# Quantitative thermo-acoustics and related problems

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joint work with G. Bal, K. Ren and G. Uhlmann

**Conference on Inverse Problems**

**Dedicated to Gunther Uhlmann's 60th Birthday**

UC Irvine

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## Outline

# Outline

# Stand-alone medical imaging modalities

- **High contrast** modalities:
  - Optical Tomography (OT);
  - Electrical Impedance Tomography (EIT);
  - Elastographic Imaging (EI).

$\Longrightarrow$ low resolution *(Poor stability of diffusion type inverse boundary problems).*

- **High resolution** medical imaging modalities:
  - Computerized Tomography (CT);
  - Magnetic Resonance Imaging (MRI);
  - Ultrasound Imaging (UI).

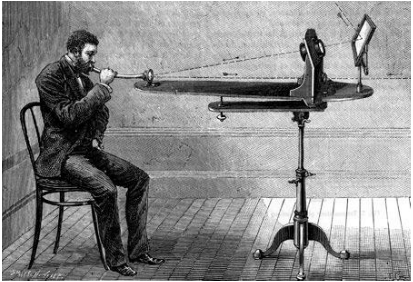
$\Longrightarrow$ sometimes low contrast.

# Multi-waves medical imaging modalities

- Physical mechanism that couples two modalities :
  - Optics/EM waves + Ultrasound: **Photo-Acoustic Tomography (PAT), Thermo-Acoustic Tomography (TAT)**; Ultrasound Modulated Optical Tomography (UMOT);
    - → *To improve resolution while keeping the high contrast capabilities of electromagnetic waves*
  - Electrical currents + Ultrasound: UMEIT;
  - Electrical currents + MRI: MREIT;
  - Elasticity + Ultrasound: TE.
  - ...etc.
- *Data fusing of independent imaging modalities.*

# Photo-Acoustic effect

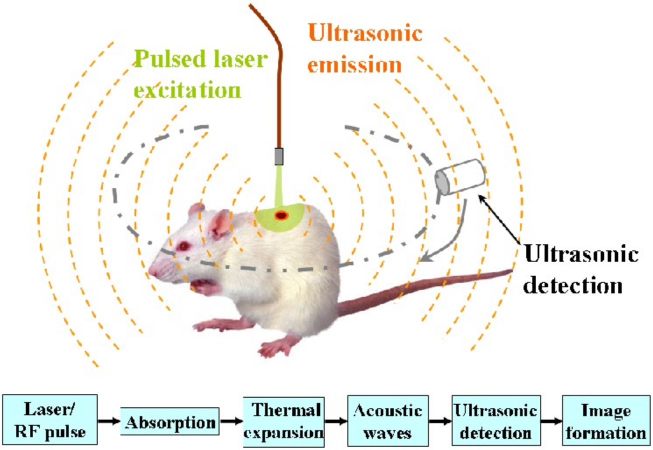
**Photoacoustic Effect**: **The sound of light** (Lightening and Thunder!)



Picture from Economist
(The sound of light)

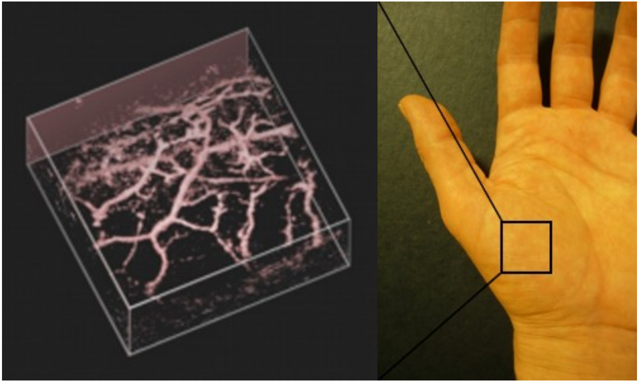
**Graham Bell**: When rapid pulses of light are incident on a sample of matter they can be absorbed and the resulting energy will then be radiated as heat. This heat causes detectable sound waves due to pressure variation in the surrounding medium.

# Photo/Thermo-Acoustic Tomography (PAT/TAT)



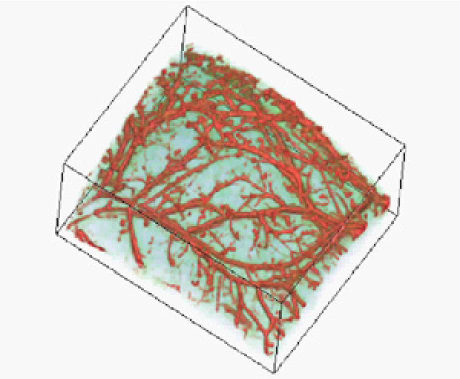

Wikipedia

# Experimental results in PAT



*Courtesy UCL (Paul Beard's Lab).*

# Experimental results in PAT



*From Lihong Wang's lab (Wash. Univ.)*

# Mathematical inverse problems

- *First step:* Inverse source problems for acoustic waves (high resolution): to reconstruct the *radiation* $H(x)$ from $p(t,x)|_{\partial\Omega}$. Here $H(x)$ is supported on a bounded domain $\Omega$.

$$\begin{aligned} (\partial_t^2 - c(x)^2\Delta)p &= 0 \quad \text{on } \mathbb{R}^n \times [0,T] \\ p(0,x) &= H(x) \quad \text{on } \mathbb{R}^n \\ \partial_t p(0,x) &= 0 \quad \text{on } \mathbb{R}^n. \end{aligned}$$

- *Second step:* **Quantitative PAT/TAT (QPAT/QTAT)**

  The outcome of the first step is the availability of special internal functionals $H(x)$ of the parameters (optical or electrical) of interest. The inverse problem of this step aims to address:

  - Which parameters can be uniquely determined;
  - With which stability (resolution)
  - Under which illumination (probing) mechanism.

## Results on the first step

### Constant Speed

KRUGER; AGRANOVSKY, AMBARTSOUMIAN, FINCH, GEORGIEVA-HRISTOVA, JIN, HALTMEIER, KUCHMENT, NGUYEN, PATCH, QUINTO, RAKESH, WANG, XU . . .

### Variable Speed

ANASTASIO ET. AL., BURGHOLZER, COX ET. AL., GEORGIEVA-HRISTOVA, GRUN, HALTMEIR, HOFER, KUCHMENT, NGUYEN, PALTAUFF, WANG, XU, STEFANOV-UHLMANN (A modified time reversal) . . .

### Discontinuous Speed (Brain Imaging)

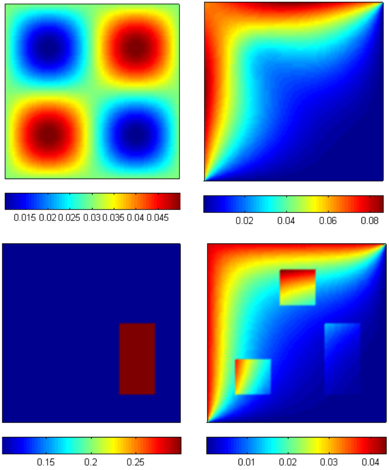
WANG, STEFANOV-UHLMANN

### Partial Data

FINCH, PATCH AND RAKESH, STEFANOV-UHLMANN.

## Motivation of the second step

# Photo-Acoustic Imaging: Qualitative vs. Quantitative




Left: True absorption coefficient $\sigma(x)$; Right: Radiation $H(x) = \Gamma(x)\sigma(x)u(x)$.

## Results on the second step of QPAT

- QPAT modeling (diffusive regime)

$$\boxed{\begin{aligned} -\nabla\cdot\gamma(x)\nabla u+\sigma(x)u &= 0, \quad \text{in } \Omega \\ u|_{\partial\Omega} &= f. \end{aligned}}$$

- $f$ is the boundary **illumination**;
- **Internal measurements** (absorption):

$$\boxed{H(x)=\Gamma(x)\sigma(x)u(x)} \quad \text{for } x\in\Omega.$$

- *Inverse problem*: to reconstruct $\gamma(x)$, $\sigma(x)$ and $\Gamma(x)$ from $H(x)$.
- *Results*:
  - Two measurements $H_1(x)$ and $H_2(x)$ uniquely and stably determine two out of three parameters (Bal-Uhlmann).
  - Results with partial boundary illuminations (Chen-Yang)

# Outline

# QTAT–Modeling (system)

- *Low frequency radiation (deeper penetration)* in QTAT is modeled by Maxwell's equations:

$$\begin{aligned} -\nabla \times \nabla \times \mathbf{E} + k^2\mathbf{E} + ik\sigma(x)\mathbf{E} &= 0, \quad \text{in } \Omega \\ \nu \times \mathbf{E}|_{\partial\Omega} &= f \end{aligned}$$

- *Internal measurements*: the map of absorbed electromagnetic radiation is

$$H(x) = \sigma(x)|\mathbf{E}|^2(x)$$

- *Inverse problem*: to reconstruct $\sigma(x)$ from $H(x)$.

## Results for ME systems (small $\sigma$)

**Theorem (Bal-Ren-Uhlmann-Z)**

Let $0 < \sigma_1(x), \sigma_2(x) \leq \sigma_M$ for a.e. $x \in \Omega$. Then for $\sigma_M < \alpha$ sufficiently small, we have that

(i). *Uniqueness*: if $H_1 = H_2$ a.e., in $\Omega$, then $\sigma_1(x) = \sigma_2(x)$ a.e. in $\Omega$ where $H_1 = H_2 > 0$.

(ii). *Stability*: moreover, we have

$$\boxed{\|w_1(\sqrt{\sigma_1} - \sqrt{\sigma_2})\|_{\mathcal{H}} \leq C\|w_2(\sqrt{H_1} - \sqrt{H_2})\|_{\mathcal{H}}},$$

for some universal constant $C$ and for positive weights given by

$$w_1^2(x) = \frac{|u_1 u_2|}{\sqrt{\sigma_1 \sigma_2}}(x), \quad w_2(x) = \frac{\max(\sigma_1^{1/2}, \sigma_2^{1/2}) + \max(\sigma_1^{-1/2}, \sigma_2^{-1/2})}{\alpha - sup_{x\in\Omega}\sqrt{\sigma_1\sigma_2}}.$$

- Denote operator $P := \frac{1}{ik}(\nabla \times \nabla \times -k^2)$, then $\alpha > 0$ is such that $(Pu, u)_{L^2} \geq \alpha\|u\|_{L^2}$. The result extends to operators with the same property.

## Proof:

Denote $\widehat{E}_j = E_j/|E_j|$, then

$$p(E_1 - E_2) = \sqrt{\sigma_1\sigma_2}(|E_2|\widehat{E}_1 - |E_1|\widehat{E}_2) + (\sqrt{H_1} - \sqrt{H_2})(\sqrt{\sigma_1}\widehat{E}_1 + \sqrt{\sigma_2}\widehat{E}_2),$$

$$\big||E_2|\widehat{E}_1 - |E_1|\widehat{E}_2\big| = |E_2 - E_1|.$$

Therefore,

$$(\alpha - \sup_{x\in\Omega}\sqrt{\sigma_1\sigma_2})\|E_1 - E_2\|_{L^2}^2 \leq \Big((\sqrt{H_1} - \sqrt{H_2})(\sqrt{\sigma_1}\widehat{E}_1 - \sqrt{\sigma_2}\widehat{E}_2), E_1 - E_2\Big)_{L^2}.$$

$$\begin{aligned}\text{LHS} \geq &(\alpha - \sup_{x\in\Omega}\sqrt{\sigma_1\sigma_2})\||E_1| - |E_2|\|_{L^2}^2 \\ \geq &(\alpha - \sup_{x\in\Omega}\sqrt{\sigma_1\sigma_2})\Big(\|w_1(\sqrt{\sigma_2} - \sqrt{\sigma_1})\|_{L^2}^2 \\ &+ \Big\|\frac{\sqrt{H_1} - \sqrt{H_2}}{H_1^{1/4} + H_2^{1/4}}(\frac{H_1^{1/4}}{\sqrt{\sigma_1}} + \frac{H_2^{1/4}}{\sqrt{\sigma_2}})\Big\|_{L^2}\Big).\end{aligned}$$

## QTAT–Modeling (scalar)

- We also consider the scalar model of Helmholtz equations

$$\boxed{\begin{aligned}(\Delta + k^2 + ik\sigma(x))u =& 0, \quad \text{in } \Omega \\ u|_{\partial\Omega} =& f\end{aligned}}$$

- *Internal measurements*:

$$\boxed{H(x) = \sigma(x)|u|^2(x)}$$

- *Inverse problem*: to reconstruct $\sigma(x)$ from $H(x)$.

## Results for scalar models

- For a given boundary illumination $f$, denote $H(x)$ and $\tilde{H}(x)$ the internal measurements for the solutions $u$ and $\tilde{u}$ to the equations with conductivities $\sigma(x)$ and $\tilde{\sigma}(x)$, respectively.
- Denote $Y := H^s(\Omega)$ where $s > n/2$ the parameters and measurements space.
- Denote $\mathcal{M} := \{\sigma \in Y : \|\sigma\|_Y \leq M\}$.

**Theorem (Bal-Ren-Uhlmann-Z)**

There is an open set of illuminations $f$ such that $H(x) = \tilde{H}(x)$ in $Y$ implies that $\sigma(x) = \tilde{\sigma}(x)$ in $Y$. Moreover, there exists a constant $C$ independent of $\sigma$ and $\tilde{\sigma}$ such that

$$\boxed{\|\sigma - \tilde{\sigma}\|_Y \leq C\|H - \tilde{H}\|_Y}.$$

## Results for scalar models (Reconstruction)

### A fixed point reconstruction scheme

More precisely, we can reconstruct $\sigma$ as finding the unique fixed point to the equation

$$\boxed{\sigma = e^{-(\rho+\overline{\rho})\cdot x}H(x) - \mathcal{H}_f[\sigma](x)} \quad \text{in } \; Y,$$

where $\rho \in \mathbb{C}^n$, $\rho \cdot \rho = 0$, the functional $\mathcal{H}_f[\sigma](x)$ defined as

$$\mathcal{H}_f[\sigma](x) := \sigma(x)(\psi_f(x) + \overline{\psi_f}(x) + |\psi_f(x)|^2)$$

is a **contraction map** for $f$ in the open set of illuminations, and $\psi_f(x)$ is the solution to

$$(\Delta + 2\rho \cdot \nabla)\psi_f = -(k^2 + ik\sigma)(1 + \psi_f) \quad \text{in } \; \Omega, \quad \psi_f|_{\partial\Omega} = e^{-\rho \cdot x} f - 1.$$

## Proof:

- Key ingredient: **Complex Geometric Optics (CGO) solutions**

$$\boxed{u(x) = e^{x\cdot\rho}\,(1+\psi_\sigma(x))} \quad \rho\in\mathbb{C}^n,\ \rho\cdot\rho = 0$$

to $(\Delta + k^2 + ik\sigma)u = 0$ in $\Omega$. Then $\psi_\sigma(x)$ satisfies

$$(\Delta + 2\rho\cdot\nabla)\psi_\sigma = -(k^2+ik\sigma)(1+\psi_\sigma),$$

and

$$\boxed{|\rho|\|\psi_\sigma\|_{H^{n/2+k+\varepsilon}(\Omega)} + \|\psi_\sigma\|_{H^{n/2+k+1+\varepsilon}(\Omega)} \lesssim \|q\|_{H^{n/2+k+\varepsilon}(\Omega)}}$$

where $q = k^2 + ik\sigma$. Moreover, we have

**Lemma**

Suppose $\sigma,\tilde{\sigma}\in\mathcal{M}$. Then for $|\rho|$ large enough, we have

$$\boxed{\|\psi_\sigma - \psi_{\tilde{\sigma}}\|_Y \le \frac{C}{|\rho|}\|\sigma-\tilde{\sigma}\|_Y}$$

## Proof (Cont'd)

- Plugging the CGO solution into $H(x) = \sigma(x)|u(x)|^2$,

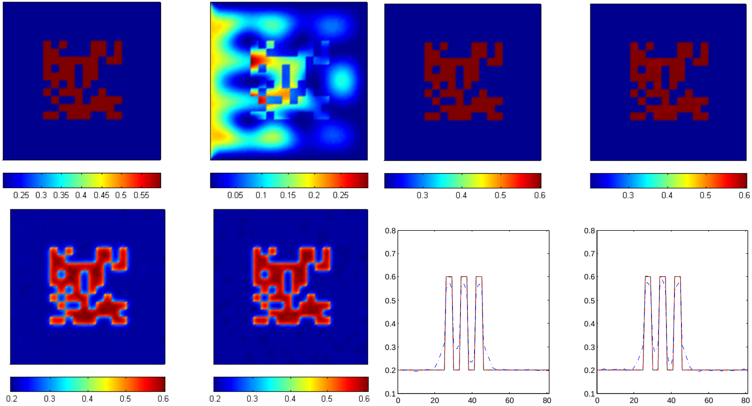
$$H(x) = e^{(\rho+\overline{\rho})\cdot x}\big(\sigma + \mathcal{H}[\sigma](x)\big)$$

where $\mathcal{H}[\sigma](x) = \sigma(\psi_\sigma + \overline{\psi_\sigma} + |\psi_\sigma|^2)$. By the previous lemma, $\mathcal{H}[\sigma]$ is a **contraction map**

$$\|\mathcal{H}[\sigma] - \mathcal{H}[\tilde{\sigma}]\|_Y \leq \frac{C}{|\rho|}\|\sigma - \tilde{\sigma}\|_Y.$$

- Above is valid provided the boundary illumination open set of $f$ is sufficiently close to the traces of CGO solutions.

# Numerical results–Discontinuous conductivity in TAT



# Outline

# Maxwell's equations (full model)

Consider Maxwell's equations:

$$\nabla \times \mathbf{E} = i\omega\mu(x)\mathbf{H}, \quad \nabla \times \mathbf{H} = -i\omega\gamma(x)\mathbf{E} \quad \text{in} \;\; \Omega$$

where $\gamma(x) := \varepsilon(x) - \frac{i}{\omega}\sigma(x)$ with boundary illumination

$$\nu \times \mathbf{E}|_{\partial\Omega} = f.$$

**Internal data**:

$$H(x) = \sigma(x)|\mathbf{E}(x)|^2.$$

- Reduction to matrix Schrödinger equations [Ola-Somersalo]:

  Set scalar potentials $\Phi := \frac{i}{\omega}\nabla\cdot(\gamma\mathbf{E})$, $\Psi := \frac{i}{\omega}\nabla\cdot(\mu\mathbf{H})$. Let $X = \left(\frac{1}{\gamma\mu^{1/2}}\Phi,\ \gamma^{1/2}\mathbf{E},\ \mu^{1/2}\mathbf{H},\ \frac{1}{\gamma^{1/2}\mu}\Psi\right)^T \in (\mathcal{D}')^8$. Under some assumptions on $\Phi$ and $\Psi$,

$$\text{Maxwell's equations} \ \Leftrightarrow\ \boxed{(P(i\nabla) - k + V)X = 0}.$$

  - $P(i\nabla)^2 = \Delta\mathbf{1}_8$;
  - $\boxed{(P(i\nabla) - k + V)(P(i\nabla) + k - V^T) = -(\Delta + k^2)\mathbf{1}_8 + Q}$ is a Schrödinger operator ;

# CGO solutions to matrix Schrödinger equations

- Define $Y$ by $\boxed{X = (P(i\nabla) + k - V^T)Y}$ Then
$$(-\Delta - k^2 + Q)Y = 0 \quad \text{in} \quad \Omega.$$
- CGO solutions of $(-\Delta - k^2 + Q)$: Given $\rho \in \mathbb{C}^3$ with $\rho \cdot \rho = k^2$ and a constant field $y_{0,\rho} \in \mathbb{C}^8$,
$$\boxed{Y_\rho(x) = e^{x\cdot\rho}(y_{0,\rho} - v_\rho(x))}$$
where $v_\rho \in H^s(\Omega)^8$ for $0 \le s \le 2$, and
$$\|v_\rho\|_{H^s} \le C|\rho|^{s-1}.$$

## CGO solutions to Maxwell's equations

- CGO solutions to Maxwell's equations:

$$X_\rho = (P(i\nabla) + k - V^T)Y_\rho$$
$$= \boxed{e^{x\cdot\rho}\left((P(-\rho)+k)y_{0,\rho} + r_\rho(x)\right) := e^{x\cdot\rho}\left(x_{0,\rho} + r_\rho(x)\right)}$$

where $\|r_\rho\|_{L^2(\Omega)} \leq C; \quad \|r_\rho\|_{H^1(\Omega)} \leq C|\rho|.$

- Choice of $y_{0,\rho}$:
  - For $\mathbf{E} = \gamma^{-1/2}(X_\rho)_2$ and $\mathbf{H} = \mu^{-1/2}(X_\rho)_3$ to be solutions of Maxwell's equations,

$$\boxed{(x_{0,\rho})_1 = (x_{0,\rho})_8 = 0.}$$

  - For $|\rho| \gg 1$,

$$\boxed{(x_{0,\rho})_2 = \mathcal{O}(|\rho|), \quad (x_{0,\rho})_3 = \mathcal{O}(1).}$$

# CGO solutions to Maxwell's equations

Let $\rho = \tau\zeta + i\sqrt{\tau^2 + k^2}\zeta^{\perp}$, where $\zeta \in \mathbb{S}^{n-1}$ with $\zeta \cdot \zeta^{\perp} = 0$.

## CGO solutions for Maxwell's equations

Assume that $\omega$ is not a resonant frequency. Fix $t > 0$ and let $\rho$ as above. For $\tau$ large enough, there exists a unique CGO solution $(\mathbf{E}, \mathbf{H})$ for Maxwell's equations of the form

$$\mathbf{E} = \gamma^{1/2} e^{x\cdot\rho}\big(\eta + R_\sigma(x)\big); \quad \mathbf{H} = \mu^{1/2} e^{x\cdot\rho}\big(\theta + T_\sigma(x)\big)$$

where

$$\boxed{\eta := (x_{0,\rho})_2 = \mathcal{O}(\tau); \quad \theta := (x_{0,\rho})_3 = \mathcal{O}(1)} \quad \text{for} \quad \tau \gg 1,$$

and $R_\sigma = (r_\rho)_2$ and $T_\sigma = (r_\rho)_3$ have bounded $L^2$ norms in $\tau$.

## More on CGO solutions to Maxwell's equations

*Suppose* $\mu(x) = \varepsilon(x) = 1$, *the dependence of* $R_\sigma$ *on* $\sigma(x)$ *is*

**Proposition**

For $\sigma, \tilde{\sigma} \in \mathcal{M}$, the CGO electric field $\mathbf{E} = e^{x\cdot\rho}\big(\eta + R_\sigma(x)\big)$ has the remainder function $R_\sigma$ satisfying

$$\|R_\sigma\|_{H^{n/2+k+\epsilon}(\Omega)} \leq C\|\sigma\|_{H^{n/2+k+2+\epsilon}(\Omega)}$$

and

$$\|R_\sigma - R_{\tilde{\sigma}}\|_{H^{n/2+k+\epsilon}(\Omega)} \leq C\|\sigma - \tilde{\sigma}\|_{H^{n/2+k+2+\epsilon}(\Omega)}$$

for $\tau \gg 1$.

## Proof of the proposition

- The proof is based on the equation satisfied by $R_\sigma$:

$$\boxed{\begin{aligned}(\Delta + 2\rho\cdot\nabla)R_\sigma = &-\gamma^{1/2}\alpha\times Q_\sigma - (R_\sigma\cdot\nabla)\alpha - k^2(\gamma-1)R_\sigma + qR_\sigma \\ &+\frac{1}{2}(\alpha\cdot\alpha)\eta - \nabla(\alpha\cdot\eta) - \frac{\Delta\gamma^{-1/2}}{\gamma^{-1/2}}\eta - k^2(\gamma-1)\eta. \\ (\Delta + 2\rho\cdot\nabla)Q_\sigma = &-k^2\gamma^{1/2}\alpha\times(\eta+R_\sigma) - k^2(\gamma-1)Q_\sigma.\end{aligned}}$$

where $\alpha = \nabla\gamma/\gamma$, $q = \Delta\gamma^{1/2}/\gamma^{1/2}$.

# Difficulty in applying the fixed point argument

Plugging the CGO solutions

$$\mathbf{E}(x) = \gamma^{1/2}\left(\eta + R_\sigma(x)\right), \quad |\eta| \sim \tau,$$

into the internal data $H(x) = \sigma(x)|\mathbf{E}(x)|^2$,

$$\frac{H(x)e^{-(\rho+\overline{\rho})}}{\tau^2} = \frac{|\eta|^2}{\tau^2}\frac{\sigma}{|\gamma|} + \mathcal{H}[\sigma](x)$$

where

$$\mathcal{H}[\sigma](x) = \frac{\sigma}{|\gamma|}\left(\frac{\eta\cdot\overline{R_\sigma}}{\tau^2} + \frac{\overline{\eta}\cdot R_\sigma}{\tau^2} + \frac{|R_\sigma|^2}{\tau^2}\right).$$

However, for $\tau \gg 1$, we can only show for $s > n/2$

$$\boxed{\|\mathcal{H}[\sigma] - \mathcal{H}[\tilde{\sigma}]\|_{H^s(\Omega)} \lesssim \frac{C}{\tau}\|\sigma - \tilde{\sigma}\|_{H^{s+2}(\Omega)}}$$

**Not a contraction in $Y$!**

# Linearized Inverse Problems

Consider Maxwell's equations

$$L_{n,\sigma}\mathbf{E} := -\nabla \times \nabla \times \mathbf{E} + (k^2 n(x) + ik\sigma(x))\mathbf{E} = 0 \quad \text{in } \Omega$$

with boundary illumination $\nu \times \mathbf{E}|_{\partial\Omega} = f$.

*Inverse Problem:* Reconstruct $(n, \sigma)$ from $H_f(x) = \sigma|\mathbf{E}|^2$.

*Linearization:*

$$L_{n_0,\sigma_0}\delta E = -(k^2\delta_n + ik\delta_\sigma)E_0, \quad \delta E(\delta_n, \delta_\sigma) \in H_0^1(\Omega)$$

where $L_{n_0,\sigma_0}E_0 = 0$ and $E_0|_{\partial\Omega} = f$.

$$dH_f(\delta_n, \delta_\sigma) = 2\sigma_0\overline{E_0} \cdot \delta E(\delta_n, \delta_\sigma) + \delta_\sigma|E_0|^2.$$

## Single measurement

Denote $\mu_0 = k^2 n_0 + ik\sigma_0$, then

$$\begin{aligned}\tilde{L}_0 \delta E :=& \Delta\delta E + (\frac{\nabla\mu_0}{\mu_0} \otimes \nabla)^T \delta E - (\frac{\nabla\mu_0}{\mu_0} \otimes \frac{\nabla\mu_0}{\mu_0})\delta E + \frac{1}{\mu_0}(\nabla^{\otimes 2}\mu_0)^T \delta E + \mu_0 \delta E \\ =& \tilde{M}(x, D)(k^2\delta_n + ik\delta_\sigma)\end{aligned}$$

where

$$\tilde{M}(x,\xi) := \left(\frac{1}{\mu_0}(\xi \otimes \xi)\sharp E_0 + i\left(\frac{\nabla\mu_0}{\mu_0^2} \otimes \xi\right)\sharp E_0 - E_0\right)$$

Let $\tilde{Q}(x,\xi)$ be the parametrix of $\tilde{L}_0$. We have

$$\delta E(\delta_n, \delta_\sigma) = \tilde{Q}(x, D)\tilde{M}(x, D)(k^2\delta_n + ik\delta_\sigma).$$

Therefore,

$$dH_f(\delta_n, \delta_\sigma) = \tilde{A}(x, D)\delta_n + \tilde{B}(x, D)\delta_\sigma + (l.o.t.)_{-1}$$

with zeroth order symbols (NOT elliptic!)

$$\tilde{A}(x,\xi) = -\frac{2k^4 n_0 \sigma_0}{|\mu_0|^2}|E_0 \cdot \hat{\xi}|^2, \quad \tilde{B}(x,\xi) = |E_0|^2 - \frac{2k^2\sigma_0^2}{|\mu_0|^2}|E_0 \cdot \hat{\xi}|^2$$

## Multiple measurements

*Illumination set:* $I := \{f_1, f_2, f_1 + f_2, f_1 + if_2\}$.

*Available internal data:*

$$\vec{H}(x) := \{H_1 = \sigma|E_1|^2, H_2 = \sigma|E_2|^2, H_{12} = \sigma\overline{E_1} \cdot E_2\}.$$

Then the linearized internal functional

$$d\vec{H}(\delta_n, \delta_\sigma) = \begin{pmatrix} \tilde{A}_1(x,D) & \tilde{B}_1(x,D) \\ \tilde{A}_2(x,D) & \tilde{B}_2(x,D) \\ \tilde{A}_{12}(x,D) & \tilde{B}_{12}(x,D) \end{pmatrix} \begin{pmatrix} \delta_n \\ \delta_\sigma \end{pmatrix} + (l.o.t.)_{-1}$$

$$:= \Psi(x,D)_{3\times 2}(\delta_n, \delta_\sigma) + (l.o.t.)_{-1}$$

$$\tilde{A}_{12}(x,\xi) = -\frac{2k^4 n_0 \sigma_0}{|\mu_0|^2}\left(\overline{E_0^{(1)}} \cdot \hat{\xi}\right)\left(E_0^{(2)} \cdot \hat{\xi}\right),$$

$$\tilde{B}_{12}(x,\xi) = (\overline{E_0^{(1)}} \cdot E_0^{(2)}) - \frac{2k^2\sigma_0^2}{|\mu_0|^2}\left(\overline{E_0^{(1)}} \cdot \hat{\xi}\right)\left(E_0^{(2)} \cdot \hat{\xi}\right).$$

## Multiple measurements (cont'd)

*Claims* [Bal-Z]:

- At least one of the $2 \times 2$ subdeterminants is nonzero when $E_0^{(1)}$ and $E_0^{(2)}$ are linearly independent.
- Consider $\Psi^*(x, D) d\vec{H}(\delta_n, \delta_\sigma)$. The principle symbol $\Psi^*(x, \xi)\Psi(x, \xi)$ is **elliptic**. Then the operator $d\vec{H}(\delta_n, \delta_\sigma)$ is *semi-Fredholm* with a finite dimensional kernel. (following the argument of [Kuchment-Steinhauer])

## Restore locality (A boundary-value-problem point of view)

Basically, eliminate the $|\xi|^{-2}$ in the principle symbol $\Psi(x,\xi)$ by applying $\Delta$ to obtain differential operator (leading term). Instead, we apply

$$\tilde{P}(x,D) := \Delta - 2\frac{\nabla\sigma_0}{\sigma_0}\cdot\nabla + \left(2\left|\frac{\nabla\sigma_0}{\sigma_0}\right|^2 - \frac{\Delta\sigma_0}{\sigma_0}\right)$$

Rewrite $d\vec{H}(\delta_n,\delta_\sigma) := d\vec{H}(\delta_\mu,\delta_\mu^*)$ where $\delta_\mu := k^2\delta_n + ik\delta_\sigma$. Then

$$\tilde{P}(x,D)d\vec{H}(\delta_\mu,\delta_\mu^*) = \Gamma(x,D)_{3\times 2}(\delta_\mu,\delta_\mu^*) + (l.o.t.)_1$$

where $\Gamma(x,D)$ is a $3\times 2$ second order differential operator

$$\Gamma_{j1}(x,\xi) = -\frac{1}{2ik}|E_0^{(j)}|^2|\xi|^2 + \frac{\sigma_0}{\mu_0}|E_0^{(j)}\cdot\xi|^2, \quad \Gamma_{j2}(x,\xi) = \overline{\Gamma_{j1}(x,\xi)}, \quad j=1,2.$$

$$\Gamma_{31}(x,\xi) = -\frac{1}{2ik}({E_0^{(1)}}^*\cdot E_0^{(2)})|\xi|^2 + \frac{\sigma_0}{\mu_0}({E_0^{(1)}}^*\cdot\xi)(E_0^{(2)}\cdot\xi)$$

$$\Gamma_{32}(x,\xi) = \frac{1}{2ik}({E_0^{(1)}}^*\cdot E_0^{(2)})|\xi|^2 + \frac{\sigma_0}{\mu_0^*}({E_0^{(1)}}^*\cdot\xi)(E_0^{(2)}\cdot\xi)$$

## A boundary value problem: Ellipticity

- Consider a fourth order $2 \times 2$ system

$$(\Gamma^*\Gamma)(x,D)(\delta_\mu,\delta_\mu^*) + (l.o.t.)_3 = \Gamma(x,D)^*\tilde{P}(x,D)d\vec{H} \quad \text{in } \Omega. \qquad (1)$$

  The leading term is an elliptic fourth order differential operator with symbol $\Gamma(x,\xi)^*\Gamma(x,\xi)$. However, the lower order term $(l.o.t.)_3$ is still a non-local pseudo-differential operator.

- Impose the elliptic boundary (Lopatinskii) condition

$$\delta_\mu|_{\partial\Omega} = 0, \quad \partial_\nu\delta_\mu|_{\partial\Omega} = 0. \qquad (2)$$

**Ellipticity [Bal-Z]**

The boundary value problem (1) + (2) is elliptic and Fredholm.

## A boundary value problem: Invertibility in a small domain

- We freeze parameters $(n_0, \sigma_0)$ and vector fields $E_0^{(1)}$ and $E_0^{(2)}$. One will have
$$(\Delta + \mu_0)(\Delta + \mu_0^*)d\vec{H}(\delta_\mu, \delta_\mu^*) := \Lambda(x, D)(\delta_\mu, \delta_\mu^*)$$
to be a $3 \times 2$ fourth order differential equation.
- From frozen parameters and vector fields to the case of domains small enough.

**Invertibility [Bal-Z]**

With Cauchy boundary condition, the eighth order differential equation

$$\Lambda(x, D)^*\Lambda(x, D)(\delta_\mu, \delta_\mu^*) = \Lambda(x, D)^*(\Delta + \mu_0)(\Delta + \mu_0^*)d\vec{H}(\delta_\mu, \delta_\mu^*)$$

admits a unique solution.

❦❦❦ Happy Birthday, Gunther! ❦❦❦