Eigenfunctions and nodal sets (real and complex)

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Nodal sets of eigenfunctions

Let $(M, g)$ be a compact Riemannian manifold and let

$$
\Delta_g = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( g^{ij} \sqrt{g} \frac{\partial}{\partial x_j} \right).
$$

be its Laplace operator.

Let $\{\varphi_j\}$ be an orthonormal basis of eigenfunctions

$$
\Delta \varphi_j = \lambda_j^2 \varphi_j, \quad \langle \varphi_j, \varphi_k \rangle = \delta_{jk}
$$

If $\partial M \neq \emptyset$ we impose Dirichlet or Neumann boundary conditions. The NODAL SET of $\varphi_j$ is its zero set:

$$
Z_{\varphi_j} = \{ x : \varphi_j(x) = 0 \}.
$$

A NODAL DOMAIN is a connected component of $M\setminus Z_{\varphi_j}$. 
Some Intuition about nodal sets

- Algebraic geometry: Eigenfunctions of eigenvalue $\lambda^2$ are analogues on $(M, g)$ of polynomials of degree $\lambda$. Their nodal sets are analogues of (real) algebraic varieties of this degree. The $\lambda_j \to \infty$ is the high degree limit or high complexity limit. This analogy is best if $(M, g)$ is real analytic.

- Quantum mechanics: $|\varphi_j(x)|^2 dV_g(x)$ is the probability density of a quantum particle of energy $\lambda_j^2$ being at $x$. Nodal sets are the least likely places for a quantum particle in the energy state $\lambda_j^2$ to be. The $\lambda_j \to \infty$ limit is the high energy or semi-classical limit.
Problems

- How many nodal domains? (Courant: the nth eigenfunction has \( \leq n \) nodal domains. No lower bound in general; Lewy: can be just two). How many connected components of \( Z_{\varphi_j} \)?

- How ‘long’ are nodal sets, i.e. the total length (or hypersurface volume in higher dimensions?)

- How are nodal sets distributed on the manifold?

- **HOW DO ANSWERS DEPEND ON BEHAVIOR OF GEODESIC FLOW?**
Nodal domains for $\Re Y^\ell_m$ spherical harmonics: geodesic flow integrable: Eigenfunctions coming from separation of variables
Chladni diagrams: Integrable case
High energy nodal set: E. J. Heller, random spherical harmonic: dimension of space of spherical harmonics of degree $N$ has dim $2N + 1$
High energy nodal set: Chaotic billiard flow
High energy nodal set: Alex Barnett / Each nodal domain is colored a random color; most are small but some are super-big (macroscopic)
Volumes of nodal hypersurfaces: real analytic case

Even the hypersurface volume is hard to study rigorously. There only exist sharp bounds in the analytic case:

**Theorem**
*(Donnelly-Fefferman, 1988)* Suppose that \((M, g)\) is real analytic and \(\Delta \varphi_\lambda = \lambda^2 \varphi_\lambda\). Then

\[
c_1 \lambda \leq H^{n-1}(Z_{\varphi_\lambda}) \leq C_2 \lambda.
\]
Distribution of nodal hypersurfaces

How do nodal hypersurfaces wind around on $M$?
We put the natural Riemannian hyper-surface measure $d\mathcal{H}^{n-1}$ to consider the nodal set as a *current of integration* $Z_{\varphi_j}$: for $f \in C(M)$ we put

$$\langle [Z_{\varphi_j}], f \rangle = \int_{Z_{\varphi_j}} f(x) d\mathcal{H}^{n-1}.$$ 

Problems:

- How does $\langle [Z_{\varphi_j}], f \rangle$ behave as $\lambda_j \to \infty$.
- If $U \subset M$ is a nice open set, find the total hypersurface volume $\mathcal{H}^{n-1}(Z_{\varphi_j} \cap U)$ as $\lambda_j \to \infty$.
- How does it reflect dynamics of the geodesic flow?
Conjecture

Let $(M, g)$ be a real analytic Riemannian manifold with ergodic geodesic flow, and let $\{\varphi_j\}$ be the density one sequence of ergodic eigenfunctions. Then,

$$\frac{1}{\lambda_j} \langle [Z_{\varphi_j}], f \rangle \sim \frac{1}{\text{Vol}(M, g)} \int_M f \text{dVol}_g.$$ 

Evidence: it follows from the “random wave model”, i.e. the conjecture that eigenfunctions in the ergodic case resemble Gaussian random waves of fixed frequency.
Quantum ergodicity

- Classical ergodicity: $G^t$ preserves the unit cosphere bundle $S^*_{g} M$. Ergodic $=$ almost all orbits are uniformly dense.
- On the quantum level, ergodicity of $G^t$ implies that eigenfunctions become uniformly distributed in phase space (Shnirelman; Z, Colin de Verdière, Zworski-Z) . This is a key ingredient in structure of nodal sets. Namely,

$$\int_{E} \varphi_j^2 dV_g \to \frac{Vol(E)}{Vol(M)}, \quad \forall E \subset M : Vol(\partial E) = 0.$$

- Equidistribution actually holds in phase space $S^* M$.
- Random wave model (Berry conjecture): when $G^t$ is chaotic, eigenfunctions of $\Delta_g$ behave like random waves.
Intensity plot of a chaotic eigenfunction in the Bunimovich stadium
Nodal domains for a random spherical harmonics
Equidistribution in the complex domain

We want to understand equidistribution of nodal sets. Clearly not feasible for general $C^\infty$ metrics. So we study:

- Equi-distribution theory of “complexified nodal sets” for real analytic $(M, g)$—i.e. complex zeros of analytic continuations of eigenfunctions into the complexification of $M$.
- Intersections of nodal lines and geodesics on surfaces (in the complex domain); intersection with the boundary when $\partial M \neq \emptyset$;
- The equi-distribution depends upon DYNAMICS OF GEODESIC FLOW
The only rigorous results on distribution of nodal sets (and level sets) of eigenfunctions concern the complex zeros of analytic continuations:

$$Z_{\varphi_j^C} = \{ \zeta \in M_C : \varphi_j^C(\zeta) = 0 \},$$

where $\varphi_j^C$ is the analytic continuation of $\varphi_j$ to the complexification $M_C$ of $M$. 
Theorem

(Z, 2007) Assume \((M, g)\) is real analytic and that the geodesic flow of \((M, g)\) is ergodic. Let \(\varphi^{C}_{\lambda j}\) be the analytic continuation to phase space of the eigenfunction \(\varphi_{\lambda j}\), and let \(Z_{\varphi^{C}_{\lambda j}}\) be its complex zero set in phase space \(B^*M\). Then for all but a sparse subsequence of \(\lambda_j\),

\[
\frac{1}{\lambda_j} \int_{Z_{\varphi^{C}_{\lambda j}}} f \omega_{g}^{n-1} \to \frac{i}{\pi} \int_{M_T} f \overline{\partial} \partial \sqrt{\rho} \wedge \omega_{g}^{n-1}
\]

As usual in quantum ergodicity, we may have to delete a sparse subsequence of exceptional eigenvalues.
Grauert tube radius $\sqrt{\rho}$

Given real analytic $(M, g)$, complexify $M \to M_\mathbb{C}$.

- Complexify $r^2(x, y) \to r^2(\zeta, \bar{\zeta})$. Grauert tube function $= \sqrt{\rho} := \sqrt{-r^2(\zeta, \bar{\zeta})}$.

Measures how deep into the complexification $\zeta \in M_\mathbb{C}$ is.
Examples: Torus

- Complexification of $\mathbb{R}^n/\mathbb{Z}^n$ is $\mathbb{C}^n/\mathbb{Z}^n$.
- Grauert tube function: $r(x, y) = |x - y|$ and $r_C(z, w) = \sqrt{(z - w)^2}$. Then

  $$\sqrt{\rho(z)} = \sqrt{-(z - \bar{z})^2} = 2|\Im z| = 2|\xi|.$$ 

- The complexified exponential map is:

  $$\exp_{\mathbb{C}x}(i\xi) = x + i\xi.$$
Kähler metric on Grauert tube

- $\rho(\zeta) = -r^2_C(\zeta, \bar{\zeta})$ is the Kähler potential of the Kähler metric $\omega_g = i\partial\bar{\partial}\rho$.
- $\sqrt{\rho}$ is singular at $\rho = 0$ (i.e. on $M_\mathbb{R}$):

$$\left(i\partial\bar{\partial}\sqrt{\rho}\right)^n = \delta_{M_\mathbb{R}}, \quad \text{i.e.} \quad \int_{M_\epsilon} f \left(i\partial\bar{\partial}\sqrt{\rho}\right)^n = \int_M f dV_g.$$
Limit distribution of zeros is singular along zero section

- The Kaehler structure on $M_\mathbb{C}$ is $\overline{\partial} \partial \rho$. But the limit current is $\overline{\partial} \partial \sqrt{\rho}$. The latter is singular along the real domain.

- The reason for the singularity is that the zero set is invariant under the involution $\zeta \to \overline{\zeta}$, since the eigenfunction is real valued on $M$. The fixed point set is $M$ and is also where zeros concentrate.
Example: the unit circle $S^1$

- The (real) eigenfunctions are $\cos k\theta$, $\sin k\theta$ on a circle.
- The complexification is the cylinder $S^1_C = S^1 \times \mathbb{R}$.
- The complexified configuration space is similar to the phase space $T^*S^1$. This is always true.
- The holomorphically extended eigenfunctions are $\cos kz$, $\sin kz$. 
Simplest case: $S^1$

The zeros of $\sin 2\pi k z$ in the cylinder $\mathbb{C}/\mathbb{Z}$ all lie on the real axis at the points $z = \frac{n}{2k}$. Thus, there are $2k$ real zeros. The limit zero distribution is:

$$
\lim_{k \to \infty} \frac{i}{2\pi k} \partial \bar{\partial} \log |\sin 2\pi k|^2 = \lim_{k \to \infty} \frac{1}{k} \sum_{n=1}^{2k} \delta \frac{n}{2k}
$$

$$
= \frac{1}{\pi} \delta_0(\xi) dx \wedge d\xi.
$$

On the other hand,

$$
\frac{i}{\pi} \partial \bar{\partial} |\xi| = \frac{i}{\pi} \frac{d^2}{4d\xi^2} |\xi| \quad \frac{2}{i} dx \wedge d\xi
$$

$$
= \frac{i}{\pi} \frac{1}{2} \delta_0(\xi) \frac{2}{i} dx \wedge d\xi.
$$
Ergodicity of eigenfunctions in the complex domain

Ergodic eigenfunctions in the complex domain:

- Have extremal growth—$\frac{1}{\lambda} \log |\varphi^C_\lambda|^2$ is like Siciak’s maximal plurisubharmonic function on $\mathbb{C}^n$;
- Have maximal growth rate of zeros
Work in Progress: Intersections of nodal lines and geodesics

To get closer to real zeros, we “magnify” the singularity in the real domain by intersecting nodal lines and geodesics on surfaces $\dim M = 2$.

Let $\gamma \subset M^2$ be geodesic arc on a real analytic Riemannian surface. We identify it with a real analytic arc-length parameterization $\gamma : \mathbb{R} \rightarrow M$. For small $\epsilon$, $\exists$ analytic continuation

$$
\gamma_{C} : S_{\tau} := \{ t + i\tau \in \mathbb{C} : |\tau| \leq \epsilon \} \rightarrow M_{\tau}.
$$

Consider the restricted (pulled back) eigenfunctions

$$
\gamma_{C}^{*} \varphi_{\lambda_{j}}^{C} \text{ on } S_{\tau}.
$$
Intersections of nodal lines and geodesics

Let

$$\mathcal{N}^\gamma_{\lambda_j} := \{(t + i\tau : \gamma^*_H \varphi^C_{\lambda_j}(t + i\tau) = 0\}$$

(1)

be the complex zero set of this holomorphic function of one complex variable. Its zeros are the intersection points. Then as a current of integration,

$$[\mathcal{N}^\gamma_{\lambda_j}] = i\partial\bar{\partial}_{t+i\tau} \log \left| \gamma^* \varphi^C_{\lambda_j}(t + i\tau) \right|^2.$$
Equidistribution of intersections

**Theorem/Conjecture**

Let $(M, g)$ be real analytic with ergodic geodesic flow. Then there exists a subsequence of eigenvalues $\lambda_{jk}$ of density one such that

$$
\frac{i}{\pi \lambda_{jk}} \partial \bar{\partial}_{t+i\tau} \log \left| \gamma^* \varphi^C \lambda_{jk} (t + i\tau) \right|^2 \to \delta_{\tau=0} ds.
$$

The convergence is weak* convergence on $C_c(S_\epsilon)$.

Thus, intersections of (complexified) nodal sets and geodesics concentrate in the real domain— and are distributed by arc-length measure on the real geodesic.

(Proof seems complete for periodic geodesics on surfaces when the geodesic satisfies a generic asymmetry condition; also for “random” geodesics in all dimensions)
Ideas of proofs

We now explain:

- Why it helps to work in the complex domain;
- How we relate nodal sets and geodesic flow;
- How to study intersections of nodal lines and geodesics in the ergodic case.
Why it helps to work in $M_\mathbb{C}$

In the complex domain we have:

1. Poincaré-Lelong formula: $Z_{\phi_j} = \frac{i}{2\pi} \partial \bar{\partial} \log |\phi_j^\mathbb{C}|^2$.

2. Compactness in $L^1$ of the PSH functions

\[
\left\{ \frac{i}{\lambda_j} \partial \bar{\partial} \log |\phi_j^\mathbb{C}|^2 \right\}.
\]

3. $L^2$ norm of $|\phi_j^\mathbb{C}(\zeta)|$ on Grauert tube $M_\tau$ is $e^{\lambda_j \tau}$. Easy to see from Poisson-wave kernel.

4. Control over weak* limits of $|\phi_j^\mathbb{C}|^2$ when geodesic flow is ergodic (quantum ergodicity).
Step I: Ergodicity of complexified eigenfunctions

The first step is to prove quantum ergodicity of the complexified eigenfunctions:

**Theorem**

Assume the geodesic flow of $(M, g)$ is ergodic. Then

$$
\frac{|\varphi_\lambda^\epsilon(z)|^2}{||\varphi_\lambda^\epsilon||^2_{L^2(\partial M_\epsilon)}} \rightarrow 1, \text{ weakly in } L^1(M_\epsilon),
$$

along a density one subsequence of $\lambda_j$.

This is the analogue of what can be proved for the real eigenfunctions (Shnirelman, SZ, Colin de Verdiere).
Nodal sets (related: Shiffman-Z, Nonnenmacher)

**Lemma**

We have:

\[
\frac{1}{\lambda_j} \log |\varphi^\epsilon(z)|^2 \to \sqrt{\rho}, \quad \text{in } L^1(M_\epsilon).
\]

Combine with Poincare- Lelong:

\[
[\tilde{Z}_j] = \partial \bar{\partial} \log |\tilde{\varphi}_j^\C|^2
\]

to get

\[
\frac{1}{\lambda_j} [\tilde{Z}_j] \to i\partial \bar{\partial} \sqrt{\rho}.
\]

The exponential growth of \( |\varphi^\C_j(\zeta)| \) comes directly from the eigenvalue equation

\[
U(i\tau)_\C \varphi_j = e^{-\lambda_j \sqrt{\rho}(\zeta)} \varphi_j^\C.
\]
Equi-distribution of intersections

So far:

\[
\frac{1}{\lambda_j} \int_{Z} f \omega_{g}^{n-1} \rightarrow \frac{i}{\pi} \int_{M_{\tau}} f \bar{\partial} \partial \sqrt{\rho} \wedge \omega_{g}^{n-1}
\]

Intersections with typical geodesic:

\[
\gamma_{c} : S_{\tau} := \{ t + i\tau \in \mathbb{C} : |\tau| \leq \epsilon \} \rightarrow M_{\tau}.
\]

Then:

\[
\frac{i}{\pi \lambda_{j_k}} \partial \bar{\partial}_{t+i\tau} \log \left| \gamma^{*} \phi_{c_{j_k}}^{j_k} (t + i\tau) \right|^2 \rightarrow \delta_{\tau=0} ds.
\]

The convergence is weak* convergence on \( C_c(S_\epsilon) \).
New ingredient: quantum ergodic restriction theorem

In the real domain:

**Theorem**

(J. Toth and S. Z 2010-2011; Dyatlov-Zworski, 2012) If $G^t$ is ergodic and a geodesic $H$ is “asymmetric” then the restrictions of \{\varphi_j\} to $H$ are quantum ergodic on $H$ in the sense that

$$\lim_{{\lambda_j \to \infty}} \langle \text{Op}_{\lambda_j}(a_0)\varphi_{\lambda_j}|_H, \varphi_{\lambda_j}|_H \rangle_{L^2(H)} = c_n \int_{B^*H} a(s, \tau) \rho^H_{\partial \Omega}(s, \tau) \, ds d\tau$$

for a certain measure $\rho^H_{\partial \Omega}(s, \tau) \, ds d\tau$. 
Intersections of complex zeros and geodesics

To analyze intersections of nodal lines and geodesics, we need a quantum ergodic restriction in the complex domain. It’s completely different! Analytic continuation decouples modes:

Example: Round $S^2$. Let $Y^N_m$ be the usual joint eigenfunctions of $\Delta$ and rotation around the z-axis, with $Y^N_m$ transforming by $e^{im\theta}$ under rotation. Any eigenfunction is $\varphi_N = \sum_{m=-N}^{N} a_{Nm} Y^N_m$.

Restrict to equator: $\varphi_N|_{\varphi=0} = \sum_{m=-N}^{N} a_{Nm} P^N_m(1)e^{im\theta}$.

Analytically continue to complex equator:

$$\varphi^C_N|_{\gamma^C} = \sum_{m=-N}^{N} a_{mN} P^N_m(1)e^{im(\theta+i\eta)}.$$  

Term with top $m$ dominates! Ergodicity (or random-ness): the $a_{NN} \neq 0$, $a_N,-N \neq 0$. Equipartition of energy.
Complexified Poisson kernel

To connect eigenfunctions and geodesic flow, we use the Poisson kernel

\[ U(i\tau, x, y) = \sum_{j=0}^{\infty} e^{-\tau \lambda_j} \varphi_j(x) \varphi_j(y). \]

It admits a holomorphic extension to \( M_\mathbb{C} \) in \( x \to \zeta \) when \( \sqrt{\rho}(\zeta) < \tau \).

**Theorem**

(Hadamard, Mizohata; Boutet de Monvel; SZ 2011, M. Stenzel 2012) \( U(i\epsilon, z, y) : L^2(M) \to H^2(\partial M_\epsilon) \) is a complex Fourier integral operator of order \( -\frac{m-1}{4} \) quantizing the complexified exponential map \( \exp_y(i\epsilon\eta/|\eta|) \).
Euclidean case

On $\mathbb{R}^n$:

$$U(t, x, y) = \int_{\mathbb{R}^n} e^{it|\xi|} e^{i\langle \xi, x-y \rangle} d\xi.$$ 

Its analytic continuation to $t + i\tau, \zeta = x + ip$ is given by

$$U(t + i\tau, x + ip, y) = \int_{\mathbb{R}^n} e^{i(t+i\tau)|\xi|} e^{i\langle \xi, x+ip-y \rangle} d\xi,$$

which converges absolutely for $|p| < \tau$.

Key point:

$$U(i\tau)\varphi_{\lambda_j} = e^{-\tau\lambda_j}\varphi_{\lambda_j}^\mathbb{C}.$$

But $U(i\tau)\varphi_{\lambda_j}$ only changes $L^2$ norms by powers of $\lambda_j$. So exponential growth $= e^{\tau\lambda_j}$. 