c} \delta \gamma \\ \delta \sigma \end{array} \right), \left( \begin{array}{c} v \\ w \end{array} \right)\right) = \int_{\Omega} F \cdot \Gamma^T\left( \begin{array}{c} v \\ w \end{array} \right)\]

\[\forall (v, w) \in H^2_0(\Omega) \times H^2_0(\Omega)\]

Where \(L\) is a lower order operator

(generally nonlocal)
• B is clearly bounded above on $H_0^2(\Omega) \times H_0^2(\Omega)$
• Know elliptic, can get coercivity bounds explicitly in some cases
Case $n=2$, constant $\sigma_0, \gamma_0$

- Have the two background solutions

\[
\begin{align*}
    u_1^0 &= e^{\sqrt{\frac{\sigma_0}{\gamma_0}} x_1} \\
    u_2^0 &= e^{\sqrt{\frac{\sigma_0}{\gamma_0}} x_2},
\end{align*}
\]

- Which give

\[
\theta_i = e_i \text{ and } d_i = \sqrt{\frac{\gamma_0}{\sigma_0}}.
\]
\[
\Gamma = \begin{pmatrix} A_{ij} & B_{ij} \end{pmatrix}
\]

Corresponding to \((i,j) = (1,1), (1,2), (2,2)\) where

\[
A_{11} = \partial_{yy} - \partial_{xx}
\]
\[
A_{12} = -2\partial_{xy}
\]
\[
A_{22} = \partial_{xx} - \partial_{yy}
\]

\[
B := B_{11} = B_{12} = B_{22} = \eta \frac{\gamma_0}{\sigma_0} \Delta.
\]
\[ B \left( \begin{pmatrix} v \\ w \end{pmatrix}, \begin{pmatrix} v \\ w \end{pmatrix} \right) = \int_{\Omega} 2(v_{xx})^2 + 2(v_{yy})^2 + 3\eta^2 \frac{\gamma^2_0}{\sigma^2_0} (\Delta w)^2 - 2\eta \frac{\gamma_0}{\sigma_0} v_{xy} \Delta w \]

Use Cauchy’s inequality

\[ |v_{xy} \Delta w| \leq \epsilon v_{xy}^2 + \frac{(\Delta w)^2}{4\epsilon} \]

and integration by parts

\[ \int_{\Omega} v_{xy}^2 = \int_{\Omega} v_{xx} v_{yy} \]
\[ B \left( \left( \begin{array}{c} v \\ w \end{array} \right), \left( \begin{array}{c} v \\ w \end{array} \right) \right) \]

\[ \geq \int_{\Omega} \left( \frac{2}{\gamma_0 \sigma_0} - |\eta| \frac{\gamma_0}{2 \sigma_0} \epsilon \right) \left( \begin{array}{c} v_{xx} \\ v_{yy} \end{array} \right) \cdot \left( \begin{array}{c} v_{xx} \\ v_{yy} \end{array} \right) + \left( 3\eta^2 \frac{\gamma_0^2}{\sigma_0^2} - |\eta| \frac{\gamma_0}{2\epsilon \sigma_0} \right) (\Delta w)^2 \]

choose \[ \epsilon = \frac{\sigma_0}{\gamma_0 |\eta|} \]. To get

\[ \geq \|v_{xx}\|_{L^2}^2 + \|v_{yy}\|_{L^2}^2 + \frac{3}{2} \frac{\gamma_0^2}{\sigma_0^2} \eta^2 \|\Delta w\|_{L^2}^2. \]
• If we have injectivity, this means that the linearized solutions

$$\|\hat{\delta}\gamma\|_{H_0^2(\Omega)}, \|\hat{\delta}\sigma\|_{H_0^2(\Omega)} \leq C\|F\|_{L^2(\Omega)}$$

• and we have explicit knowledge of $C$
• System is elliptic- but don’t yet know if injective.

• But since problem is square:

\[ \int_{\Omega} \Gamma \left( \begin{array}{c} v \\ w \end{array} \right) \cdot \Gamma \left( \begin{array}{c} v \\ w \end{array} \right) = 0 \Rightarrow \Gamma \left( \begin{array}{c} v \\ w \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \]
Case where domain is small

- If the domain is small, $\nabla u_i^0$ and $u_i^0$ are close to constants

$$dH_{ij}(\delta \gamma, \delta \sigma) = \delta \gamma \nabla u_i^0 \cdot \nabla u_j^0 + \gamma_0 \nabla \delta u_i \cdot \nabla u_j^0 + \gamma_0 \nabla u_i^0 \cdot \nabla \delta u_j + \eta \delta \sigma u_i^0 u_j^0 + \eta \sigma_0 \delta u_i u_j^0 + \eta \sigma_0 u_i^0 \delta u_j$$

So when we take $L_0$ of data, lower order terms are differential operators.

$$L_0 = -\nabla \cdot \gamma_0 \nabla + \sigma$$
• if \( \Gamma \) is a differential operator and
\[
\Gamma \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

• since
\[
v = \frac{\partial v}{\partial \nu} = w = \frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega
\]

We can get that \( v = w = 0 \) from Holmgren’s theorem.
Conclusions/Future

- Have ellipticity for linearized system
- Have injectivity with boundary data if the domain is small enough (by variational formulation and Holmgren’s theorem)
- So for small domains, can extend to local nonlinear injectivity/inversion
- Still to do: injectivity for more general domains