# Eigenfunctions and nodal sets (real and complex)

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# Nodal sets of eigenfunctions

Let $(M, g)$ be a compact Riemannian manifold and let

$$\Delta_g = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( g^{ij} \sqrt{g} \frac{\partial}{\partial x_j} \right).$$

be its Laplace operator.
Let $\{\varphi_j\}$ be an orthonormal basis of eigenfunctions

$$\Delta \varphi_j = \lambda_j^2 \varphi_j, \quad \langle \varphi_j, \varphi_k \rangle = \delta_{jk}$$

If $\partial M \neq \emptyset$ we impose Dirichlet or Neumann boundary conditions.
The NODAL SET of $\varphi_j$ is its zero set:

$$Z_{\varphi_j} = \{x : \varphi_j(x) = 0\}.$$

A NODAL DOMAIN is a connected component of $M \backslash Z_{\varphi_j}$

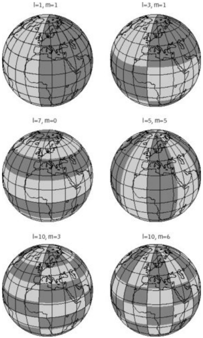
# Some Intuition about nodal sets

- Algebraic geometry: Eigenfunctions of eigenvalue $\lambda^2$ are analogues on $(M, g)$ of polynomials of degree $\lambda$. Their nodal sets are analogues of (real) algebraic varieties of this degree. The $\lambda_j \to \infty$ is the high degree limit or high complexity limit. This analogy is best if $(M, g)$ is real analytic.
- Quantum mechanics: $|\varphi_j(x)|^2 dV_g(x)$ is the probability density of a quantum particle of energy $\lambda_j^2$ being at $x$. Nodal sets are the least likely places for a quantum particle in the energy state $\lambda_j^2$ to be. The $\lambda_j \to \infty$ limit is the high energy or semi-classical limit.

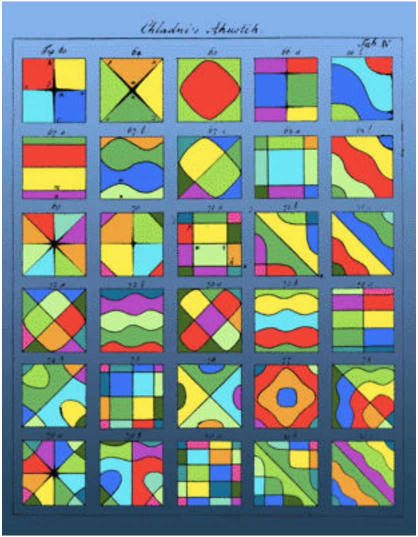
# Problems

- How many nodal domains? (Courant: the nth eigenfunction has $\leq n$ nodal domains. No lower bound in general; Lewy: can be just two). How many connected components of $Z_{\varphi_j}$?
- How 'long' are nodal sets, i.e. the total length (or hypersurface volume in higher dimensions?)
- How are nodal sets distributed on the manifold?
- HOW DO ANSWERS DEPEND ON BEHAVIOR OF GEODESIC FLOW?

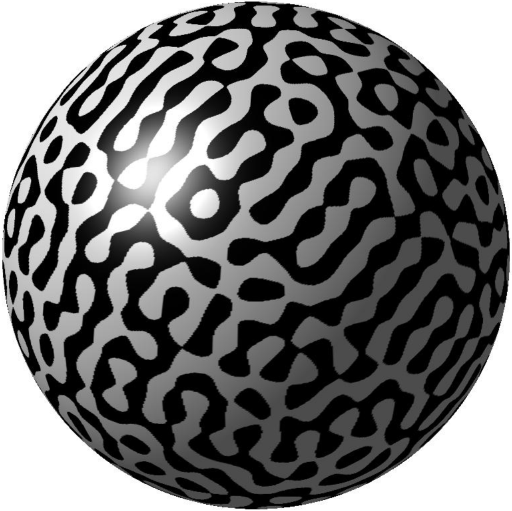
# Nodal domains for $\Re Y_m^\ell$ spherical harmonics: geodesic flow integrable: Eigenfunctions coming from separation of variables



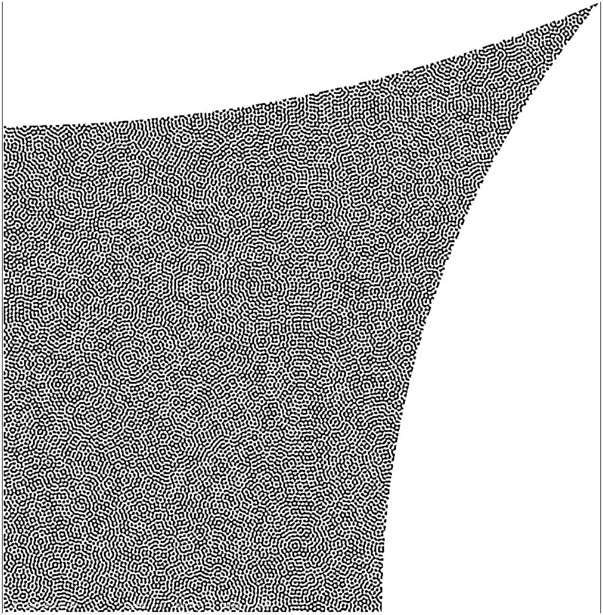
# Chladni diagrams: Integrable case



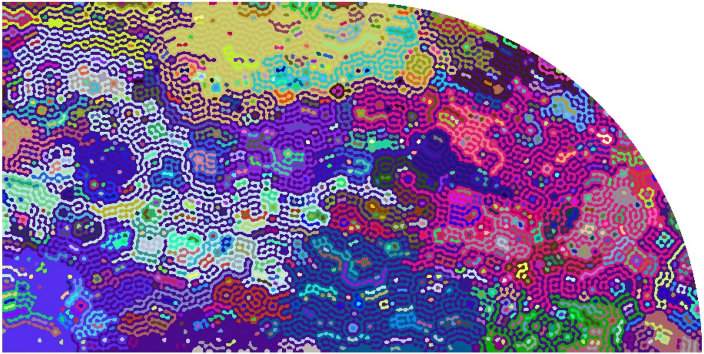
High energy nodal set: E. J. Heller, random spherical harmonic: dimension of space of spherical harmonics of degree $N$ has dim $2N+1$



# High energy nodal set: Chaotic billiard flow



High energy nodal set: Alex Barnett// Each nodal domain is colored a random color; most are small but some are super-big (macroscopic)



# Volumes of nodal hypersurfaces: real analytic case

Even the hypersurface volume is hard to study rigorously. There only exist sharp bounds in the analytic case:

**Theorem**

*(Donnelly-Fefferman, 1988) Suppose that $(M, g)$ is real analytic and $\Delta\varphi_\lambda = \lambda^2\varphi_\lambda$. Then*

$$c_1\lambda \leq \mathcal{H}^{n-1}(Z_{\varphi_\lambda}) \leq C_2\lambda.$$

# Distribution of nodal hypersurfaces

How do nodal hypersurfaces wind around on $M$.?
We put the natural Riemannian hyper-surface measure $d\mathcal{H}^{n-1}$ to consider the nodal set as a *current of integration* $Z_{\varphi_j}$]: for $f \in C(M)$ we put

$$\langle [Z_{\varphi_j}], f \rangle = \int_{Z_{\varphi_j}} f(x) d\mathcal{H}^{n-1}.$$

**Problems**:

- How does $\langle [Z_{\varphi_j}], f \rangle$ behave as $\lambda_j \to \infty$.
- If $U \subset M$ is a nice open set, find the total hypersurface volume $\mathcal{H}^{n-1}(Z_{\varphi_j} \cap U)$ as $\lambda_j \to \infty$.
- How does it reflect dynamics of the geodesic flow?

# Physics conjecture on real nodal hypersurface: ergodic case

CONJECTURE

*Let $(M, g)$ be a real analytic Riemannian manifold with ergodic geodesic flow, and let $\{\varphi_j\}$ be the density one sequence of ergodic eigenfunctions. Then,*

$$\frac{1}{\lambda_j}\langle [Z_{\varphi_j}], f\rangle \sim \frac{1}{Vol(M,g)}\int_M f dVol_g .$$

Evidence: it follows from the "random wave model", i.e. the conjecture that eigenfunctions in the ergodic case resemble Gaussian random waves of fixed frequency.

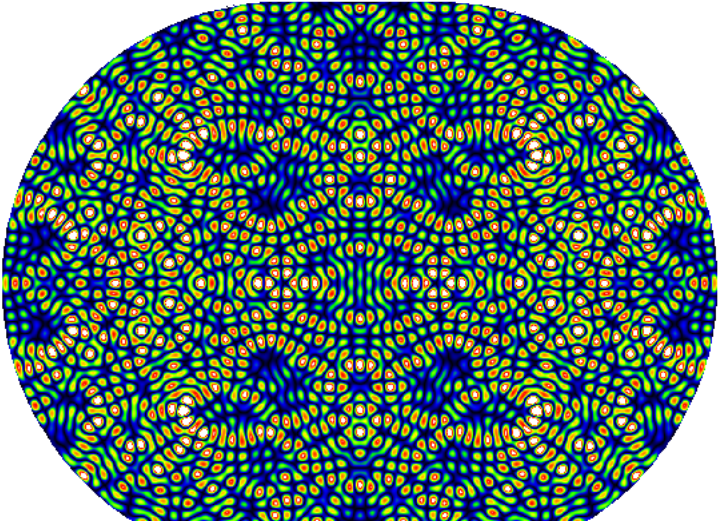
# Quantum ergodicity

- Classical ergodicity: $G^t$ preserves the unit cosphere bundle $S^*_g M$. Ergodic = almost all orbits are uniformly dense.
- On the quantum level, ergodicity of $G^t$ implies that eigenfunctions become uniformly distributed in phase space (Shnirelman; Z, Colin de Verdière, Zworski-Z) . This is a key ingredient in structure of nodal sets. Namely,

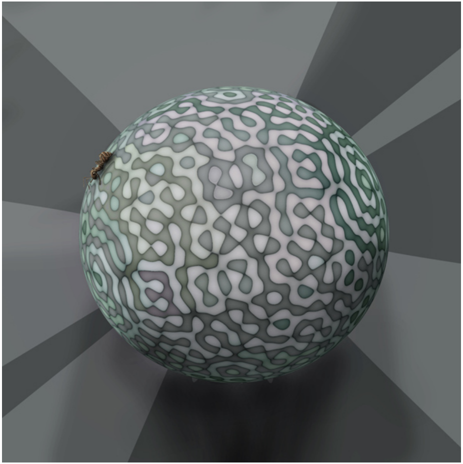
$$\int_E \varphi_j^2 dV_g \to \frac{Vol(E)}{Vol(M)}, \quad \forall E \subset M : Vol(\partial E) = 0.$$

- Equidistribution actually holds in phase space $S^* M$.
- Random wave model (Berry conjecture): when $G^t$ is chaotic, eigenfunctions of $\Delta_g$ behave like random waves.

# Intensity plot of a chaotic eigenfunction in the Bunimovich stadium



# Nodal domains for a random spherical harmonics



# Equidistribution in the complex domain

We want to understand equidistribution of nodal sets. Clearly not feasible for general $C^\infty$ metrics. So we study:

- Equi-distribution theory of "complexified nodal sets" for real analytic $(M, g)$– i.e. complex zeros of analytic continuations of eigenfunctions into the complexification of $M$.
- Intersections of nodal lines and geodesics on surfaces (in the complex domain); intersection with the boundary when $\partial M \neq \emptyset$;
- The equi-distribution depends upon DYNAMICS OF GEODESIC FLOW

# Real versus complex nodal hypersurfaces

The only rigorous results on distribution of nodal sets (and level sets) of eigenfunctions concern the complex zeros of analytic continuations:

$$Z_{\varphi_j^{\mathbb{C}}} = \{\zeta \in M_{\mathbb{C}} : \varphi_j^{\mathbb{C}}(\zeta) = 0\},$$

where $\varphi_j^{\mathbb{C}}$ is the analytic continuation of $\varphi_j$ to the complexification $M_{\mathbb{C}}$ of $M$.

# Equi-distribution of complex nodal sets in the ergodic case

THEOREM

*(Z, 2007) Assume $(M, g)$ is real analytic and that the geodesic flow of $(M, g)$ is ergodic. Let $\varphi_{\lambda_j}^{\mathbb{C}}$ be the analytic continuation to phase space of the eigenfunction $\varphi_{\lambda_j}$, and let $Z_{\varphi_{\lambda_j}^{\mathbb{C}}}$ be its complex zero set in phase space $B^*M$. Then for all but a sparse subsequence of $\lambda_j$,*

$$\frac{1}{\lambda_j} \int_{Z_{\varphi_{\lambda_j}^{\mathbb{C}}}} f \omega_g^{n-1} \to \frac{i}{\pi} \int_{M_\tau} f \bar{\partial} \partial \sqrt{\rho} \wedge \omega_g^{n-1}$$

As usual in quantum ergodicity, we may have to delete a sparse subsequence of exceptional eigenvalues.

# Grauert tube radius $\sqrt{\rho}$

Given real analytic $(M, g)$, complexify $M \to M_{\mathbb{C}}$.

- Complexify $r^2(x, y) \to r^2(\zeta, \bar{\zeta})$. Grauert tube function $=$

$$\sqrt{\rho} := \sqrt{-r^2(\zeta, \bar{\zeta})}.$$

Measures how deep into the complexification $\zeta \in M_{\mathbb{C}}$ is.

# Examples: Torus

- Complexification of $\mathbb{R}^n/\mathbb{Z}^n$ is $\mathbb{C}^n/\mathbb{Z}^n$.
- Grauert tube function: $r(x,y) = |x-y|$ and $r_{\mathbb{C}}(z,w) = \sqrt{(z-w)^2}$. Then

$$\sqrt{\rho}(z) = \sqrt{-(z-\bar{z})^2} = 2|\Im z| = 2|\xi|.$$

- The complexified exponential map is:

$$\exp_{\mathbb{C}x}(i\xi) = x + i\xi.$$

# Kähler metric on Grauert tube

- $\rho(\zeta) = -r_{\mathbb{C}}^2(\zeta, \bar{\zeta})$ is the Kähler potential of the Kähler metric $\omega_g = i\partial\bar{\partial}\rho$.
- $\sqrt{\rho}$ is singular at $\rho = 0$ (i.e. on $M_{\mathbb{R}}$):

$$(i\partial\bar{\partial}\sqrt{\rho})^n = \delta_{M_{\mathbb{R}}}, \quad i.e. \int_{M_\epsilon} f(i\partial\bar{\partial}\sqrt{\rho})^n = \int_M f dV_g.$$

# Limit distribution of zeros is singular along zero section

- The Kaehler structure on $M_{\mathbb{C}}$ is $\overline{\partial}\partial\rho$. But the limit current is $\overline{\partial}\partial\sqrt{\rho}$. The latter is singular along the real domain.
- The reason for the singularity is that the zero set is invariant under the involution $\zeta \to \bar{\zeta}$, since the eigenfunction is real valued on $M$. The fixed point set is $M$ and is also where zeros concentrate.

# Example: the unit circle $S^1$

- The (real) eigenfunctions are $\cos k\theta, \sin k\theta$ on a circle.
- The complexification is the cylinder $S^1_{\mathbb{C}} = S^1 \times \mathbb{R}$.
- The complexified configuration space is similar to the phase space $T^*S^1$. This is always true.
- The holomorphically extended eigenfunctions are $\cos kz, \sin kz$.

# Simplest case: $S^1$

The zeros of $\sin 2\pi kz$ in the cylinder $\mathbb{C}/\mathbb{Z}$ all lie on the real axis at the points $z = \frac{n}{2k}$. Thus, there are $2k$ real zeros. The limit zero distribution is:

$$\begin{aligned} \lim_{k\to\infty} \frac{i}{2\pi k} \partial\bar{\partial} \log |\sin 2\pi k|^2 &= \lim_{k\to\infty} \frac{1}{k} \sum_{n=1}^{2k} \delta_{\frac{n}{2k}} \\ &= \frac{1}{\pi} \delta_0(\xi) dx \wedge d\xi. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{i}{\pi} \partial\bar{\partial} |\xi| &= \frac{i}{\pi} \frac{d^2}{4d\xi^2} |\xi| \ \ \frac{2}{i} dx \wedge d\xi \\ &= \frac{i}{\pi} \ \frac{1}{2} \ \delta_0(\xi) \ \ \frac{2}{i} dx \wedge d\xi. \end{aligned}$$

# Ergodicity of eigenfunctions in the complex domain

Ergodic eigenfunctions in the complex domain:

- Have extremal growth– $\frac{1}{\lambda} \log |\varphi_\lambda^{\mathbb{C}}|^2$ is like Siciak's maximal plurisubharmonic function on $\mathbb{C}^n$;
- Have maximal growth rate of zeros

# Work in Progress: Intersections of nodal lines and geodesics

To get closer to real zeros, we "magnify" the singularity in the real domain by intersecting nodal lines and geodesics on surfaces $\dim M = 2$.

Let $\gamma \subset M^2$ be geodesic arc on a real analytic Riemannian surface. We identify it with a a real analytic arc-length parameterization $\gamma : \mathbb{R} \to M$. For small $\epsilon$, $\exists$ analytic continuation

$$\gamma_{\mathbb{C}} : S_\tau := \{t + i\tau \in \mathbb{C} : |\tau| \leq \epsilon\} \to M_\tau.$$

Consider the restricted (pulled back) eigenfunctions

$$\gamma_{\mathbb{C}}^* \varphi_{\lambda_j}^{\mathbb{C}} \text{ on } S_\tau.$$

# Intersections of nodal lines and geodesics

Let

$$\mathcal{N}_{\lambda_j}^{\gamma} := \{(t + i\tau : \gamma_H^* \varphi_{\lambda_j}^{\mathbb{C}}(t + i\tau) = 0\} \tag{1}$$

be the complex zero set of this holomorphic function of one complex variable. Its zeros are the intersection points.
Then as a current of integration,

$$[\mathcal{N}_{\lambda_j}^{\gamma}] = i\partial\bar{\partial}_{t+i\tau} \log \left|\gamma^* \varphi_{\lambda_j}^{\mathbb{C}}(t + i\tau)\right|^2. \tag{2}$$

# Equidistribution of intersections

THEOREM/CONJECTURE

*Let $(M, g)$ be real analytic with ergodic geodesic flow. Then there exists a subsequence of eigenvalues $\lambda_{j_k}$ of density one such that*

$$\frac{i}{\pi\lambda_{j_k}}\partial\bar{\partial}_{t+i\tau}\log\left|\gamma^*\varphi^{\mathbb{C}}_{\lambda_{j_k}}(t+i\tau)\right|^2 \to \delta_{\tau=0}ds.$$

*The convergence is weak* convergence on $C_c(S_\epsilon)$.*

Thus, intersections of (complexified) nodal sets and geodesics concentrate in the real domain– and are distributed by arc-length measure on the real geodesic.
(Proof seems complete for periodic geodesics on surfaces when the geodesic satisfies a generic asymmetry condition; also for "random" geodesics in all dimensions)

# Ideas of proofs

We now explain:

- Why it helps to work in the complex domain;
- How we relate nodal sets and geodesic flow;
- How to study intersections of nodal lines and geodesics in the ergodic case.

# Why it helps to work in $M_{\mathbb{C}}$

In the complex domain we have:

1. Poincaré-Lelong formula: $Z_{\varphi_j} = \frac{i}{2\pi} \partial\bar{\partial} \log |\varphi_j^{\mathbb{C}}|^2$.
2. Compactness in $L^1$ of the PSH functions
$$\{\frac{i}{\lambda_j} \partial\bar{\partial} \log |\varphi_j^{\mathbb{C}}|^2\}.$$
3. $L^2$ norm of $|\varphi_j^{\mathbb{C}}(\zeta)|$ on Grauert tube $M_\tau$ is $e^{\lambda_j \tau}$. Easy to see from Poisson-wave kernel.
4. Control over weak* limits of $|\varphi_j^{\mathbb{C}}|^2\}$ when geodesic flow is ergodic (quantum ergodicity).

# Step I: Ergodicity of complexified eigenfunctions

The first step is to prove quantum ergodicity of the complexified eigenfunctions:

**Theorem**

*Assume the geodesic flow of $(M, g)$ is ergodic. Then*

$$\frac{|\varphi_\lambda^\epsilon(z)|^2}{||\varphi_\lambda^\epsilon||^2_{L^2(\partial M_\epsilon)}} \to 1, \quad \textit{weakly in } \; L^1(M_\epsilon),$$

*along a density one subsquence of $\lambda_j$.*

This is the analogue of what can be proved for the real eigenfunctions (Shnirelman, SZ, Colin de Verdiere).

# Nodal sets (related: Shiffman-Z, Nonnenmacher)

**Lemma**

*We have:*

$$\frac{1}{\lambda_j} \log |\varphi_\lambda^\epsilon(z)|^2 \to \sqrt{\rho}, \quad in\ L^1(M_\epsilon).$$

Combine with Poincare- Lelong:

$$[\tilde{Z}_j] = \partial\bar{\partial} \log |\tilde{\varphi}_j^{\mathbb{C}}|^2$$

to get

$$\frac{1}{\lambda_j}[\tilde{Z}_j] \to i\partial\bar{\partial}\sqrt{\rho}.$$

The exponential growth of $|\varphi_j^{\mathbb{C}}(\zeta)|$ comes directly from the eigenvalue equation

$$U(i\tau)_{\mathbb{C}}\varphi_j = e^{-\lambda_j\sqrt{\rho}(\zeta)}\varphi_j^{\mathbb{C}}.$$

# Equi-distribution of intersections

So far:

$$\frac{1}{\lambda_j}\int_{Z_{\varphi^{\mathbb{C}}_{\lambda_j}}} f\omega_g^{n-1} \to \frac{i}{\pi}\int_{M_\tau} f\bar{\partial}\partial\sqrt{\rho}\wedge\omega_g^{n-1}$$

Intersections with typical geodesic:

$$\gamma_{\mathbb{C}} : S_\tau := \{t + i\tau \in \mathbb{C} : |\tau| \leq \epsilon\} \to M_\tau.$$

Then:

$$\frac{i}{\pi\lambda_{j_k}}\partial\bar{\partial}_{t+i\tau}\log\left|\gamma^*\varphi^{\mathbb{C}}_{\lambda_{j_k}}(t+i\tau)\right|^2 \to \delta_{\tau=0}ds.$$

The convergence is weak* convergence on $C_c(S_\epsilon)$.

# New ingredient: quantum ergodic restriction theorem

In the real domain:

THEOREM

*(J. Toth and S. Z 2010-2011; Dyatlov-Zworski, 2012) If $G^t$ is ergodic and a geodesic $H$ is "asymmetric" then the restrictions of $\{\varphi_j\}$ to $H$ are quantum ergodic on $H$ in the sense that*

$$\lim_{\lambda_j \to \infty; j \in S} \langle Op_{\lambda_j}(a_0) \varphi_{\lambda_j}|_H, \varphi_{\lambda_j}|_H \rangle_{L^2(H)}$$

$$= c_n \int_{B^* H} a(s, \tau)\, \rho^H_{\partial\Omega}(s, \tau)\, ds d\tau$$

*for a certain measure $\rho^H_{\partial\Omega}(s, \tau)\, ds d\tau$.*

# Intersections of complex zeros and geodesics

To analyze intersections of nodal lines and geodesics, we need a quantum ergodic restriction in the complex domain. It's completely different ! Analytic continuation decouples modes:

Example: Round $S^2$. Let $Y^N_m$ be the usual joint eigenfunctions of $\Delta$ and rotation around the z-axis, with $Y^N_m$ transforming by $e^{im\theta}$ under rotation. Any eigenfunction is $\varphi_N = \sum_{m=-N}^N a_{Nm} Y^N_m$. Restrict to equator: $\varphi_N|_{\varphi=0} = \sum_{m=-N}^N a_{Nm} P^N_m(1) e^{im\theta}$. Analytically continue to complex equator:

$$\varphi_N^{\mathbb{C}}|_{\gamma\mathbb{C}} = \sum_{m=-N}^{N} a_{mN} P^N_m(1) e^{im(\theta+i\eta)}.$$

Term with top $m$ dominates! Ergodicity (or random-ness): the $a_{NN} \neq 0$, $a_{N,-N} \neq 0$. Equipartition of energy.

# Complexified Poisson kernel

To connect eigenfunctions and geodesic flow, we use the Poisson kernel

$$U(i\tau, x, y) = \sum_{j=0}^{\infty} e^{-\tau\lambda_j} \varphi_j(x)\varphi_j(y).$$

It admits a holomorphic extension to $M_{\mathbb{C}}$ in $x \to \zeta$ when $\sqrt{\rho}(\zeta) < \tau$.

THEOREM

*(Hadamard, Mizohata; Boutet de Monvel; SZ 2011, M. Stenzel 2012) $U(i\epsilon, z, y) : L^2(M) \to H^2(\partial M_\epsilon)$ is a complex Fourier integral operator of order $-\frac{m-1}{4}$ quantizing the complexified exponential map $\exp_y(i\epsilon)\eta/|\eta|)\}$.*

# Euclidean case

On $\mathbb{R}^n$:

$$U(t,x,y) = \int_{\mathbb{R}^n} e^{it|\xi|} e^{i\langle \xi, x-y\rangle} d\xi.$$

Its analytic continuation to $t + i\tau, \zeta = x + ip$ is given by

$$U(t+i\tau, x+ip, y) = \int_{\mathbb{R}^n} e^{i(t+i\tau)|\xi|} e^{i\langle \xi, x+ip-y\rangle} d\xi,$$

which converges absolutely for $|p| < \tau$.
Key point:

$$U(i\tau)\varphi_{\lambda_j} = e^{-\tau\lambda_j}\varphi^{\mathbb{C}}_{\lambda_j}.$$

But $U(i\tau)\varphi_{\lambda_j}$ only changes $L^2$ norms by powers of $\lambda_j$. So exponential growth $= e^{\tau\lambda_j}$.