

LOCAL RIGIDITY OF CONFORMALLY EUCLIDEAN METRICS FOR THE ANISOTROPIC CALDERÓN PROBLEM

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ABSTRACT. We prove local rigidity of every smooth conformally Euclidean metric for the anisotropic Calderón problem on smooth compact domains $M \subset \mathbb{R}^n$, $n \geq 3$. Any smooth Riemannian metric g sufficiently close to a fixed background $g_0 = e^{2c}\mathbf{e}$ in $H^s(M)$, with integer $s > n/2 + 1$, having the same Dirichlet-to-Neumann map satisfies $g = \Phi^*g_0$ for a smooth diffeomorphism Φ fixing the boundary pointwise.

1. INTRODUCTION

Let $M \subset \mathbb{R}^n$, $n \geq 3$, be a smooth compact domain, and let $\mathbf{e}$ denote the Euclidean metric. For a smooth Riemannian metric g on M , the Dirichlet-to-Neumann (DN) map is defined, for $f \in C^\infty(\partial M)$, by

$$\Lambda_g f = \partial_{\nu_g} u|_{\partial M},$$

where u is the unique solution of the Dirichlet problem

$$\begin{cases} \Delta_g u = 0 & \text{in } M, \\ u = f & \text{on } \partial M. \end{cases}$$

Here, ν_g is the outward unit normal with respect to g . We prove that for every smooth conformal factor c in M , if g is sufficiently close to $g_0 := e^{2c}\mathbf{e}$ in $H^s(M)$, with integer $s > n/2 + 1$, and if $\Lambda_g = \Lambda_{g_0}$, then g is isometric to g_0 by a diffeomorphism fixing the boundary pointwise. We present the proof for $c = 0$ first because it is simpler, and because its proof allows the smallness parameter $\varepsilon_0 > 0$ to be bounded from below in an explicit way, in principle.

Theorem 1. *Let $M \subset \mathbb{R}^n$, $n \geq 3$, be a smooth compact domain. Fix an integer $s > n/2 + 1$. There exists $\varepsilon_0 > 0$, depending only on M and s , such that for every smooth Riemannian metric g on M , the conditions*

$$(1.1) \quad \Lambda_g = \Lambda_{\mathbf{e}}, \quad \|g - \mathbf{e}\|_{H^s(M)} < \varepsilon_0$$

imply that

$$g = y^*\mathbf{e}$$

for a diffeomorphism $y: M \rightarrow M$ fixing ∂M pointwise. The map $x \mapsto y$ is given by the g -harmonic extensions

$$\Delta_g y^j = 0 \quad \text{in } M, \quad y^j|_{\partial M} = x^j|_{\partial M}, \quad j = 1, \dots, n.$$

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The idea is the following. We first put g in harmonic coordinates fixing the boundary. Writing $\bar{g} = y_*g$, the tensor

$$B = \sqrt{\det \bar{g}} \bar{g}^{-1} - \text{Id}$$

is then divergence-free (which reminds us of the solenoidal gauge in boundary rigidity but this time, it is about the conductivity tensor). Equality (1.1) of the DN maps, together with boundary determination in [20], implies that $\bar{g} - \mathbf{e}$ has a vanishing boundary jet. We can therefore extend $\bar{g}$ and B smoothly outside M by the Euclidean metric and zero, respectively.

For the Euclidean metric, we have the harmonic functions used by Calderón in his original paper [6]:

$$(1.2) \quad u_{\mathbf{e}}(x, \zeta) = e^{x \cdot \zeta}, \quad \zeta \in \mathbb{C}^n, \quad \zeta \cdot \zeta = 0.$$

Equality of the DN maps also allows us to construct global $\bar{g}$ -harmonic solutions of the form $e^{x \cdot \zeta}(1 + r_\zeta)$, where r_ζ is smooth and supported in M , as shown in Proposition 2.5 below. The remainder r_ζ is controlled when $|\zeta| \|B\|_{L^\infty} \ll 1$ by a known Carleman estimate. An orthogonality identity of Alessandrini type, combined with the divergence-free condition, then yields a quadratic estimate for $\widehat{B}$ in a ball in the frequency space with radius at most $C_0/\|B\|_{L^\infty}$, see (2.13). We choose its radius as a fractional power of $\|B\|_{L^2}^{-1}$ and control the higher frequencies using the a priori H^s bound supplied by the theorem's smallness assumption (1.1). We use interpolation to ensure that this radius would lie within the allowed range, and smallness allows us to absorb both contributions, proving $B = 0$.

We emphasize that we do not take $|\zeta| \rightarrow \infty$ for a fixed metric. Our solutions are $\bar{g}$ -harmonic extensions of the boundary values of the Euclidean solutions (1.2). We keep the Euclidean exponential factor $e^{x \cdot \zeta}$, although for a general anisotropic metric its phase does not satisfy the eikonal equation $\bar{g}^{ij} \zeta_i \zeta_j = 0$ in the metric. This representation does not determine the interior phase, since the factor $1 + r_\zeta$ may contribute to it. Instead of taking the usual $|\zeta| \rightarrow \infty$ asymptotics of the Sylvester–Uhlmann construction in [30], which force $r_\zeta \rightarrow 0$ in their setting, we obtain in Proposition 2.5 an estimate for r in terms of the perturbation, uniformly over the allowed frequency range. Although this range grows as the perturbation tends to zero, the rigidity argument requires no large-frequency limit for a fixed nonzero perturbation.

Our main result, which implies Theorem 1 but with a different proof, is to allow g_0 to be any fixed smooth conformally Euclidean background metric.

Theorem 2. *Let $M \subset \mathbb{R}^n$, $n \geq 3$, be a smooth compact domain, let $g_0 = e^{2c} \mathbf{e}$ with $c \in C^\infty(M)$, and fix an integer $s > n/2 + 1$. There exists $\varepsilon_0 > 0$, depending on M , g_0 , s , such that every smooth Riemannian metric g on M satisfying*

$$\|g - g_0\|_{H^s(M)} < \varepsilon_0$$

and

$$(1.3) \quad \Lambda_g = \Lambda_{g_0}$$

has the form $g = \Phi^* g_0$ for a smooth diffeomorphism $\Phi: M \rightarrow M$ fixing ∂M pointwise.

The proof of Theorem 2 replaces the harmonic coordinates of Theorem 1 by a harmonic embedding in a higher-dimensional Euclidean space to construct the gauge. We replace the Euclidean exponentials (1.2) by background C.G.O. solutions. For the conformal background $g_0 = e^{2c} \mathbf{e}$, we use the normalized conductivity perturbation

$$B = e^{-(n-2)c} \sqrt{\det \bar{g}} \bar{g}^{-1} - \text{Id},$$

where $\bar{g}$ is the metric in the harmonic gauge. This agrees with the preceding definition when $c = 0$, and for uniqueness it is enough to prove $B = 0$. We do not have $\text{div } B = 0$ anymore; instead,

we prove that $\|\operatorname{div} B\|_{L^2} \leq C\|B\|_{L^2}$. Assuming that the uniqueness does not hold in this gauge, we get a normalized sequence of perturbations for which we establish compactness. Passing to a subsequence, we obtain a nonzero element of the kernel of the linearization. We show that this kernel consists of potential tensors, and that our choice of gauge forces the limiting tensor to vanish. This contradiction completes the argument. Since we prove Theorem 2 by a compactness argument, an explicit lower bound for ε_0 is not readily provided.

Previous work. Calderón's problem has a rich history starting with the original work by Calderón [6] who studied isotropic conductivities (corresponding to metrics conformal to $\mathbf{e}$ when $n \geq 3$ in metrics language), and proved injectivity of the linearization using the solutions (1.2). In their seminal paper [30], Sylvester and Uhlmann proved global uniqueness for the nonlinear problem in dimensions $n \geq 3$, for isotropic conductivities, introducing complex geometric optics (C.G.O.) solutions. Those solutions created a powerful technique used in many other problems. In [23], Nachman proposed a constructive reconstruction method.

Since the paper [30] appeared, it was conjectured that a similar result should hold for general anisotropic conductivities (general metrics in the metric language), up to an action of a diffeomorphism fixing ∂M pointwise. The anisotropic version turned out to be much harder. The two-dimensional case is more approachable, but not easy by any means; and there one has conformal invariance as well, in addition to the diffeomorphism one. It has been solved, see [3, 7, 14, 19, 29].

The $n \geq 3$ anisotropic case is mostly open. The main difficulty, when trying to adapt the method in [30], is the inability to construct C.G.O. solutions. Under an analyticity assumption of the metric class, uniqueness modulo the gauge has been established in increased generality in [17–20]. Carleman estimates and complex geometric optics constructions have yielded uniqueness results for conformal factors on conformally transversally anisotropic manifolds, which have conformally product types of metrics. Dos Santos Ferreira, Kenig, Salo, and Uhlmann [9] characterized the geometric restrictions imposed by the existence of limiting Carleman weights and constructed complex geometric optics solutions on admissible manifolds, obtaining uniqueness within a fixed conformal class. The approach was extended to more general transversal geometries in [10]. Until very recently, there was no result for an open set of metrics, and even the linearization about a fixed anisotropic metric was not known to be injective modulo the linearized gauge. Even if it were, the inherent instability of the problem [1, 22] would not yield local uniqueness by linearization, see also [28]. Here we prove local rigidity near each fixed smooth conformally Euclidean metric.

Our first theorem belongs to a family of rigidity results saying that a metric which looks flat from boundary measurements must be flat, up to the natural diffeomorphism gauge. In boundary rigidity, Gromov [13], among many other results, established rigidity of Euclidean regions. A Lorentzian counterpart was obtained by Oksanen, Rakesh, and Salo [24]: a globally hyperbolic metric agreeing with the Minkowski metric outside a compact set and having the same hyperbolic DN map is isometric to the Minkowski metric. Those results have been generalized. Burago and Ivanov [5] proved boundary rigidity for metrics sufficiently close to Euclidean ones. In subsequent work [25], Oksanen, Rakesh, and Salo extended their rigidity result to semiglobal uniqueness, allowing the background metric to be a sufficiently small perturbation of the Minkowski metric. Theorem 2 establishes local rigidity for each fixed smooth conformally Euclidean background and nearby anisotropic metrics.

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Note. After the first version of this paper, containing Theorem 1, was posted, Yi-Hsuan Lin informed P.S. that he had independently obtained the same result as that theorem by a related argument. Lin's preprint was subsequently posted as [21].

2. PROOF OF THEOREM 1

All metrics, functions, and tensors in the rest of the paper are smooth unless stated otherwise. We work in this section under the assumptions of Theorem 1, choosing ε_0 sufficiently small as required below. We write $h = g - \mathbf{e}$ for the original metric perturbation. Sobolev norms are taken in a Euclidean way. For tensor fields, we use the square root of the sum of the squared component Sobolev norms, and pointwise norms are Frobenius norms.

2.1. The harmonic gauge. The idea to use harmonic coordinates was suggested by John Lee to the authors of [31]; harmonic coordinates were also used in [27]. Harmonic coordinates were also used in the Poisson embedding approach in [17]. In two dimensions, they were also used in [2], and in [8].

In harmonic coordinates $y = (y^1, \dots, y^n)$, the Laplacian has no first-order terms. Indeed, write $\bar{g} = y_*g$ for the metric in these coordinates. The equations $\Delta_g y^j = 0$ imply $\partial_{y^i}(\sqrt{\det \bar{g}} \bar{g}^{ij}) = 0$, and therefore

$$\Delta_{\bar{g}} = \frac{1}{\sqrt{\det \bar{g}}} \partial_{y^i} (\sqrt{\det \bar{g}} \bar{g}^{ij} \partial_{y^j}) = \bar{g}^{ij} \partial_{y^i} \partial_{y^j}.$$

We next construct such coordinates globally and establish their compatibility with the boundary jets.

Lemma 2.1. *Let y be the harmonic extension map defined in Theorem 1. For $\varepsilon_0 > 0$ sufficiently small, $y: M \rightarrow M$ is a diffeomorphism fixing ∂M pointwise, and*

$$\|y - \text{Id}\|_{H^{s+1}(M)} + \|y_*g - \mathbf{e}\|_{H^s(M)} \leq C \|h\|_{H^s(M)}.$$

Here C depends on M and s only.

Proof. Set $w^j(x) = y^j(x) - x^j$, $j = 1, \dots, n$, and $A^{ij} = \sqrt{\det g} g^{ij}$, where x are the original coordinates. The equation for w^j is, with $\partial_i := \partial_{x^i}$,

$$\Delta w^j = -\partial_i(A^{ij} - \delta^{ij}) - \partial_i((A^{ik} - \delta^{ik})\partial_k w^j), \quad w^j|_{\partial M} = 0.$$

A standard estimate for the Dirichlet realization of Δ , combined with the fact that $H^s(M)$ is an algebra for $s > n/2$, yields

$$\|w\|_{H^{s+1}(M)} \leq C \|A - \text{Id}\|_{H^s(M)} (1 + \|w\|_{H^{s+1}(M)}).$$

We have $\|A - \text{Id}\|_{H^s(M)} \leq C \|h\|_{H^s(M)}$ with C depending only on M and s , uniformly for ε_0 sufficiently small. By the smallness assumption in (1.1), choosing ε_0 sufficiently small allows us to absorb w on the right and prove the estimate for it. Decreasing ε_0 further if necessary, we get $\|Dw\|_{L^\infty(M)} \leq 1/2$ by Sobolev embedding.

We will show that $x \mapsto y$ is a diffeomorphism on M now. Set $d(x) = \text{dist}(x, \partial M)$. Integrating Dw along a segment to a nearest boundary point yields $|w(x)| \leq \frac{1}{2}d(x)$, thus $y(x) = x + w(x)$ maps M into itself. To prove surjectivity, for $z \in M^{\text{int}}$, consider the map $T_z(x) = z - w(x)$. We want to solve $x = T_z(x)$, equivalent to $x = z - w(x)$, uniquely. The map T_z is a contraction on the closed ball $\overline{B(z, d(z))} \subset M$, mapping that ball into itself because

$$|T_z(x) - z| = |w(x)| \leq \frac{1}{2}d(x) \leq \frac{1}{2}(d(z) + |x - z|) \leq d(z).$$

Moreover, every solution of $y(x) = z$ lies in this ball, since

$$|x - z| = |w(x)| \leq \frac{1}{2}(d(z) + |x - z|) \implies |x - z| \leq d(z).$$

Banach's contraction theorem therefore implies that every interior point has exactly one preimage. Together with the condition $y|_{\partial M} = \text{Id}$, this proves that $x \mapsto y$ is a bijection. Since y is smooth up to the boundary, and $Dy = \text{Id} + Dw$ is everywhere invertible, the inverse function theorem, including at boundary points, shows that y^{-1} is smooth up to the boundary. Thus y is a diffeomorphism of M onto itself.

Since y is close to Id in $H^{s+1}(M)$ and $\det(Dy) \geq 2^{-n}$, the Sobolev inversion, composition, and product estimates in [16] yield

$$\|y_*g - \mathbf{e}\|_{H^s(M)} \leq C(\|h\|_{H^s(M)} + \|w\|_{H^{s+1}(M)}) \leq C\|h\|_{H^s(M)},$$

with C depending only on M, s . $\square$

Set $\bar{g} = y_*g$. In the next lemma, “flat” means zero jet at ∂M , not to be confused with zero curvature.

Lemma 2.2. *The metric perturbation $\bar{g} - \mathbf{e}$ is flat at ∂M .*

Proof. We use the boundary recovery result [20, Propositions 1.1–1.2]. The DN map there $f \mapsto (\Lambda_g f) dS_g$ maps functions to forms. It implies that the principal symbol of Λ_g with our definition is $\sigma_1(\Lambda_g)(x', \xi') = |\xi'|_{g|_{T\partial M}}$. Thus $\Lambda_g = \Lambda_{g_0}$ implies $g|_{T\partial M} = g_0|_{T\partial M}$. This allows us to remove the determinant factor incorporated in dS_g . The smooth boundary determination result in [20, Proposition 1.3] therefore recovers the full boundary jet of the metric in boundary normal coordinates from our DN map. Note that even though [20] is about analytic metrics, the proof there requires no analyticity assumptions. By the DN equality in (1.1), there is therefore a diffeomorphism ψ defined in a boundary collar, fixing ∂M pointwise, such that $\hat{g} := \psi^*g$ and $\mathbf{e}$ have the same boundary jets. Set $Y = y \circ \psi$ in the collar. Clearly, $\Delta_{\hat{g}}Y^j = (\Delta_g y^j) \circ \psi = 0$, and having the same DN maps leads to

$$\partial_{\nu_{\hat{g}}}Y^j = \partial_{\nu_g}y^j = \Lambda_g(x^j|_{\partial M}) = \Lambda_{\mathbf{e}}(x^j|_{\partial M}) = \partial_{\nu_{\mathbf{e}}}x^j \quad \text{on } \partial M.$$

Since $\hat{g} = \mathbf{e}$ at ∂M , the unit normals of those two metrics agree there. Thus $w^j := Y^j - x^j$ satisfies

$$(2.1) \quad \Delta_{\hat{g}}w^j = (\Delta_{\mathbf{e}} - \Delta_{\hat{g}})x^j \quad \text{in the collar}, \quad w^j|_{\partial M} = \partial_{\nu_{\hat{g}}}w^j|_{\partial M} = 0.$$

The r.h.s. of the PDE in (2.1) is flat at ∂M . In Euclidean boundary normal coordinates, the vanishing of w^j and its first normal derivative implies the vanishing of all their tangential derivatives at the boundary. Problem (2.1) is non-characteristic, so the equation implies that the second normal derivative also vanishes. Repeated normal differentiation of the equation then shows inductively that all higher normal derivatives vanish: at each step, the remaining terms involve only tangential derivatives of normal derivatives already known to vanish. Thus w^j is flat at ∂M for every j , and $Y - \text{Id}$ has zero boundary jet.

Near ∂M , we can pass back to the harmonic coordinates y to get $Y_*\hat{g} = y_*g = \bar{g}$. Since $\hat{g} - \mathbf{e}$ and $Y - \text{Id}$ are flat at ∂M , so is $\bar{g} - \mathbf{e}$. $\square$

Lemma 2.3. *The metric*

$$\tilde{g} = \begin{cases} \bar{g} = y_*g, & \text{in } M, \\ \mathbf{e}, & \text{in } \mathbb{R}^n \setminus M \end{cases}$$

is smooth on $\mathbb{R}^n$, and its Cartesian coordinate functions are $\tilde{g}$ -harmonic.

Proof. Smoothness follows from Lemma 2.2. Next,

$$\Delta_{\tilde{g}}x^j = (\Delta_g y^j) \circ y^{-1} = 0 \quad \text{in } M.$$

Outside M , the metric is Euclidean, so $\Delta_{\tilde{g}}x^j = 0$ there as well. Since $\tilde{g}$ and x^j are smooth on $\mathbb{R}^n$, this identity holds across ∂M . $\square$

2.2. Main identity. From now on, we denote $\tilde{g}$ by g . We have g defined in $\mathbb{R}^n$ now, with its Euclidean coordinates g -harmonic. Since y fixes ∂M pointwise, the DN equality in (1.1) continues to hold for this metric. The smallness assumption in (1.1) refers to the original perturbation h . We use the notation

$$A^{ij} = \sqrt{\det g} g^{ij}, \quad B^{ij} = A^{ij} - \delta^{ij}$$

introduced earlier, now used throughout $\mathbb{R}^n$. Then we have in $\mathbb{R}^n$,

$$(2.2) \quad \partial_i B^{ij} = 0, \quad \text{supp } B \subset M.$$

Also,

$$(2.3) \quad \sqrt{\det g} \Delta_g = \Delta + B^{ij} \partial_i \partial_j.$$

Integrating by parts and using the DN equality in (1.1), we get Alessandrini's identity [1] in this case:

$$(2.4) \quad \int B^{ij} \partial_i u_g \partial_j v_e \, dx = 0$$

for all g -harmonic u_g and $\mathbf{e}$ -harmonic v_e , both in $C^1(M)$. By (2.2), the integral in (2.4) extends to $\mathbb{R}^n$.

Since B is divergence-free, an integration by parts in (2.4) yields

$$\int_M B^{ij} u_g \partial_i \partial_j v_e \, dx = 0.$$

Plugging the exponential solutions (1.2) for v_e , we get

$$(2.5) \quad \int_M e^{x \cdot \zeta} B^{ij}(x) \zeta_i \zeta_j u_g(x) \, dx = 0, \quad \zeta \cdot \zeta = 0,$$

for every $C^1(M)$ g -harmonic function u_g . This equality is our main orthogonal identity which we will use to prove that $B = 0$. Once we do that, taking determinants in $\sqrt{\det g} g^{-1} = \text{Id}$ yields

$$1 = \det(\sqrt{\det g} g^{-1}) = (\det g)^{(n-2)/2}.$$

Since $n \geq 3$, it follows that $\det g = 1$, and then $g = \mathbf{e}$.

2.3. Construction of C.G.O.-like solutions related to g . We use the C.G.O. ansatz as in [30]. As explained in the Introduction, we construct solutions with frequencies bounded by $C/\|B\|_{L^\infty}$, assuming $B \not\equiv 0$. To reiterate, they *are not* extensions of the C.G.O. solutions in [30] to anisotropic metrics.

For $\zeta \neq 0$ such that $\zeta \cdot \zeta = 0$, write, in “polar coordinates,”

$$\zeta = \tau(a + ib) \quad \text{with } a \perp b \text{ real and unit, and } \tau > 0.$$

Conjugate $\Delta_{\mathbf{e}}$ with $e^{\zeta \cdot x}$ to get

$$P_\zeta := e^{-\zeta \cdot x} \Delta_{\mathbf{e}} e^{\zeta \cdot x} = \Delta_{\mathbf{e}} + 2\zeta \cdot \nabla.$$

We write $\langle \tau \rangle = (1 + \tau^2)^{1/2}$, $\langle \zeta \rangle = (1 + |\zeta|^2)^{1/2}$ in what follows.

Lemma 2.4. *For $v \in H^2(\mathbb{R}^n)$ supported in a fixed ball, and for every $\tau > 0$, we have*

$$(2.6) \quad \|v\|_{L^2} + \langle \tau \rangle^{-1} \|\nabla v\|_{L^2} + \langle \tau \rangle^{-2} \|\nabla^2 v\|_{L^2} \leq C \langle \tau \rangle^{-1} \|P_\zeta v\|_{L^2}.$$

All norms are over $\mathbb{R}^n$, and C depends only on the fixed ball and n . In particular,

$$\|v\|_{H^2} \leq C \|P_\zeta v\|_{L^2}, \quad 0 < \tau \leq 1.$$

Proof. Choose a fixed ball whose interior contains the prescribed support ball. It suffices to consider smooth v compactly supported in this larger ball. We use the Carleman estimate in [9, Theorem 4.1], see also [11] and [26, Theorem 5.3 and its proof], applied to $e^{i\tau b \cdot x} v$ with the linear Carleman weight $-a \cdot x$ and small parameter $h = \tau^{-1}$, to get

$$(2.7) \quad \|v\|_{L^2} \leq C \tau^{-1} \|P_\zeta v\|_{L^2}.$$

Here the potential is zero, so this estimate holds for every $\tau > 0$. For completeness, we provide the short proof in this case following the proof of [26, Theorem 5.3]. Set $w = e^{i\tau b \cdot x} v$. Then

$$e^{i\tau b \cdot x} P_\zeta v = (\Delta + \tau^2 + 2\tau a \cdot \nabla) w.$$

The self-adjoint operator $\Delta + \tau^2$ commutes with the skew-adjoint operator $2\tau a \cdot \nabla$, thus integration by parts yields

$$\|P_\zeta v\|_{L^2}^2 = \|(\Delta + \tau^2)w\|_{L^2}^2 + 4\tau^2 \|a \cdot \nabla w\|_{L^2}^2.$$

The Poincaré inequality along lines parallel to a gives $\|a \cdot \nabla w\|_{L^2}^2 \geq C^{-1} \|w\|_{L^2}^2 = C^{-1} \|v\|_{L^2}^2$, with C depending only on the fixed ball. This proves (2.7).

The constant in (2.7) is independent of the orthonormal pair a, b , by rotational invariance of the Euclidean Laplacian. For $0 < \tau \leq 1$, set $u = e^{x \cdot \zeta} v$, so that $\Delta u = e^{x \cdot \zeta} P_\zeta v$. The functions $e^{\pm x \cdot \zeta}$ are uniformly bounded on the chosen ball in this range. Let $\lambda_1 > 0$ be the first eigenvalue of the Dirichlet Laplacian $-\Delta$ on that ball. Since u vanishes near its boundary, the spectral theorem yields

$$\|v\|_{L^2} \leq C \|u\|_{L^2} \leq C \lambda_1^{-1} \|\Delta u\|_{L^2} \leq C \|P_\zeta v\|_{L^2}.$$

Combining the two bounds, we obtain, for all $\tau > 0$,

$$\|v\|_{L^2} \leq C \langle \tau \rangle^{-1} \|P_\zeta v\|_{L^2}.$$

Integration by parts or using the Fourier transform yields

$$(2.8) \quad \|\nabla v\|_{L^2}^2 \leq \|v\|_{L^2} \|\Delta v\|_{L^2}, \quad \|\nabla^2 v\|_{L^2} = \|\Delta v\|_{L^2}.$$

Since $\Delta v = P_\zeta v - 2\zeta \cdot \nabla v$ and $|\zeta| = \sqrt{2}\tau$,

$$\begin{aligned} \|\Delta v\|_{L^2} &\leq \|P_\zeta v\|_{L^2} + 2\sqrt{2}\tau \|\nabla v\|_{L^2} \\ &\leq \|P_\zeta v\|_{L^2} + \frac{1}{2} \|\Delta v\|_{L^2} + 4\tau^2 \|v\|_{L^2}, \end{aligned}$$

where we used (2.8) and the inequality $ab \leq (a^2 + b^2)/2$. Absorbing the Laplacian term and using the L^2 estimate for v , we obtain, for all $\tau > 0$,

$$\|\nabla^2 v\|_{L^2} = \|\Delta v\|_{L^2} \leq C (\|P_\zeta v\|_{L^2} + \tau^2 \|v\|_{L^2}) \leq C \langle \tau \rangle \|P_\zeta v\|_{L^2}.$$

Inequality (2.8) then implies

$$\|\nabla v\|_{L^2}^2 \leq \|v\|_{L^2} \|\Delta v\|_{L^2} \leq C \|P_\zeta v\|_{L^2}^2.$$

Combining these estimates proves (2.6). $\square$

We construct our g -harmonic solutions now. We continue to use the convention $|\zeta| = \sqrt{2}\tau$.

Proposition 2.5. *For ζ such that $\zeta \cdot \zeta = 0$, let u solve*

$$\Delta_g u = 0 \quad \text{in } M, \quad u|_{\partial M} = e^{x \cdot \zeta}.$$

Then there exist constants $C_0 > 0$, $C > 0$, depending only on M and n , so that $u = e^{x \cdot \zeta}(1 + r)$ with

$$(2.9) \quad \|r\|_{L^2(M)} + \langle \tau \rangle^{-1} \|\nabla r\|_{L^2(M)} + \langle \tau \rangle^{-2} \|\nabla^2 r\|_{L^2(M)} \leq C \frac{\tau^2}{\langle \tau \rangle} \|B\|_{L^2(M)}$$

when

$$\|B\|_{L^\infty} < C_0, \quad 0 < \tau < C_0 / \|B\|_{L^\infty}.$$

Moreover, extending u by $e^{x \cdot \zeta}$ outside M results in a smooth g -harmonic function U on $\mathbb{R}^n$. The corresponding zero extension of r satisfies $r = e^{-x \cdot \zeta} U - 1 \in C_0^\infty(\mathbb{R}^n)$ and $\text{supp } r \subset M$.

Proof. Set $E(x) = e^{x \cdot \zeta}$, so that $\Delta E = 0$. Since $g = \mathbf{e}$ on ∂M , the unit normals for the two metrics agree there. By the DN equality in (1.1), we get

$$\partial_\nu u = \Lambda_g(E|_{\partial M}) = \Lambda_{\mathbf{e}}(E|_{\partial M}) = \partial_{\nu_{\mathbf{e}}} E \quad \text{on } \partial M.$$

Extend u to U on $\mathbb{R}^n$ by setting $U = E$ outside M . The conormal flux $\sqrt{\det g} g^{ij} \nu_i \partial_j u$ equals $\partial_\nu u$ on ∂M , since $g = \mathbf{e}$ there. Thus the boundary values and conormal fluxes match across ∂M , so integration by parts shows that $\Delta_g U = 0$ in the distributional sense in $\mathbb{R}^n$. Elliptic regularity implies that U is smooth. Therefore

$$r = e^{-x \cdot \zeta} U - 1 \in C_0^\infty(\mathbb{R}^n), \quad \text{supp } r \subset M.$$

Conjugating (2.3) by E and using $\zeta \cdot \zeta = 0$ and the symmetry of B , we obtain on $\mathbb{R}^n$

$$(2.10) \quad P_\zeta r = -B^{ij} \zeta_i \zeta_j - B^{ij} (\partial_i \partial_j r + 2\zeta_i \partial_j r + \zeta_i \zeta_j r).$$

Let $X_\tau(r)$ be the norm on the left-hand side of (2.9), where all norms are over $\mathbb{R}^n$. Since $|\zeta| = \sqrt{2}\tau$, equation (2.10) implies

$$\begin{aligned} \|P_\zeta r\|_{L^2} &\leq C\tau^2 \|B\|_{L^2} + C\|B\|_{L^\infty} (\|\nabla^2 r\|_{L^2} + \tau \|\nabla r\|_{L^2} + \tau^2 \|r\|_{L^2}) \\ &\leq C\tau^2 \|B\|_{L^2} + C\langle \tau \rangle^2 \|B\|_{L^\infty} X_\tau(r). \end{aligned}$$

Apply Lemma 2.4 in a fixed ball containing M to the solution r of (2.10). For every $\tau > 0$ it follows that

$$X_\tau(r) \leq C\langle \tau \rangle^{-1} \|P_\zeta r\|_{L^2} \leq C \frac{\tau^2}{\langle \tau \rangle} \|B\|_{L^2} + C\langle \tau \rangle \|B\|_{L^\infty} X_\tau(r).$$

Choose $C_0 > 0$ so that $2CC_0 \leq 1/2$. If $\|B\|_{L^\infty} < C_0$ and $\tau\|B\|_{L^\infty} < C_0$, required by Proposition 2.5, then

$$C\langle \tau \rangle \|B\|_{L^\infty} < 2CC_0 \leq \frac{1}{2}.$$

Thus the last term can be absorbed into the left-hand side, yielding

$$X_\tau(r) \leq 2C \frac{\tau^2}{\langle \tau \rangle} \|B\|_{L^2}.$$

Both r and B are supported in M , so their norms over $\mathbb{R}^n$ equal their norms over M . This proves (2.9). $\square$

By Lemma 2.1 and the Sobolev product and embedding estimates, choosing ε_0 in Theorem 1 sufficiently small ensures $\|B\|_{L^\infty} < C_0$, which we use for the rest of the argument.

We return to the orthogonal identity (2.5). Let $u_\zeta = e^{x \cdot \zeta}(1 + r_\zeta)$ be the solution provided by Proposition 2.5. Choose η with $\eta \cdot \eta = 0$ so that $\zeta + \eta = -i\xi$, $\xi \in \mathbb{R}^n$; see (2.12) below for a specific choice. Taking $u_e = e^{x \cdot \eta}$, we get

$$\int_M e^{-ix \cdot \xi} B^{ij}(x) \eta_i \eta_j \, dx = - \int_M e^{-ix \cdot \xi} B^{ij}(x) \eta_i \eta_j r_\zeta(x) \, dx.$$

Since $\operatorname{Re} \eta = -\operatorname{Re} \zeta$ and both vectors are null, $|\eta| = |\zeta| = \sqrt{2}\tau$. By the Cauchy inequality,

$$\left| \int_M e^{-ix \cdot \xi} B^{ij}(x) \eta_i \eta_j \, dx \right| \leq |\eta|^2 \|B\|_{L^2(M)} \|r_\zeta\|_{L^2(M)}.$$

Using (2.9) and $\tau^2 / \langle \tau \rangle \leq \tau$, we obtain, under the conditions on ξ , η above,

$$(2.11) \quad |\hat{B}^{ij}(\xi) \eta_i \eta_j| \leq C |\eta|^3 \|B\|_{L^2(M)}^2 \quad \text{for } 0 < |\eta| < \frac{\sqrt{2} C_0}{\|B\|_{L^\infty}}.$$

2.4. Estimating the full tensor. The divergence-free condition (2.2) implies $\hat{B}(\xi)\xi = 0$. Our first goal is to combine this with polarization identity techniques to estimate all components of $\hat{B}$ in (2.11).

Write $\xi = \rho\omega$, $|\omega| = 1$. Suppose $\rho > 0$ and take $\tau = \rho$. Following a now standard construction [30], choose orthonormal $a, b \in \omega^\perp$, which is possible since $n \geq 3$, and set

$$(2.12) \quad \eta_\pm = \rho \left(-a + i \left(-\frac{\omega}{2} \pm \frac{\sqrt{3}}{2} b \right) \right), \quad \zeta_\pm = \rho \left(a + i \left(-\frac{\omega}{2} \mp \frac{\sqrt{3}}{2} b \right) \right).$$

These vectors satisfy $\zeta_\pm + \eta_\pm = -i\xi$ and $\eta_\pm \cdot \eta_\pm = \zeta_\pm \cdot \zeta_\pm = 0$, with the parameter $\tau = \rho$. Using the divergence-free condition $\hat{B}\omega = 0$, we get

$$\hat{B}(\eta_\pm/\rho, \eta_\pm/\rho) = \hat{B}(a, a) - \frac{3}{4} \hat{B}(b, b) \mp i\sqrt{3} \hat{B}(a, b),$$

where, for fixed ξ , we regard $\hat{B}$ as the complex symmetric bilinear form $\hat{B}(v, w) := \hat{B}^{ij}(\xi) v_i w_j$. For each fixed $\xi = \rho\omega \neq 0$ in the allowed range, $0 < |\xi| = \rho < C_0/\|B\|_{L^\infty}$, we solve the following system for the entries of $\hat{B}(\xi)$. Since $|\eta_\pm| = \sqrt{2}\rho$, dividing (2.11) by ρ^2 yields

$$\left| \hat{B}(\eta_\pm/\rho, \eta_\pm/\rho) \right| \leq C \rho \|B\|_{L^2(M)}^2,$$

uniformly in ρ, ω , and the orthonormal pair a, b . Taking the sum and difference estimates $\hat{B}(a, a) - \frac{3}{4} \hat{B}(b, b)$ and $\hat{B}(a, b)$ by the same bound. Exchanging a, b also estimates $\hat{B}(b, b) - \frac{3}{4} \hat{B}(a, a)$. The system for the two diagonal entries has a constant coefficient non-singular matrix. Its inverse is therefore bounded independently of ξ and the choice of a, b . Taking pairs from an orthonormal basis of $\omega^\perp$, and using $\hat{B}(\xi)\omega = 0$, bounds all entries of $\hat{B}(\xi)$ in an orthonormal basis of $\mathbb{R}^n$. Since the Frobenius tensor norm is invariant under orthogonal changes of basis, we obtain

$$(2.13) \quad |\hat{B}(\xi)| \leq C |\xi| \|B\|_{L^2(M)}^2 \quad \text{when } 0 < |\xi| < C_0/\|B\|_{L^\infty}$$

(but we can also use the component-wise norm here). The constant in (2.13) is uniform as $\xi \rightarrow 0$: the remainder estimate (2.9) is uniform for small $\tau = \rho$, and the system above introduces no further dependence on ρ . By continuity, (2.13) extends to $\xi = 0$, giving $\hat{B}(0) = 0$.

If we ignore for a moment the different norms of B appearing above, this estimate is not very useful at its limiting frequency $|\xi| = C_0/\|B\|_{L^\infty}$. It does imply $\hat{B}(0) = 0$ however and is meaningful

at intermediate frequencies. What we do below is to use a frequency cutoff equal to a fractional power of $C/\|B\|_{L^2}$, and control the higher frequencies by the a priori bound on $\|B\|_{H^s}$.

2.5. Splitting the frequencies. Since B is flat at ∂M , its extension as zero is smooth. For the integer s fixed in Theorem 1, the Sobolev product estimates and Lemma 2.1 therefore give

$$\|B\|_{H^s(\mathbb{R}^n)} \leq C\|\bar{g} - \mathbf{e}\|_{H^s(M)} \leq C\|h\|_{H^s(M)},$$

where C is uniform for ε_0 sufficiently small. Thus, by (1.1), we may set

$$m = \|B\|_{L^2}, \quad \text{then} \quad m \leq \|B\|_{H^s} \leq C'\varepsilon_0,$$

where $C' > 0$ depends only on M and s . All norms below are over $\mathbb{R}^n$. Assume $m > 0$, since otherwise $B = 0$ and there is nothing to prove. Since $s - 1 > n/2$, by Sobolev embedding and interpolation,

$$\|B\|_{L^\infty} \leq C\|B\|_{H^{s-1}} \leq C\|B\|_{L^2}^{1/s}\|B\|_{H^s}^{1-1/s} \leq Cm^{1/s}\varepsilon_0^{1-1/s}.$$

Thus any R satisfying

$$0 < R < cm^{-1/s}\varepsilon_0^{-(1-1/s)},$$

where $c > 0$ depends only on M, s and is sufficiently small, satisfies $R\|B\|_{L^\infty} < C_0$. Hence (2.13) holds for $|\xi| \leq R$ for such R . We split the frequencies at $|\xi| = R$ now; for a similar argument, see [1] but the difference here is that estimate (2.13) is quadratic in B :

$$m^2 = (2\pi)^{-n} \left(\int_{|\xi| \leq R} |\widehat{B}(\xi)|^2 d\xi + \int_{|\xi| > R} |\widehat{B}(\xi)|^2 d\xi \right).$$

For the low frequencies, by (2.13),

$$(2\pi)^{-n} \int_{|\xi| \leq R} |\widehat{B}(\xi)|^2 d\xi \leq Cm^4 \int_{|\xi| \leq R} |\xi|^2 d\xi \leq CR^{n+2}m^4.$$

For the high frequencies, the a priori Sobolev bound implies

$$(2\pi)^{-n} \int_{|\xi| > R} |\widehat{B}(\xi)|^2 d\xi \leq R^{-2s} (2\pi)^{-n} \int_{\mathbb{R}^n} |\xi|^{2s} |\widehat{B}(\xi)|^2 d\xi \leq CR^{-2s} \|B\|_{H^s}^2 \leq C\varepsilon_0^2 R^{-2s}.$$

Combining the two estimates, we obtain

$$(2.14) \quad m^2 \leq CR^{n+2}m^4 + C\varepsilon_0^2 R^{-2s}, \quad 0 < R < cm^{-1/s}\varepsilon_0^{-(1-1/s)},$$

where C and c are independent of R, m , and ε_0 .

2.6. Optimizing the frequency cutoff and completing the proof. It remains to choose an admissible cutoff R in (2.14) and complete the proof. Choose

$$R = L(\varepsilon_0/m)^{1/s},$$

where $L \geq \max\{1, (4C)^{1/(2s)}\}$ is fixed, with C the constant in (2.14). The second term in (2.14) satisfies

$$C\varepsilon_0^2 R^{-2s} = CL^{-2s} m^2 \leq \frac{1}{4} m^2.$$

Since $m \leq C'\varepsilon_0$ and $2 - (n+2)/s > 0$ by the choice of s in Theorem 1, the first term in (2.14) satisfies, enlarging C if necessary,

$$CR^{n+2}m^4 = CL^{n+2}\varepsilon_0^{(n+2)/s} m^{2-(n+2)/s} m^2 \leq CL^{n+2}\varepsilon_0^2 m^2.$$

For this choice of R , the upper bound in the second inequality in (2.14) is equivalent to $L\varepsilon_0 < c$. We first fix L as above and then choose ε_0 sufficiently small that

$$CL^{n+2}\varepsilon_0^2 \leq \frac{1}{4}, \quad L\varepsilon_0 < c,$$

where C is the constant in the low-frequency estimate. The cutoff is then allowed in (2.14), and both terms in (2.14) are at most $m^2/4$. Hence (2.14) yields $m^2 \leq m^2/2$, contradicting $m > 0$. Thus $B = 0$ and $g = \mathbf{e}$ in the harmonic gauge, completing the proof of Theorem 1.

3. PROOF OF THEOREM 2

Preliminaries. We fix M, g_0, s as in Theorem 2 and assume that g is a smooth Riemannian metric satisfying (1.3) and sufficiently close to g_0 in $H^s(M)$.

For a symmetric matrix field A , write $L_A u = \partial_i(A^{ij}\partial_j u)$. The conductivity associated with a metric g is

$$A_g = \sqrt{\det g} g^{-1}, \quad L_{A_g} = \sqrt{\det g} \Delta_g.$$

For the background $g_0 = e^{2c}\mathbf{e}$, set

$$a = e^{(n-2)c/2}, \quad A_0 := A_{g_0} = a^2 \text{Id}.$$

Extend a smoothly to a neighborhood of M where it remains positive, and choose a smooth cutoff $0 \leq \chi \leq 1$ compactly supported in that neighborhood with $\chi = 1$ near M . Replace the extension by $1 + \chi(a - 1)$ in that neighborhood and by 1 outside it. We continue to denote the resulting positive extension by a , and extend g_0 by $a^{4/(n-2)}\mathbf{e}$. Then $a = 1$ outside a fixed ball, and

$$(3.1) \quad q = \frac{\Delta a}{a} \in C_0^\infty(\mathbb{R}^n),$$

where Δ is the Euclidean Laplacian.

As in the proof of Lemma 2.2, the DN equality (1.3) implies equality of the induced boundary metrics. The integration by parts used in Section 2.2 therefore yields

$$(3.2) \quad \int_M (A_g - A_0)^{ij} \partial_i u \partial_j v dx = 0$$

for $u, v \in C^\infty(M)$ satisfying $\Delta_g u = 0$ and $\Delta_{g_0} v = 0$ in M .

3.1. The harmonic embedding. In Section 2.1, the identity map provides global harmonic coordinates for the Euclidean background, and Lemma 2.1 constructs the gauge by taking their g -harmonic extensions. For a general background, we use a harmonic embedding into a higher-dimensional Euclidean space. The following lemma applies to any (smooth) background metric.

Lemma 3.1. *Let g_0 be a Riemannian metric on M . There exist an open neighborhood $U \subset \mathbb{R}^n$ of M , a smooth Riemannian extension of g_0 to U , an integer $N \geq n$, and real-valued functions $F^1, \dots, F^N \in C^\infty(U)$ such that*

$$\Delta_{g_0} F^\alpha = 0 \quad \text{in } U, \quad \alpha = 1, \dots, N,$$

and $F = (F^1, \dots, F^N): U \rightarrow \mathbb{R}^N$ is a smooth embedding, that is, an injective immersion which is a homeomorphism onto its image.

Proof. Extend g_0 smoothly to a connected open neighborhood V of M , and choose a compact set $K \subset V$ with $M \subset \text{int } K$. By [12, Section 2, Lemma 4], finitely many real-valued g_0 -harmonic functions on V define a map F which is injective on K and satisfies $\text{rank } DF = n$ there. Since K is compact, $F|_K$ is a homeomorphism onto its image. Restricting to $U = \text{int } K$ proves the lemma. $\square$

Note that one can take $N = 2n + 1$ by the main result in [12]. The embedding $F|_M$ can also be viewed as a finite-dimensional projection of the Poisson embedding in [17], obtained by selecting finitely many boundary data.

3.2. The harmonic gauge. The next lemma extends Lemma 2.1. In the Euclidean case, the harmonic extension of the identity is the change of coordinates itself. The g -harmonic extension of the embedding F from Lemma 3.1 may not take values in $F(U)$, so we first project it onto $F(U)$ before applying F^{-1} . After changing coordinates, the remaining discrepancy is normal to $F(U)$; this is the field Z below. For $F = \text{Id}$ and $N = n$, the projection is the identity and $Z = 0$.

Lemma 3.2. *Let F be the harmonic embedding of Lemma 3.1 for g_0 . For g sufficiently close to g_0 in $H^s(M)$, there is a diffeomorphism $\Phi: M \rightarrow M$ fixing ∂M pointwise such that, writing*

$$(3.3) \quad \bar{g} = \Phi_* g, \quad B = a^{-2}(A_{\bar{g}} - A_0),$$

there exists $Z \in C^\infty(M; \mathbb{R}^N)$ satisfying

$$(3.4) \quad L_{A_{\bar{g}}}(F + Z) = 0, \quad Z|_{\partial M} = 0, \quad \partial_k F \cdot Z = 0, \quad k = 1, \dots, n.$$

Moreover,

$$(3.5) \quad \begin{aligned} \|\Phi - \text{Id}\|_{H^{s+1}} + \|B\|_{H^s} + \|Z\|_{H^{s+1}} &\leq C\|g - g_0\|_{H^s}, \\ \|Z\|_{H^1} &\leq C\|B\|_{L^2}. \end{aligned}$$

All norms in this lemma are over M , and C depends on M, g_0, s and the fixed embedding F .

Proof. For $\alpha = 1, \dots, N$, let F_g^α solve

$$\Delta_g F_g^\alpha = 0 \quad \text{in } M, \quad F_g^\alpha|_{\partial M} = F^\alpha|_{\partial M}.$$

Elliptic regularity up to the boundary implies that $F_g \in C^\infty(M; \mathbb{R}^N)$. Since $L_{A_0} F = 0$, the difference $F_g - F$ satisfies

$$L_{A_0}(F_g - F) = -\text{div}((A_g - A_0)\nabla F_g), \quad (F_g - F)|_{\partial M} = 0.$$

Standard elliptic Dirichlet estimates and the algebra property of $H^s(M)$ imply

$$\begin{aligned} \|F_g - F\|_{H^{s+1}} &\leq C\|(A_g - A_0)\nabla F_g\|_{H^s} \\ &\leq C\|A_g - A_0\|_{H^s}(1 + \|F_g - F\|_{H^{s+1}}), \end{aligned}$$

where we used that F is fixed and smooth. For g sufficiently close to g_0 , we can absorb the last term and obtain

$$(3.6) \quad \|F_g - F\|_{H^{s+1}} \leq C\|A_g - A_0\|_{H^s} \leq C\|g - g_0\|_{H^s}.$$

By Sobolev embedding,

$$\|F_g - F\|_{C^0(M)} \leq C\|g - g_0\|_{H^s}.$$

Choose a neighborhood $\mathcal{T}$ of $F(M)$ in $\mathbb{R}^N$ on which the Euclidean nearest-point projection $\pi: \mathcal{T} \rightarrow F(U)$ is smooth. Compactness of $F(M)$ and the preceding estimate ensure that $F_g(M) \subset \mathcal{T}$ for g sufficiently close to g_0 . We can therefore define

$$(3.7) \quad \Phi = F^{-1} \circ \pi \circ F_g: M \rightarrow U.$$

Since $F_g = F$ on ∂M , the map Φ fixes the boundary pointwise. The smooth composition estimate in [16, Proposition 2.20], together with $(F^{-1} \circ \pi) \circ F = \text{Id}$, yields

$$(3.8) \quad \begin{aligned} \|\Phi - \text{Id}\|_{H^{s+1}} &= \|(F^{-1} \circ \pi) \circ F_g - (F^{-1} \circ \pi) \circ F\|_{H^{s+1}} \\ &\leq C\|F_g - F\|_{H^{s+1}}. \end{aligned}$$

To apply that estimate, one may extend $F^{-1} \circ \pi$ smoothly to $\mathbb{R}^N$ using a cutoff equal to one on a smaller neighborhood of $F(M)$ containing $F_g(M)$.

By (3.6), (3.8), and Sobolev embedding,

$$\|\Phi - \text{Id}\|_{C^1(M)} \leq C\|g - g_0\|_{H^s(M)}.$$

For g sufficiently close to g_0 , this bound is less than $1/2$. Since Φ fixes ∂M pointwise, the argument of Lemma 2.1 proves that Φ is a diffeomorphism of M onto itself, smooth with smooth inverse up to the boundary.

To apply the estimates of [16] on $\mathbb{R}^n$, extend $\Phi - \text{Id}$ by a fixed bounded Sobolev extension operator and a cutoff. Its small H^{s+1} norm ensures that adding Id defines a global diffeomorphism restricting to Φ on M . The composition and inversion estimates, together with $\Phi^{-1} - \text{Id} = -(\Phi - \text{Id}) \circ \Phi^{-1}$, imply

$$\|\Phi^{-1} - \text{Id}\|_{H^{s+1}} \leq C\|\Phi - \text{Id}\|_{H^{s+1}} \leq C\|g - g_0\|_{H^s}.$$

To estimate B , use the conductivity transformation law

$$A_{\bar{g}} = \left(\frac{D\Phi A_g D\Phi^\top}{\det D\Phi} \right) \circ \Phi^{-1}.$$

By (3.3), this implies

$$a^2 B = \left(\frac{D\Phi A_g D\Phi^\top}{\det D\Phi} - A_0 \right) \circ \Phi^{-1} + A_0 \circ \Phi^{-1} - A_0.$$

Since $\det D\Phi \geq 2^{-n}$ and a is fixed, smooth, and positive, the Sobolev product and composition estimates in [16, Lemmas 2.7 and 2.11, Proposition 2.20] yield

$$\begin{aligned} \|B\|_{H^s} &\leq C(\|A_g - A_0\|_{H^s} + \|D\Phi - \text{Id}\|_{H^s} + \|\Phi^{-1} - \text{Id}\|_{H^s}) \\ &\leq C(\|g - g_0\|_{H^s} + \|\Phi - \text{Id}\|_{H^{s+1}}) \\ &\leq C\|g - g_0\|_{H^s}. \end{aligned}$$

Now set

$$(3.9) \quad Z = F_g \circ \Phi^{-1} - F.$$

Invariance of the Laplace–Beltrami operator under diffeomorphisms implies $L_{A_{\bar{g}}}(F + Z) = 0$. Also, $Z = 0$ on ∂M because $F_g = F$ there and Φ fixes the boundary. By (3.7),

$$\pi(F_g(x)) = F(\Phi(x)).$$

The defining property of the nearest-point projection therefore implies

$$F_g(x) - F(\Phi(x)) \perp T_{F(\Phi(x))}F(U).$$

Setting $y = \Phi(x)$ and using (3.9), we obtain

$$Z(y) \perp T_{F(y)}F(U) = \text{span}\{\partial_1 F(y), \dots, \partial_n F(y)\},$$

which proves the normality condition in (3.4). Moreover,

$$Z = (F_g - F) \circ \Phi^{-1} + F \circ \Phi^{-1} - F.$$

The Sobolev composition estimates, (3.6), and the bound for $\Phi^{-1} - \text{Id}$ imply

$$\|Z\|_{H^{s+1}} \leq C(\|F_g - F\|_{H^{s+1}} + \|\Phi^{-1} - \text{Id}\|_{H^{s+1}}) \leq C\|g - g_0\|_{H^s}.$$

Finally, $L_{A_0}F = 0$ and (3.4) imply

$$(3.10) \quad L_{A_{\bar{g}}}Z = -\text{div}(a^2 B \nabla F), \quad Z|_{\partial M} = 0.$$

Multiply by Z and integrate by parts to get

$$\|\nabla Z\|_{L^2}^2 \leq C \left| \int_M a^2 B^{ij} \partial_j F \cdot \partial_i Z dx \right| \leq C \|B\|_{L^2} \|\nabla Z\|_{L^2}.$$

The Poincaré inequality proves the H^1 bound in (3.5). $\square$

3.3. Boundary jets.

Lemma 3.3. *The tensor $\bar{g} - g_0$ is flat at ∂M .*

Proof. As in Lemma 2.2, the boundary determination in [20, Proposition 1.3] provides a local diffeomorphism ψ in a collar fixing ∂M pointwise, such that $\hat{g} = \psi^* g$ and g_0 have equal boundary jets. Set $W = F_g \circ \psi - F$. Then

$$(3.11) \quad \begin{aligned} \Delta_{\hat{g}} W^\alpha &= (\Delta_{g_0} - \Delta_{\hat{g}}) F^\alpha, \\ W^\alpha|_{\partial M} &= \partial_{\nu_{\hat{g}}} W^\alpha|_{\partial M} = 0, \quad \alpha = 1, \dots, N. \end{aligned}$$

The Neumann data in (3.11) follow from DN equality (1.3) and $\hat{g} = g_0$ on ∂M . The right side is flat; ellipticity makes the boundary noncharacteristic, so the induction in Lemma 2.2 shows that W is flat.

The new step in the proof is to use the projection in (3.7):

$$\Phi \circ \psi = F^{-1} \circ \pi \circ (F + W).$$

Since $F^{-1} \circ \pi$ is smooth near $F(M)$ and $F^{-1} \circ \pi \circ F = \text{Id}$, flatness of W at ∂M implies that $\Phi \circ \psi - \text{Id}$ is flat there as well. Since $(\Phi \circ \psi)_* \hat{g} = \bar{g}$, the claim follows. $\square$

Lemma 3.3 allows us to extend $\bar{g}$ by g_0 outside M and B by zero, so that $B \in C_0^\infty(\mathbb{R}^n)$. By (3.5),

$$(3.12) \quad \|B\|_{H^s(\mathbb{R}^n)} \leq C \|g - g_0\|_{H^s(M)}.$$

The following estimate replaces the divergence-free identity (2.2).

Lemma 3.4. *For B as above,*

$$(3.13) \quad \|\text{div } B\|_{L^2(\mathbb{R}^n)} \leq C \|B\|_{L^2(\mathbb{R}^n)}.$$

Proof. We estimate $\text{div}(a^2 B)$ by taking the tangential components of the equation for Z ; the normality condition eliminates the second derivatives of Z from those components.

We have in M ,

$$(3.14) \quad L_{A_{\bar{g}}} Z = -L_{A_{\bar{g}}} F = -\partial_i(a^2 B^{ij} \partial_j F) = -\partial_i(a^2 B^{ij}) \partial_j F - a^2 B^{ij} \partial_i \partial_j F,$$

where in the first equality we used Lemma 3.2, and in the second the definition of B and the fact that F is g_0 -harmonic. We also have

$$(3.15) \quad L_{A_{\bar{g}}} \partial_k F = (\partial_i A_0^{ij} + \partial_i(a^2 B^{ij})) \partial_j \partial_k F + A_{\bar{g}}^{ij} \partial_i \partial_j \partial_k F.$$

Applying $L_{A_{\bar{g}}}$ to $\partial_k F \cdot Z = 0$, we obtain

$$(3.16) \quad 0 = \partial_k F \cdot L_{A_{\bar{g}}} Z + Z \cdot L_{A_{\bar{g}}} \partial_k F + 2A_{\bar{g}}^{ij} \partial_i \partial_k F \cdot \partial_j Z.$$

Using (3.14), (3.15), and (3.16), and reordering terms, we find that

$$\begin{aligned} (\partial_k F \cdot \partial_j F - Z \cdot \partial_j \partial_k F) \partial_i(a^2 B^{ij}) &= -a^2 B^{ij} \partial_k F \cdot \partial_i \partial_j F \\ &\quad + Z \cdot (\partial_i A_0^{ij} \partial_j \partial_k F + A_{\bar{g}}^{ij} \partial_i \partial_j \partial_k F) \\ &\quad + 2A_{\bar{g}}^{ij} \partial_i \partial_k F \cdot \partial_j Z. \end{aligned}$$

By Lemma 3.1, the matrix $\partial_k F \cdot \partial_j F$ is uniformly positive definite on M . Furthermore, by Lemma 3.2 and the embedding $H^{s+1}(M) \hookrightarrow C^0(M)$, we have

$$\|Z\|_{L^\infty(M)} \leq C\|g - g_0\|_{H^s(M)}.$$

Thus, taking g close enough to g_0 , we see that $\partial_k F \cdot \partial_j F - Z \cdot \partial_j \partial_k F$ is still uniformly positive definite and therefore uniformly invertible. Since $A_{\bar{g}}$ is uniformly bounded and a, F are fixed and smooth, it follows that

$$\|\operatorname{div}(a^2 B)\|_{L^2(\mathbb{R}^n)} \leq C(\|B\|_{L^2(\mathbb{R}^n)} + \|Z\|_{H^1(M)}) \leq C\|B\|_{L^2(\mathbb{R}^n)},$$

where the second inequality follows from Lemma 3.2. Finally, $\operatorname{div} B = a^{-2} \operatorname{div}(a^2 B) - 2B \nabla \log a$, which proves (3.13). $\square$

3.4. The Liouville transform. By (3.3), the extended conductivity is $A_{\bar{g}} = a^2(\operatorname{Id} + B)$ on $\mathbb{R}^n$, and it equals A_0 outside M . Set

$$(3.17) \quad b_2 = \|B\|_{L^2}, \quad b_s = \|B\|_{H^s}, \quad b_\infty = \|B\|_{L^\infty}.$$

All norms below are over $\mathbb{R}^n$ unless another domain is indicated. Since $s - 1 > n/2$, Sobolev embedding implies

$$(3.18) \quad b_\infty + \|\operatorname{div} B\|_{L^\infty} \leq C b_s.$$

Furthermore, the interpolation used in Section 2 yields

$$(3.19) \quad b_\infty \leq C\|B\|_{H^{s-1}} \leq C b_2^{1/s} b_s^{1-1/s}.$$

The three quantities in (3.17) have different roles: b_2 measures the perturbation in its “natural” norm, b_s controls its regularity and high-frequency behavior, and b_∞ will be used to determine the admissible frequency range. As in Proposition 2.5, the continuation argument below will require a condition of the form $\tau b_\infty \leq c_0$; (3.19) will allow us to choose the frequency cutoff within that range.

The substitution $u = v/a$ removes the background conformal factor from the principal part of the equation $L_{A_{\bar{g}}} u = 0$. The background creates the potential q in (3.1), while the additional coefficients are controlled by B .

Lemma 3.5. *Set*

$$(3.20) \quad q_B = a^{-1} \operatorname{div}(B \nabla a) = \nabla \log a \cdot \operatorname{div} B + a^{-1} B^{ij} \partial_i \partial_j a.$$

Then, for $v \in C^\infty(\mathbb{R}^n)$,

$$(3.21) \quad a^{-1} L_{A_{\bar{g}}}(v/a) = (\Delta - q)v + \operatorname{div}(B \nabla v) - q_B v.$$

Furthermore,

$$(3.22) \quad \|\operatorname{div} B\|_{L^2} + \|q_B\|_{L^2} \leq C b_2, \quad \|\operatorname{div} B\|_{L^\infty} + \|q_B\|_{L^\infty} \leq C b_s.$$

The constant C depends only on the fixed data, including the extension of a and the harmonic embedding F .

Proof. Identity (3.20) follows by expanding the divergence. Since $A_0 = a^2 \operatorname{Id}$,

$$(3.23) \quad \frac{1}{a} \operatorname{div} \left(a^2 \nabla \left(\frac{v}{a} \right) \right) = \Delta v - \frac{\Delta a}{a} v.$$

Using that B is symmetric, we also have

$$(3.24) \quad \frac{1}{a} \operatorname{div} \left(a^2 B \nabla \left(\frac{v}{a} \right) \right) = \operatorname{div}(B \nabla v) - \frac{v}{a} \operatorname{div}(B \nabla a).$$

Adding (3.23) and (3.24) proves (3.21). The L^2 bounds in (3.22) follow from

$$\|q_B\|_{L^2} \leq C(\|\operatorname{div} B\|_{L^2} + \|B\|_{L^2})$$

and Lemma 3.4. The same expansion of q_B , together with (3.18), proves the L^∞ bounds. $\square$

Identity (3.21) replaces (2.3) from Section 2: when $a = 1$, we have $q = q_B = 0$, and the Euclidean harmonic gauge makes $\operatorname{div} B = 0$. Here the bounds in (3.22) control the additional first-order and zeroth-order terms without losing a derivative in L^2 .

3.5. Background C.G.O. solutions. We first construct exact g_0 -harmonic solutions to replace the Euclidean exponentials (1.2). We keep q fixed. The next lemma is essentially the construction in [30] written in the form we need it here. The remainder estimates are also proved in [4, Propositions 3.1 and 3.3].

Lemma 3.6. *Fix a closed ball $\mathcal{B}$ containing M in its interior and an integer $\ell \geq 0$. There are constants $T_*, C > 0$ such that, if $\zeta \in \mathbb{C}^n$, $\zeta \cdot \zeta = 0$, and $|\zeta| \geq T_*$, then there is a smooth global g_0 -harmonic function*

$$(3.25) \quad u_\zeta^0 = a^{-1} e^{x \cdot \zeta} (1 + r_\zeta^0)$$

satisfying

$$(3.26) \quad \|r_\zeta^0\|_{C^\ell(\mathcal{B})} \leq \frac{C}{|\zeta|}.$$

The constants depend on the fixed extended background, $\mathcal{B}$, and ℓ , and are uniform in the direction of ζ .

Proof. We use the Faddeev solution operator G_ζ for $P_\zeta = \Delta + 2\zeta \cdot \nabla$, so that $P_\zeta G_\zeta f = f$; see [11, Proposition 4.5.4 and Remark 4.5.12]. Fix $-1 < \sigma < 0$ and an integer $j > \ell + n/2$. Let $H_\sigma^j(\mathbb{R}^n)$ have norm

$$\|f\|_{H_\sigma^j} = \sum_{|\alpha| \leq j} \|\langle x \rangle^\sigma \partial^\alpha f\|_{L^2}, \quad \langle x \rangle = (1 + |x|^2)^{1/2}.$$

The weighted L^2 estimate and the commutation of G_ζ with derivatives imply

$$(3.27) \quad \|G_\zeta f\|_{H_\sigma^j} \leq \frac{C}{|\zeta|} \|f\|_{H_{\sigma+1}^j}.$$

Since $q \in C_0^\infty(\mathbb{R}^n)$ is fixed, multiplication by q maps H_σ^j to $H_{\sigma+1}^j$ continuously, with norm depending on q, j, σ . Thus (3.27) implies that $f \mapsto G_\zeta(qf)$ has operator norm at most $C/|\zeta|$ on H_σ^j . Increasing T_* makes this norm at most $1/2$. We seek r_ζ^0 as the solution of

$$(3.28) \quad r_\zeta^0 = G_\zeta q + G_\zeta(qr_\zeta^0).$$

Equation (3.28) has a Neumann-series solution in $H_\sigma^j(\mathbb{R}^n)$, with

$$(3.29) \quad \|r_\zeta^0\|_{H_\sigma^j} \leq 2\|G_\zeta q\|_{H_\sigma^j} \leq \frac{C}{|\zeta|} \|q\|_{H_{\sigma+1}^j} \leq \frac{C}{|\zeta|},$$

uniformly in the direction of ζ . Applying P_ζ to (3.28) yields $P_\zeta r_\zeta^0 = q(1 + r_\zeta^0)$, and (3.21) with $B = 0$ proves that (3.25) is g_0 -harmonic. Smoothness follows from elliptic regularity. On the fixed ball $\mathcal{B}$, the weighted norm in (3.29) controls the ordinary H^j norm, so Sobolev embedding proves (3.26). $\square$

In the integral identities, the factor $a^2 B$ is paired with two gradients of solutions of the form (3.25). Their leading factors a^{-1} cancel a^2 ; choosing $\zeta + \eta = -i\xi$ then leads to the Fourier transform of B , as in Section 2.2, and as in [30]. The spatial derivative bounds in (3.26) will control the resulting Fourier error operators.

For $\tau \geq 1$, use the norm

$$(3.30) \quad X_\tau(v) = \|v\|_{L^2} + \tau^{-1} \|\nabla v\|_{L^2} + \tau^{-2} \|\nabla^2 v\|_{L^2},$$

equivalent to the norm in (2.9). The following consequence of Lemma 2.4 incorporates the fixed background potential.

Corollary 3.7. *There are constants $T_0 \geq 1$ and $C > 0$, depending only on $\mathcal{B}, n$ and $\|q\|_{L^\infty}$, such that, for $v \in H^2(\mathbb{R}^n)$ supported in $\mathcal{B}$ and $\zeta \in \mathbb{C}^n$ with $\zeta \cdot \zeta = 0$ and $|\zeta| = \sqrt{2}\tau$, we have*

$$(3.31) \quad X_\tau(v) \leq \frac{C}{\tau} \|(P_\zeta - q)v\|_{L^2}, \quad \tau \geq T_0.$$

Proof. For $\tau \geq 1$, Lemma 2.4 and (3.30) imply

$$\begin{aligned} X_\tau(v) &\leq \frac{C}{\tau} \|P_\zeta v\|_{L^2} \\ &\leq \frac{C}{\tau} \|(P_\zeta - q)v\|_{L^2} + \frac{C\|q\|_{L^\infty}}{\tau} X_\tau(v). \end{aligned}$$

Choose T_0 so that $C\|q\|_{L^\infty}/T_0 \leq 1/2$ and absorb the last term to obtain (3.31). $\square$

We now construct the $\bar{g}$ -harmonic extensions of the boundary values of the background C.G.O. solutions of Lemma 3.6, taken with $\ell \geq 2$, as in Proposition 2.5.

Lemma 3.8. *After increasing T_0 from Corollary 3.7, there are constants $c_0, C > 0$, depending only on the fixed data, such that the following holds. Let $\zeta \cdot \zeta = 0$, $|\zeta| = \sqrt{2}\tau$, and assume*

$$(3.32) \quad b_s \leq c_0, \quad \tau \geq T_0, \quad \tau b_\infty \leq c_0.$$

The $\bar{g}$ -harmonic solution u_ζ in M with boundary value u_ζ^0 extends by u_ζ^0 outside M to a smooth global $\bar{g}$ -harmonic function of the form

$$(3.33) \quad u_\zeta = a^{-1} e^{x \cdot \zeta} (1 + r_\zeta^0 + r_\zeta), \quad r_\zeta \in C_0^\infty(\mathbb{R}^n), \quad \text{supp } r_\zeta \subset M,$$

with

$$(3.34) \quad X_\tau(r_\zeta) \leq C\tau b_2.$$

Proof. The Dirichlet solution is smooth up to ∂M by elliptic regularity. The extension argument of Proposition 2.5 applies: (1.3), invariance of the DN map under Φ , and Lemma 3.3 imply that the boundary values and conormal fluxes of u_ζ and u_ζ^0 agree. Thus the extension is $\bar{g}$ -harmonic in the distributional sense across ∂M . The extended metric is smooth by Lemma 3.3, so elliptic regularity makes the extension smooth as well. Consequently,

$$r_\zeta = a e^{-x \cdot \zeta} (u_\zeta - u_\zeta^0) \in C_0^\infty(\mathbb{R}^n), \quad \text{supp } r_\zeta \subset M,$$

which proves (3.33).

Conjugating (3.21) by $e^{x \cdot \zeta}$ and using $(P_\zeta - q)(1 + r_\zeta^0) = 0$, we obtain

$$(3.35) \quad \begin{aligned} (P_\zeta - q)r_\zeta &= -[B^{ij}(\partial_i + \zeta_i)(\partial_j + \zeta_j) \\ &\quad + (\partial_i B^{ij})(\partial_j + \zeta_j) - q_B](1 + r_\zeta^0 + r_\zeta). \end{aligned}$$

Choose T_0 large enough that Lemma 3.6 applies with $\ell \geq 2$. Then (3.26) implies

$$\|1 + r_\zeta^0\|_{L^\infty(M)} \leq C, \quad \|\nabla r_\zeta^0\|_{L^\infty(M)} + \|\nabla^2 r_\zeta^0\|_{L^\infty(M)} \leq C\tau^{-1}.$$

Together with the L^2 bounds in (3.22), these estimates bound the part of the right side of (3.35) acting on $1 + r_\zeta^0$ by $C\tau^2 b_2$. For the part acting on r_ζ , the definition (3.30) bounds the terms with coefficient B by $C\tau^2 b_\infty X_\tau(r_\zeta)$, while the L^∞ bounds in (3.22) control those with coefficients $\operatorname{div} B$ and q_B by $C\tau b_s X_\tau(r_\zeta)$. Therefore,

$$(3.36) \quad \|(P_\zeta - q)r_\zeta\|_{L^2} \leq C\tau^2 b_2 + C(\tau^2 b_\infty + \tau b_s)X_\tau(r_\zeta).$$

Applying Corollary 3.7 to (3.36) yields

$$(3.37) \quad X_\tau(r_\zeta) \leq C\tau b_2 + C(\tau b_\infty + b_s)X_\tau(r_\zeta).$$

Compared with Proposition 2.5, the additional b_s term comes from $\operatorname{div} B$ and q_B ; the fixed potential q has already been absorbed in Corollary 3.7. Choose c_0 so that $2Cc_0 \leq 1/2$. Under (3.32), the last term in (3.37) can then be absorbed, proving (3.34). $\square$

3.6. Fourier estimates. We use the polarization construction in (2.12), retaining the components in the direction ω that vanish in Section 2 but are only estimated here by Lemma 3.4.

Lemma 3.9. *For every $\omega \in \mathbb{S}^{n-1}$, one can choose $L = L(n)$ pairs $(z_\ell, e_\ell) \in \mathbb{C}^n \times \mathbb{C}^n$ such that*

$$z_\ell \cdot z_\ell = e_\ell \cdot e_\ell = 0, \quad z_\ell + e_\ell = -i\omega, \quad |z_\ell| = |e_\ell| = \sqrt{2},$$

and, for every complex symmetric matrix S ,

$$(3.38) \quad |S| \leq C \left(\sum_{\ell=1}^L |S(z_\ell, e_\ell)| + |S\omega| \right).$$

Here $S(z, e) = S^{ij} z_i e_j$, $|S|$ is the Frobenius norm, and C depends only on n . The vectors may be chosen piecewise smoothly in ω .

Proof. Choose an orthonormal basis of $\omega^\perp$. For each ordered pair p, q of distinct basis vectors, use (2.12) with $\rho = 1$, $a = p$, and $b = q$, and write $z_\pm = \zeta_\pm$, $e_\pm = \eta_\pm$. These vectors satisfy the stated null, sum, and norm conditions. The calculation leading to (2.13), with the $S(\omega, \omega)$ term retained, reads

$$S(z_\pm, e_\pm) = -S(p, p) + \frac{3}{4}S(q, q) - \frac{1}{4}S(\omega, \omega) \pm i\sqrt{3}S(p, q).$$

Taking the difference controls $S(p, q)$; taking the sum controls $S(p, p) - \frac{3}{4}S(q, q)$ up to $C|S\omega|$. Interchanging p, q gives the same invertible two-by-two system as in Section 2, so both diagonal entries are controlled as well. Since $n \geq 3$, every basis vector has an orthogonal partner in $\omega^\perp$. Thus all entries of S restricted to $\omega^\perp$ are controlled, while the remaining entries are bounded by $|S\omega|$. This proves (3.38), uniformly in ω . Smooth local orthonormal frames on a finite cover of the sphere provide the asserted piecewise smooth choice. $\square$

For $S = \widehat{B}(\xi)$ and $\omega = \xi/|\xi|$, the additional term in (3.38) is controlled by Lemma 3.4. Indeed, $\widehat{\operatorname{div} B}(\xi) = i\widehat{B}(\xi)\xi$, so Plancherel's identity and (3.13) imply

$$(3.39) \quad \left\| \mathbf{1}_{\{|\xi| > T\}} \widehat{B}(\xi) \frac{\xi}{|\xi|} \right\|_{L^2} \leq \frac{C}{T} \|\operatorname{div} B\|_{L^2} \leq \frac{C}{T} b_2, \quad T > 0.$$

We next estimate the errors in the background C.G.O. amplitudes.

Lemma 3.10. *Let $e(x, \xi) = (e^{ij}(x, \xi))$ be measurable in ξ and C^{n+1} in x , with support in a fixed compact set in x , and suppose that*

$$(3.40) \quad |\partial_x^\alpha e(x, \xi)| \leq C|\xi|^{-1}, \quad |\alpha| \leq n+1, \quad |\xi| \geq T_0.$$

For a square-integrable matrix field B , define

$$(3.41) \quad E_T B(\xi) = \mathbf{1}_{\{|\xi| > T\}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} e^{ij}(x, \xi) B^{ij}(x) dx, \quad T \geq T_0.$$

Then

$$(3.42) \quad \|E_T B\|_{L^2} \leq \frac{C}{T} \|B\|_{L^2},$$

where C depends only on n , the fixed compact support, and the constant in (3.40).

Proof. The formal adjoint of (3.41) satisfies

$$(2\pi)^{-n} (E_T^* f)^{ij} = \text{Op}(\mathbf{1}_{\{|\xi| > T\}} \overline{e^{ij}(\bullet, \xi)}) f,$$

where Op is with the left quantization. We are not claiming that its symbol satisfies the standard symbol estimates. The compact support and (3.40) imply

$$\sup_{\xi} \sum_{|\alpha| \leq n+1} \|\partial_x^\alpha (\mathbf{1}_{\{|\xi| > T\}} \overline{e^{ij}(\bullet, \xi)})\|_{L^1} \leq \frac{C}{T}.$$

Applying [15, Theorem 18.1.11', p. 75] componentwise, followed by Plancherel's identity and duality, proves (3.42). The theorem requires no derivatives in ξ , so the sharp cutoff is allowed. $\square$

We combine the preceding estimates with the frequency splitting used in Section 2.

Lemma 3.11. *After increasing T_0 and decreasing c_0 from Lemma 3.8, if necessary, the conductivity perturbation B in (3.3) satisfies*

$$(3.43) \quad \|\mathbf{1}_{\{|\xi| > T\}} \widehat{B}\|_{L^2} \leq C(R^{n/2+1} b_2^2 + T^{-1} b_2 + R^{-s} b_s)$$

whenever

$$(3.44) \quad b_s \leq c_0, \quad R \geq T \geq T_0, \quad R b_\infty \leq c_0.$$

The constant C depends only on the fixed data and is independent of B, T, R .

Proof. Take the background solutions of Lemma 3.6 with the $C^{n+2}(\mathcal{B})$ bound in (3.26), increasing T_0 accordingly. First let $T < |\xi| \leq R$, write $\xi = \rho\omega$, and choose (z_ℓ, e_ℓ) as in Lemma 3.9. For each pair set $\zeta = \rho z_\ell$ and $\eta = \rho e_\ell$. Then $\zeta + \eta = -i\xi$, both vectors are null, and $|\zeta| = |\eta| = \sqrt{2}\rho$. Conditions (3.44) allow us to apply Lemma 3.8 with $\tau = \rho$.

The identity (3.2), applied to $\bar{g}$ using invariance of the DN map, and (3.3) imply

$$(3.45) \quad \int_M a^2 B^{ij} \partial_i u_\zeta^0 \partial_j u_\eta^0 dx = - \int_M a^2 B^{ij} \partial_i (u_\zeta - u_\zeta^0) \partial_j u_\eta^0 dx.$$

By (3.33),

$$a e^{-x \cdot \zeta} \nabla (u_\zeta - u_\zeta^0) = \nabla r_\zeta + (\zeta - \nabla \log a) r_\zeta.$$

Thus (3.34) and the background estimate (3.26) yield

$$(3.46) \quad \|a e^{-x \cdot \zeta} \nabla (u_\zeta - u_\zeta^0)\|_{L^2(M)} \leq C \rho^2 b_2, \quad \|a e^{-x \cdot \eta} \nabla u_\eta^0\|_{L^\infty(M)} \leq C \rho.$$

In (3.45), the two gradient factors contribute $a^{-2}e^{x \cdot (\zeta + \eta)} = a^{-2}e^{-ix \cdot \xi}$, so a^2 cancels and the remaining exponential has modulus one. Cauchy–Schwarz and (3.46) therefore imply

$$(3.47) \quad \rho^{-2} \left| \int_M a^2 B^{ij} \partial_i u_\zeta^0 \partial_j u_\eta^0 dx \right| \leq C \rho b_2^2.$$

For the background amplitudes, (3.25)–(3.26) imply

$$(3.48) \quad \frac{ae^{-x \cdot \zeta}}{\rho} \nabla u_\zeta^0 = \frac{\zeta}{\rho} (1 + r_\zeta^0) + \frac{\nabla r_\zeta^0 - (1 + r_\zeta^0) \nabla \log a}{\rho} = z_\ell + O_{C^{n+1}(\mathcal{B})}(\rho^{-1}),$$

and similarly with η, e_ℓ . These expansions hold for every $\rho \geq T_0$. Fix $\chi \in C_0^\infty(\mathcal{B}^{\text{int}})$ equal to one near M and set

$$e_\ell^{ij}(x, \xi) = \chi(x) \left[\rho^{-2} a^2 e^{-x \cdot (\zeta + \eta)} \partial_i u_\zeta^0 \partial_j u_\eta^0 - z_{\ell i} e_{\ell j} \right],$$

extending it by zero for $|\xi| < T_0$. By (3.48), this amplitude satisfies (3.40). It is measurable in ξ by the phase choice and the Neumann-series construction in Lemma 3.6. Let $E_{\ell, T}$ be the operator (3.41) with amplitude e_ℓ^{ij} . Since $\text{supp } B \subset M$, for $T < |\xi| \leq R$ we have

$$(3.49) \quad \rho^{-2} \int_M a^2 B^{ij} \partial_i u_\zeta^0 \partial_j u_\eta^0 dx = \widehat{B}(\xi)(z_\ell, e_\ell) + E_{\ell, T} B(\xi).$$

Combining (3.47), (3.49), and Lemma 3.10, we obtain

$$\begin{aligned} \|\mathbf{1}_{\{T < |\xi| \leq R\}} \widehat{B}(\xi)(z_\ell, e_\ell)\|_{L^2} &\leq C b_2^2 \left(\int_{T < |\xi| \leq R} |\xi|^2 d\xi \right)^{1/2} + C T^{-1} b_2 \\ &\leq C(R^{n/2+1} b_2^2 + T^{-1} b_2). \end{aligned}$$

Lemma 3.9 and (3.39) now control all components:

$$(3.50) \quad \|\mathbf{1}_{\{T < |\xi| \leq R\}} \widehat{B}\|_{L^2} \leq C(R^{n/2+1} b_2^2 + T^{-1} b_2).$$

Finally, the Sobolev bound, as in Section 2, implies

$$\|\mathbf{1}_{\{|\xi| > R\}} \widehat{B}\|_{L^2} \leq C R^{-s} \|B\|_{H^s} = C R^{-s} b_s.$$

Adding this to (3.50) proves (3.43). □

3.7. Compactness.

Lemma 3.12. *Let $B_j \neq 0$ be conductivity perturbations as in Lemma 3.11, with the same fixed data, and write $b_{2,j}, b_{s,j}, b_{\infty,j}$ for their norms as in (3.17). If $b_{s,j} \rightarrow 0$, then a subsequence of $H_j = B_j/b_{2,j}$ converges strongly in $L^2(\mathbb{R}^n)$ to a tensor H satisfying*

$$\text{supp } H \subset M, \quad \|H\|_{L^2} = 1.$$

Proof. Since $b_{2,j} \leq b_{s,j}$, the cutoffs

$$(3.51) \quad R_j = b_{s,j}^{-1/(n+2)} \left(\frac{b_{s,j}}{b_{2,j}} \right)^{1/s}$$

tend to infinity. By (3.19),

$$R_j b_{\infty,j} \leq C b_{s,j}^{1-1/(n+2)} \rightarrow 0.$$

Moreover, (3.51) and $s > n/2 + 1$ imply

$$(3.52) \quad \begin{aligned} R_j^{n/2+1} b_{2,j} &= b_{s,j}^{1/2} \left(\frac{b_{2,j}}{b_{s,j}} \right)^{1-(n/2+1)/s} \leq b_{s,j}^{1/2}, \\ R_j^{-s} \frac{b_{s,j}}{b_{2,j}} &= b_{s,j}^{s/(n+2)}. \end{aligned}$$

Fix $T \geq T_0$. For all sufficiently large j , the conditions (3.44) hold with $R = R_j$. Dividing (3.43) by $b_{2,j}$ and using (3.52), we obtain

$$(3.53) \quad \limsup_{j \rightarrow \infty} \|\mathbf{1}_{\{|\xi| > T\}} \widehat{H}_j\|_{L^2} \leq \frac{C}{T}.$$

Since $\|H_j\|_{L^2} = 1$ and $\text{supp } H_j \subset M$, a subsequence converges weakly in $L^2(\mathbb{R}^n)$ to a tensor H supported in M . For each fixed T , the map $f \mapsto \widehat{f}|_{\{|\xi| \leq T\}}$ from $L^2(M)$ to $L^2(\{|\xi| \leq T\})$ has the square-integrable kernel $e^{-ix \cdot \xi}$, so it is Hilbert–Schmidt and hence compact. Therefore, along this same subsequence,

$$\|\mathbf{1}_{\{|\xi| \leq T\}} (\widehat{H}_j - \widehat{H})\|_{L^2} \longrightarrow 0.$$

Weak lower semicontinuity and (3.53) also imply $\|\mathbf{1}_{\{|\xi| > T\}} \widehat{H}\|_{L^2} \leq C/T$. Splitting the frequencies at T thus yields

$$\limsup_{j \rightarrow \infty} \|\widehat{H}_j - \widehat{H}\|_{L^2} \leq \frac{2C}{T}.$$

Letting $T \rightarrow \infty$ and applying Plancherel’s identity proves strong convergence in L^2 . In particular, $\|H\|_{L^2} = 1$. $\square$

3.8. The linearized problem. We identify covariant and contravariant tensors using $\mathbf{e}$ and denote the Euclidean tensor by $\mathbf{e}$ in either case.

For a smooth positive function a , and a vector field X , define the normalized infinitesimal gauge by

$$\mathcal{G}_a X = DX + DX^\top - (\text{div } X + 2X \cdot \nabla \log a) \mathbf{e}.$$

Thus $a^2 \mathcal{G}_a X$ is the corresponding variation of the conductivity $A_0 = a^2 \text{Id}$.

Lemma 3.13. *Let $X \in H_0^1(M; \mathbb{R}^n)$ and let $u, v \in C^\infty(M)$ be g_0 -harmonic. Then*

$$(3.54) \quad \int_M a^2 (\mathcal{G}_a X)^{ij} \partial_i u \partial_j v dx = 0,$$

and

$$(3.55) \quad L_{A_0}(-X \cdot \nabla u) = -\text{div}(a^2 \mathcal{G}_a X \nabla u)$$

in a distributional sense. The same identities hold on $\mathbb{R}^n$ for compactly supported $X \in H^1(\mathbb{R}^n; \mathbb{R}^n)$ and smooth global solutions of $L_{A_0} u = L_{A_0} v = 0$.

Proof. First let X be smooth and compactly supported, and let φ_t be its flow. The pushed-forward conductivity is

$$A_t = \left(\frac{a^2 D \varphi_t D \varphi_t^\top}{\det D \varphi_t} \right) \circ \varphi_t^{-1}, \quad \frac{d}{dt} A_t \Big|_{t=0} = a^2 \mathcal{G}_a X.$$

Differentiating $L_{A_t}(u \circ \varphi_t^{-1}) = 0$ proves (3.55). Testing this identity against v , and using $L_{A_0} v = 0$ and compact support of $X \cdot \nabla u$, proves (3.54). Both identities extend by H^1 density to the stated classes of vector fields. $\square$

We can call $\mathcal{G}_a X$ with $X \in H_0^1(M; \mathbb{R}^n)$ *potential tensor fields*, by analogy with the X-ray geodesic transform of symmetric two-tensor fields.

In Proposition 3.16, testing the linearized identity with background C.G.O. solutions will yield the Fourier condition in the next lemma. Its decomposition writes H as $\mathcal{G}_a X$ plus a scalar tensor, which Lemma 3.15 will eliminate. It will then remain to prove that X vanishes outside M , so that H is a potential tensor.

Lemma 3.14. *Let $H \in L^2(\mathbb{R}^n)$ be a real symmetric tensor with compact support, and assume that for almost every $\xi \neq 0$ the restriction of $\widehat{H}(\xi)$ to $\xi^\perp$ is a scalar multiple of the Euclidean form. Then there are compactly supported $X \in H^1(\mathbb{R}^n; \mathbb{R}^n)$ and $\beta \in L^2(\mathbb{R}^n)$ such that*

$$(3.56) \quad H = DX + DX^\top + \beta \mathbf{e}.$$

Proof. For $\xi = \rho\omega \neq 0$, set $S = \widehat{H}(\xi)$ and define

$$(3.57) \quad \widehat{\beta}(\xi) = \frac{\text{tr } S - S(\omega, \omega)}{n-1}, \quad \widehat{X}(\xi) = \frac{S\omega - \frac{1}{2}(S(\omega, \omega) + \widehat{\beta}(\xi))\omega}{i\rho}.$$

The hypothesis implies $S = i(\xi \otimes \widehat{X} + \widehat{X} \otimes \xi) + \widehat{\beta}\mathbf{e}$, proving (3.56). Moreover, $|\widehat{\beta}| + |\xi||\widehat{X}| \leq C|\widehat{H}|$. Since H has compact support, $\widehat{H}$ is bounded, and $|\xi|^{-2}$ is integrable near zero for $n \geq 3$. These bounds imply $X \in H^1$ and $\beta \in L^2$; the reality of H implies that X, β are real.

Outside a ball containing $\text{supp } H$, taking the trace of (3.56) yields

$$\partial_i X_j + \partial_j X_i = \frac{2}{n}(\text{div } X)\delta_{ij}.$$

Differentiating and combining the three permuted identities gives, distributionally,

$$(3.58) \quad \partial_i \partial_j X_k = \frac{1}{n}(\delta_{jk} \partial_i \text{div } X + \delta_{ik} \partial_j \text{div } X - \delta_{ij} \partial_k \text{div } X).$$

Taking the divergence in k gives $(n-2)\partial_i \partial_j \text{div } X = -\delta_{ij} \Delta \text{div } X$. Tracing this identity shows that $\Delta \text{div } X = 0$, hence $D^2 \text{div } X = 0$. By (3.58), $D^3 X = 0$ there. Thus X is a polynomial of degree at most two on the connected exterior of the ball; since $X \in L^2$, it vanishes there. Equation (3.56) then shows that β vanishes there as well. $\square$

The following lemma is basically the Sylvester–Uhlmann [30] argument but with $\sigma \in L^2$ only.

Lemma 3.15. *Let $a > 0$ be smooth on $\mathbb{R}^n$ and equal to one outside a ball. If $\sigma \in L^2(\mathbb{R}^n)$ has compact support and*

$$(3.59) \quad \int_{\mathbb{R}^n} a^2 \sigma \nabla u \cdot \nabla v \, dx = 0$$

for every pair of smooth global solutions of $L_{A_0} u = L_{A_0} v = 0$, then $\sigma = 0$.

Proof. The product identity $L_{A_0}(uv) = 2a^2 \nabla u \cdot \nabla v$ and (3.59) imply

$$(3.60) \quad \langle a^{-2} L_{A_0} \sigma, (au)(av) \rangle = 0.$$

The distribution in this pairing belongs to H^{-2} and has compact support. Fix $\xi \in \mathbb{R}^n$, choose orthonormal $p, q \in \xi^\perp$, and, for $t > |\xi|/2$, set

$$(3.61) \quad \begin{aligned} \zeta_t &= tp + i \left(-\frac{\xi}{2} + \sqrt{t^2 - \frac{|\xi|^2}{4}} q \right), \\ \eta_t &= -tp + i \left(-\frac{\xi}{2} - \sqrt{t^2 - \frac{|\xi|^2}{4}} q \right). \end{aligned}$$

These vectors are null, have norm $\sqrt{2}t$, and satisfy $\zeta_t + \eta_t = -i\xi$. Apply the background construction of Lemma 3.6 with two derivatives on a ball containing $\text{supp } \sigma$. Its solutions satisfy

$$(au_{\zeta_t}^0)(au_{\eta_t}^0) = e^{-ix \cdot \xi}(1 + r_{\zeta_t}^0)(1 + r_{\eta_t}^0) \longrightarrow e^{-ix \cdot \xi}$$

in C^2 near $\text{supp } \sigma$. Passing to the limit in (3.60) shows that the Fourier transform of $a^{-2}L_{A_0}\sigma$ vanishes. Hence $L_{A_0}\sigma = 0$. Elliptic regularity makes σ smooth, and testing against $\bar{\sigma}$ gives $\int a^2|\nabla\sigma|^2 dx = 0$. Compact support then implies $\sigma = 0$. $\square$

The next proposition identifies the kernel of the linearized conductivity DN map at A_0 with the potential tensors. For smooth H , the bilinear form in (3.62) represents the linearized data in the direction a^2H ; its L^2 formulation is needed for the limit in Lemma 3.12.

Proposition 3.16. *A real symmetric tensor $H \in L^2(M)$ satisfies*

$$(3.62) \quad \int_M a^2 H^{ij} \partial_i u \partial_j v dx = 0$$

for all g_0 -harmonic $u, v \in C^\infty(M)$ if and only if

$$(3.63) \quad H = \mathcal{G}_a X \quad \text{for some } X \in H_0^1(M; \mathbb{R}^n).$$

Proof. The reverse implication is Lemma 3.13. For the forward implication, extend H by zero to $\mathbb{R}^n$. Fix $\xi \neq 0$ and orthonormal $p, q \in \xi^\perp$, and use the background C.G.O. solutions with phases (3.61) in (3.62). By (3.26), uniformly on M ,

$$t^{-1} a e^{-x \cdot \zeta_t} \nabla u_{\zeta_t}^0 \longrightarrow p + iq, \quad t^{-1} a e^{-x \cdot \eta_t} \nabla u_{\eta_t}^0 \longrightarrow -p - iq.$$

Dividing (3.62) by t^2 and letting $t \rightarrow \infty$ therefore yields

$$(3.64) \quad \widehat{H}(\xi)(p + iq, p + iq) = 0.$$

Replacing q by $-q$ in (3.64) shows that $\widehat{H}(\xi)(p, p) = \widehat{H}(\xi)(q, q)$ and $\widehat{H}(\xi)(p, q) = 0$. Thus the restriction to $\xi^\perp$ is scalar. Lemma 3.14 provides the compactly supported X, β in (3.56), and

$$(3.65) \quad H - \mathcal{G}_a X = \sigma e, \quad \sigma = \beta + \text{div } X + 2X \cdot \nabla \log a.$$

The scalar σ belongs to L^2 and has compact support. Subtracting the global identity (3.54) from (3.62) shows that (3.59) holds, so Lemma 3.15 implies $\sigma = 0$.

It remains to prove that X vanishes outside M , including any bounded complementary components. Let $\varphi \in C_0^\infty(\mathbb{R}^n \setminus M)$ be real and set $\tilde{a} = a e^\varphi$. Since $\tilde{a} = a$ on M , the global solutions for $\tilde{a}^2 \text{Id}$ restrict to g_0 -harmonic functions in M , so (3.62) also holds for this background. The decomposition (3.57) depends only on H and therefore produces the same X, β . Repeating the scalar argument in (3.65) with $\tilde{a}$ gives $H = \mathcal{G}_{\tilde{a}} X$, and hence

$$(3.66) \quad 0 = \mathcal{G}_{\tilde{a}} X - \mathcal{G}_a X = -2(X \cdot \nabla \varphi) e.$$

On any ball compactly contained in $\mathbb{R}^n \setminus M$, choose φ equal to a coordinate function near that ball. Equation (3.66) then forces each component of X to vanish there. Thus the global H^1 vector field X is zero outside M , and the trace characterization of H_0^1 on smooth domains proves (3.63). $\square$

3.9. Completion of the proof.

Proof of Theorem 2. Fix g_0 and suppose the conclusion fails. Then there are smooth metrics g_j with $\Lambda_{g_j} = \Lambda_{g_0}$ and $\|g_j - g_0\|_{H^s(M)} \rightarrow 0$ which are not related to g_0 by boundary-fixing diffeomorphisms. Apply Lemmas 3.2–3.3 and write $\bar{g}_j = (\Phi_j)_* g_j$, with perturbations B_j as in (3.3). By (3.12), $b_{s,j} \rightarrow 0$. Moreover, $B_j \neq 0$: otherwise $A_{\bar{g}_j} = A_0$, hence $\bar{g}_j = g_0$ since $n \geq 3$, contradicting the choice of g_j . Lemma 3.12 provides, after passage to a subsequence,

$$H_j = B_j/b_{2,j} \longrightarrow H \quad \text{in } L^2, \quad \|H\|_{L^2} = 1.$$

Set $A_j = A_{\bar{g}_j} = a^2(\text{Id} + B_j)$. By (3.18), $A_j \rightarrow A_0$ uniformly, and these conductivities are uniformly elliptic for large j .

Fix g_0 -harmonic $u, v \in C^\infty(M)$ and let u_j solve $L_{A_j} u_j = 0$ with boundary value u . Then

$$L_{A_j}(u_j - u) = -\text{div}(a^2 B_j \nabla u), \quad (u_j - u)|_{\partial M} = 0.$$

The energy estimate and Poincaré inequality imply

$$(3.67) \quad \|u_j - u\|_{H^1(M)} \leq C b_{2,j},$$

where C may depend on the fixed function u . Applying (3.2) to $\bar{g}_j$ and using (3.67), we obtain

$$(3.68) \quad \left| \int_M a^2 H_j^{ij} \partial_i u \partial_j v dx \right| = \frac{1}{b_{2,j}} \left| \int_M a^2 B_j^{ij} \partial_i (u_j - u) \partial_j v dx \right| \leq C b_{2,j} \longrightarrow 0.$$

Strong L^2 convergence in (3.68) proves (3.62) for H . Proposition 3.16 therefore gives $H = \mathcal{G}_a X$ for some $X \in H_0^1(M; \mathbb{R}^n)$.

Let Z_j be the normal fields from Lemma 3.2 and set $V_j = Z_j/b_{2,j}$. By (3.5), this sequence is bounded in $H_0^1(M; \mathbb{R}^N)$, and (3.4), (3.10) imply

$$(3.69) \quad L_{A_j} V_j = -\text{div}(a^2 H_j \nabla F), \quad DF^\top V_j = 0.$$

After passing to a further subsequence, $V_j \rightharpoonup V$ in H_0^1 . Since $A_j \rightarrow A_0$ uniformly and $H_j \rightarrow H$ in L^2 , passing to the weak formulation of (3.69) yields

$$(3.70) \quad L_{A_0} V = -\text{div}(a^2 H \nabla F), \quad DF^\top V = 0.$$

Each component of F is g_0 -harmonic. Thus (3.55), applied to $H = \mathcal{G}_a X$, shows that $-DFX \in H_0^1(M; \mathbb{R}^N)$ solves the same Dirichlet problem as V in (3.70). Uniqueness gives $V = -DFX$. The normality in (3.70) now implies $DF^\top DF X = 0$. Since DF has full rank by Lemma 3.1, we have $X = 0$, hence $H = 0$, contradicting $\|H\|_{L^2} = 1$. $\square$

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