L^p eigenfunction estimates and directional oscillation

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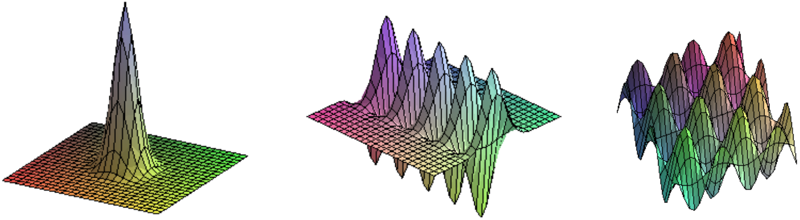
Eigenfunction Concentration

Would like to understand behaviour of eigenfunctions of Laplace-Beltrami and similar operators. Let M be a compact Riemannian manifold without boundary

$$-\Delta u_j = \lambda_j^2 u_j$$

- How large can u_j be ?
- Where can u_j be large?
- What do concentrations look like?



L^p Eigenfunction Estimates

- Seek estimates of the form

$$\|u_j\|_{L^p} \lesssim f(\lambda_j, p)\, \|u_j\|_{L^2}$$

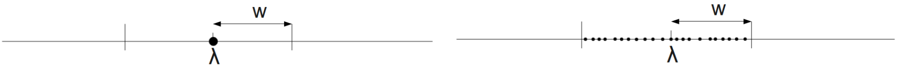
  and sharp examples
- Expect properties of classical flow to be evident in estimates.
- Not easy to study eigenfunctions directly. Therefore we will study sums (clusters) of eigenfunctions.

Spectral Windows

We study norms of spectral clusters on windows of width w

$$E_\lambda = \sum_{\lambda_j \in [\lambda - w, \lambda + w]} E_j$$

E_j projection onto λ_j eigenspace.



Obviously include eigenfunctions but also can include sums of eigenfunctions if w is large enough.
Shrinking the window avoids pollution of estimates by eigenfunctions of similar eigenvalue.

Smoothed Spectral Clusters

Pick χ smooth such that $\chi(0) = 1$ and $\hat{\chi}$ is supported in $[\epsilon, 2\epsilon]$. We will study

$$\chi_{\lambda,A} = \chi(A(\sqrt{-\Delta} - \lambda))$$

Write

$$\chi_{\lambda,A} = \int_{\epsilon}^{2\epsilon} e^{itA\sqrt{-\Delta}} e^{-itA\lambda} \hat{\chi}(t) dt$$

If we can write $e^{itA\sqrt{-\Delta}}$ as an integral operator with kernel $e(x, y, t, A)$ we can write

$$\chi_{\lambda,A} u = \int_{\epsilon}^{2\epsilon} \int_{M} e(x, y, t, A) e^{-itA\lambda} \hat{\chi}(t) u(y) dt dy$$

Spectral Window Width One

The operator $e^{it\sqrt{-\Delta}}$ is the fundamental solution to

$$\begin{cases} (i\partial_t + \sqrt{-\Delta})U(t) = 0 \\ U(0) = \delta_y \end{cases}$$

We can build a parametrix for this propagator and write its kernel as

$$e(x, y, t) = \int_0^\infty e^{i\theta(d(x,y)-t)} a(x, y, t, \theta) d\theta$$

where $a(x, y, t, \theta)$ has principal symbol

$$\theta^{\frac{n-1}{2}} a_0(x, y, t)$$

Expression for $\chi_{\lambda,1}$

Substituting into the expression for χ_λ

$$\chi_{\lambda,1} u = \int_\epsilon^{2\epsilon} \int_M \int_0^\infty e^{i\theta(d(x,y)-t)} e^{-it\lambda} \theta^{\frac{n-1}{2}} \tilde{a}(x,y,t,\theta) u(y) d\theta dy dt$$

Change of variables $\theta \to \lambda\theta$

$$\chi_{\lambda,1} u = \lambda^{\frac{n+1}{2}} \int_\epsilon^{2\epsilon} \int_M \int_0^\infty e^{i\lambda\theta(d(x,y)-t)} e^{-it\lambda} \theta^{\frac{n-1}{2}} \tilde{a}(x,y,t,\theta) u(y) d\theta dy dt$$

Now use stationary phase in (t,θ). Nondegenerate critical points when

$$d(x,y) = t \quad \theta = 1$$

$$\chi_{\lambda,1} = \lambda^{\frac{n-1}{2}} \int_M e^{i\lambda d(x,y)} a(x,y) u(y) dy$$

where $a(x,y)$ is supported away from the diagonal.

Estimates for $\chi_{\lambda,1}$

In 1988 Sogge obtained a full set of $L^2 \to L^p$ estimates for $\chi_{\lambda,1}$. Technique depends on $TT^\star$ method, need to estimate

$$\lambda^{n-1} \int_M e^{i\lambda(d(x,z)-d(z,y))} a(x,z)\bar{a}(z,y)dz$$

- Bound depends on $|x-y|$
- Interpolate with L^2 estimates and apply Hardy-Littlewood-Sobolev

Decaying Spectral Window Width

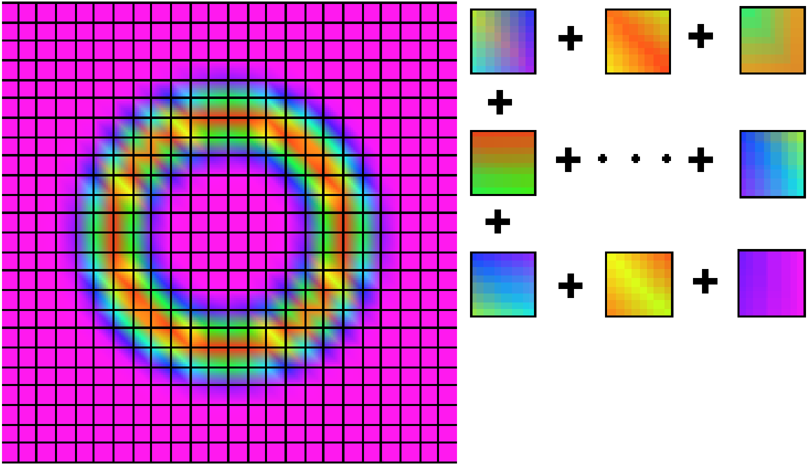
Assume the window width $w = 1/A \to 0$ as $\lambda \to \infty$. We need to evaluate

$$\int_{t<A} e^{it\sqrt{-\Delta}} e^{it\lambda} dt$$

Cannot achieve this on any manifold but for manifolds without conjugate point we can use the universal cover. If M has no conjugate points its universal cover $\widetilde{M}$ is a manifold with infinite injectivity radius. Therefore we can find a solution for

$$\begin{cases} (i\partial_t + \sqrt{-\Delta}_{\widetilde{M}})U(t) = 0 \\ U(0) = \delta_y \end{cases}$$

for all time on $\widetilde{M}$



Expression for Propagator Kernel

$e^{it\sqrt{-\Delta}}$ has kernel

$$e(x, y, t) = \sum_{\gamma \in \Gamma} \tilde{e}(x, \gamma y, t)$$

where Γ is the group of automorphisms of the covering $\pi : \widetilde{M} \to M$ and the fundamental solution of

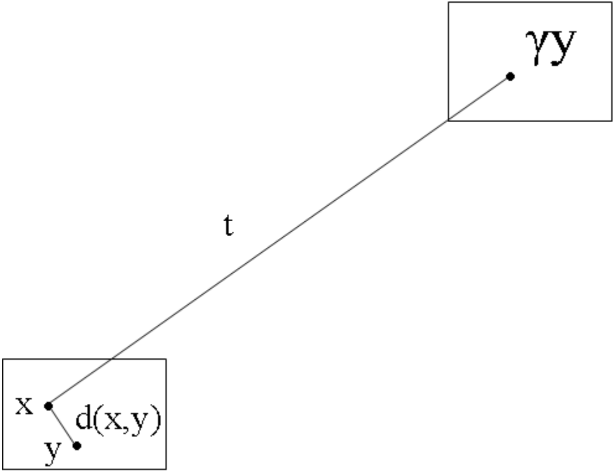
$$\begin{cases} (i\partial_t + \sqrt{-\Delta}_{\widetilde{M}})U(t) = 0 \\ U(0) = \delta_y \end{cases}$$

is given by

$$U(t)u = \int_{\widetilde{M}} \tilde{e}(x, y, t)u(y)dy$$

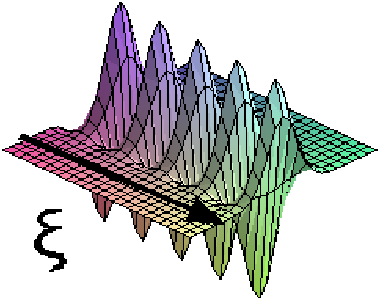
Technical Difficulties

No longer have strong relationships between distance and time.
Cannot use HLS as before.



Directionally Localised Examples

Quasimode localised in direction ξ



- Well defined direction of oscillation for short time.
- Can obtain “good” L^p bounds.
- Consider general quasimode as a sum of directionally localised ones.

Directional Localisation

Return to window width one for simplicity Let

- $\{\xi_j\}$ be a set of $\lambda^{-\frac{1}{2}}$ separated directions in S^{n-1}.
- $\zeta(\eta)$ a smooth, cut off function of scale supported when $|\eta| \leq 2\lambda^{-\frac{1}{2}}$.
- x_i a set of λ^{-1} separated points in M.
- $\beta(x)$ a cut off function supported when $|x| \leq 2\lambda^{-1}$.

$$\chi_{\lambda,1}(\xi_j, x_i) = \lambda^{\frac{n-1}{2}} \beta(x - x_i) \int_M e^{i\lambda d(x,y)} \zeta\left(\frac{x-y}{|x-y|} - \xi_j\right) a(x,y) u(y) dy$$

Interaction of Directional Oscillation

Let $K_{i,j,l,m}(x,z)$ be the kernel of $\chi_{\lambda,1}(\xi_j, x_i)(\chi_{\lambda,1}(\xi_m, x_l))^\star$.

Then for x_i far enough from x_l, $|K_{i,j,l,m}(x,z)|$ decays if

- $\frac{x_i - x_l}{|x_i - x_l|} \neq \xi_j$
- $\xi_j \neq \xi_m$

Otherwise non-stationary phase

$$d(x,z) - d(z,y)$$

Take geometric averages localised at different points and many directions.

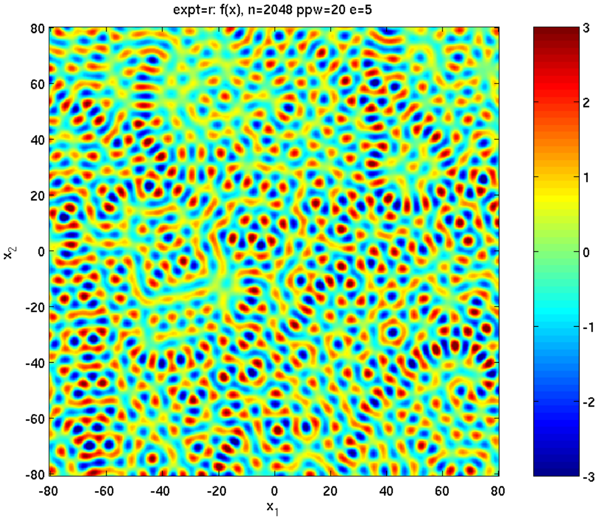
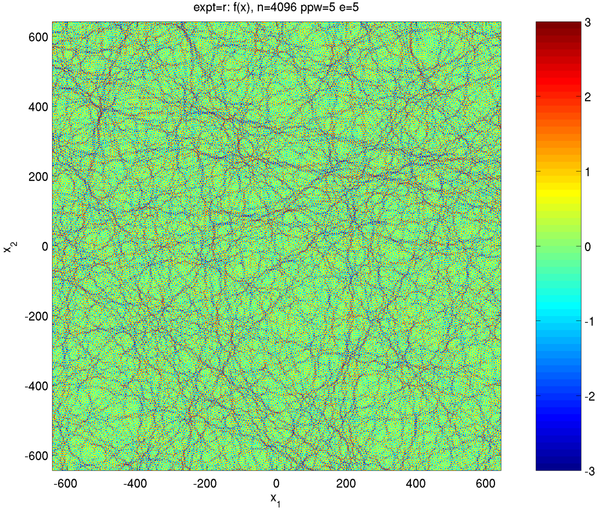


Image due to Alex Barnett



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Application to Clusters with Decaying Window Width

- Can still define directionally localised projectors and these give good $L^2 \to L^p$ estimates.
- Can get cancellation as long as frequency localisation is not too large
- Means we can run up to Ehrenfest time