LOCAL RIGIDITY OF CONFORMALLY EUCLIDEAN METRICS FOR THE ANISOTROPIC HELMHOLTZ EQUATION AT FIXED ENERGY

SEBASTIÁN MUÑOZ-THON AND PLAMEN STEFANOV

Abstract. We prove local rigidity of smooth conformally Euclidean metrics g_0 on smooth compact domains $M \subset \mathbb{R}^n$, $n \geq 3$, for the anisotropic Helmholtz equation $(-\Delta_g - k^2)u = 0$ at a fixed energy $k^2 > 0$ which is not a Dirichlet eigenvalue of $-\Delta_{g_0}$ in M. Any smooth metric sufficiently close to g_0 in $H^s(M)$, with integer $s > n/2 + 1$, and having the same Dirichlet-to-Neumann map, is its pullback by a smooth diffeomorphism fixing the boundary pointwise. The proof uses a Helmholtz embedding to fix the gauge and a compactness argument together with a characterization of the kernel of the linearization.

1. Introduction

Let $M \subset \mathbb{R}^n$, $n \geq 3$, be a smooth compact domain, and let e denote the Euclidean metric. Let $k > 0$ be fixed. For a smooth Riemannian metric g on M, we define the Dirichlet-to-Neumann (DN) map, for $f \in C^\infty(\partial M)$, by

$$\Lambda_g(k) f = \partial_{\nu_g} u\big|_{\partial M},$$

where u is the unique solution of the Dirichlet problem

$$\begin{aligned} (-\Delta_g - k^2)u &= 0 \quad \text{in } M, \\ u &= f \quad \text{on } \partial M, \end{aligned}$$

assuming that k^2 is not a Dirichlet eigenvalue of $-\Delta_g$ in M. Here, ν_g is the outward unit normal with respect to g. We prove that for every smooth conformal factor c in M, if g is sufficiently close to $g_0 := e^{2c}\mathsf{e}$ in $H^s(M)$, with integer $s > n/2 + 1$, and if $\Lambda_g(k) = \Lambda_{g_0}(k)$, then g is isometric to g_0 by a diffeomorphism fixing the boundary pointwise.

Theorem 1. *Let $M \subset \mathbb{R}^n$, $n \geq 3$, be a smooth compact domain, let $g_0 = e^{2c}\mathsf{e}$ with $c \in C^\infty(M)$, and fix an integer $s > n/2 + 1$. Fix $k \geq 0$ such that k^2 is not a Dirichlet eigenvalue for $-\Delta_{g_0}$ in M. There exists $\varepsilon_0 > 0$, depending on M, g_0, k, s, such that every smooth Riemannian metric g on M satisfying*

$$\|g - g_0\|_{H^s(M)} < \varepsilon_0$$

and

$$\Lambda_g(k) = \Lambda_{g_0}(k) \tag{1.1}$$

has the form $g = \Phi^* g_0$ for a smooth diffeomorphism $\Phi \colon M \to M$ fixing ∂M pointwise.

Theorem 1 extends the result in [7] from $k = 0$ to $k \geq 0$ with a proof following that in [7]. For a real energy λ below the lowest Dirichlet eigenvalue of $-\Delta_{g_0}$, including $\lambda < 0$, local rigidity for $-\Delta_g - \lambda$ follows from [7, Corollary 1.1]. The conformal factor supplied by that corollary equals one by unique continuation, since its difference from one satisfies a linear elliptic equation with zero Cauchy data. Recently, OpenAI announced [10] a solution of the anisotropic Calderón problem for $k = 0$ but at the time of writing this paper, the proof is still being checked for accuracy.

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Cekić, Krupchyk, Sahoo, and Uhlmann [1] showed recently that, for non-trapping metrics with strictly convex boundary, equality of Helmholtz DN maps at a sufficiently large frequency implies equality of lens data. The frequency threshold depends on the two metrics; metric uniqueness then follows under assumptions ensuring lens rigidity. In fact, under these lens rigidity assumptions, their result is equivalent to uniqueness, modulo the gauge, given the DN maps over a sequence $k_j \to \infty$.

For a smooth metric g equal to e outside a ball, let $S_g(k)$ denote the scattering matrix of $-\Delta_g$ at energy k^2, see, e.g., [4, Sections 4.1 and 4.4]. Equality of these matrices is equivalent to equality of the scattering amplitudes for all incident and observation directions. Theorem 1 has the following consequence.

Corollary 1.1. *Let $B \subset \mathbb{R}^n$, $n \geq 3$, be a ball, and let $g_0 = e^{2c}\mathsf{e}$, where $c \in C_0^\infty(B)$. Fix $k > 0$ and an integer $s > n/2 + 1$. There exists $\varepsilon_0 > 0$, depending on B, g_0, k, s, such that every smooth Riemannian metric g on $\mathbb{R}^n$ satisfying*

$$g = \mathsf{e} \quad \text{outside } B, \qquad \|g - g_0\|_{H^s(\mathbb{R}^n)} < \varepsilon_0,$$

and $S_g(k) = S_{g_0}(k)$, has the form $g = \Phi^* g_0$, where $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ is a smooth diffeomorphism equal to the identity outside B.

Novikov [8] and Ramm [11] establish uniqueness for smooth compactly supported potentials in dimensions $n \geq 3$ from the scattering amplitude at any fixed positive energy. The proof is based on the complex geometric optics solutions in [13]. By the Liouville transform, this also yields uniqueness for conformally Euclidean metrics equal to e outside a ball. Uniqueness at fixed energy was proved for real exponentially decreasing potentials in [9, 15] and for electrics potentials and magnetic fields in [5]. In Corollary 1.1, only the background g_0 is assumed conformally Euclidean; the second metric g is allowed to be an arbitrary sufficiently small anisotropic perturbation.

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2. Proof of Theorem 1

Preliminaries. We fix M, g_0, k, s as in Theorem 1 and assume that g is a smooth Riemannian metric satisfying (1.1) and sufficiently close to g_0 in $H^s(M)$. For a symmetric matrix field A, write $L_A u = \partial_i(A^{ij}\partial_j u)$. For a metric g, write $|g| = \det(g_{ij})$. Its associated conductivity is

$$A_g = |g|^{1/2}\, g^{-1}, \qquad L_{A_g} = |g|^{1/2}\, \Delta_g.$$

Fix a smooth compactly supported extension of c to $\mathbb{R}^n$ and continue to write $g_0 = e^{2c}\mathsf{e}$ for the extended background. Set

$$a = e^{(n-2)c/2}, \qquad A_0 \coloneqq A_{g_0} = a^2 \mathrm{Id}.$$

Under the DN equality (1.1), boundary determination [2, Theorem 8.4] implies equality of the induced boundary metrics. Green's formula therefore yields the Helmholtz Alessandrini identity

$$\int_M \left[(A_g - A_0)^{ij}(\partial_i u)\,(\partial_j v) - k^2\big(|g|^{1/2} - |g_0|^{1/2}\big)uv\right] \mathrm{d}x = 0 \tag{2.1}$$

for $u, v \in C^\infty(M)$ satisfying $(\Delta_g + k^2)u = 0$ and $(\Delta_{g_0} + k^2)v = 0$ in M.

2.1. **The Helmholtz embedding.** The next proposition holds for general smooth metrics g.

Proposition 2.1. *There exist an integer N, an open neighborhood U of M, and a smooth embedding $F\colon U \to \mathbb{R}^N$ satisfying*

$$(-\Delta_{g_0} - k^2)F^\alpha = 0.$$

Proof. Choose concentric balls B_1, B_2 with $M \subset B_1 \Subset B_2$ such that k^2 is not a Dirichlet eigenvalue of $-\Delta_{g_0}$ in B_2, and set $K = \overline{B_1}$. Such an outer radius exists by strict domain monotonicity of the Dirichlet eigenvalues. Fix a nonempty open subset $\Gamma \subset \partial B_2$, and let u_f solve $(-\Delta_{g_0} - k^2)u_f = 0$ in B_2 with boundary value $f \in C_0^\infty(\Gamma)$, real-valued. We adapt [6, Propositions A.1 and A.5], using the Runge approximation theorem [6, Proposition A.2] for this operator.

First we show that solutions to the Helmholtz problem separate points. Let $x \neq y$ be points in B_2. Choose disjoint small balls $D_1, D_2 \Subset B_2$ centered at x, y, with $B_2 \setminus \overline{D_1 \cup D_2}$ connected. Solving the Helmholtz equation in D_1 with boundary value 1 produces a solution v_1 with $v_1(x) \neq 0$ when D_1 is sufficiently small, as follows by rescaling. Runge approximation of v_1 on D_1 and 0 on D_2, followed by Sobolev embedding, yields $u_f(x) \neq u_f(y)$.

For the differentials, fix $p \in B_2$ and $\xi \in T_p^* B_2$. On a sufficiently small ball $B(p, \varepsilon)$, let v_ε solve the Helmholtz equation with boundary value $\xi \cdot (x - p)$. Rescaling by $\varepsilon^{-1} v_\varepsilon(p + \varepsilon y)$ makes the zeroth-order term $-\varepsilon^2 k^2$, and elliptic estimates imply

$$|\mathrm{d}v_\varepsilon(p) - \xi| \leq C(\varepsilon + \varepsilon^2 k^2)|\xi|.$$

For each fixed ε, Runge approximation in $H^{n/2+2}(B(p,\varepsilon))$, hence in C^1, approximates $\mathrm{d}v_\varepsilon(p)$ by differentials $\mathrm{d}u_f(p)$. Letting $\varepsilon \to 0$ shows that the range of $f \mapsto \mathrm{d}u_f(p)$ is dense in $T_p^* B_2$. Being a linear subspace of a finite-dimensional space, it is closed, so every ξ is attained.

Given $x \in K$, choose solutions $u_x^1, \ldots, u_x^n$ whose differentials form a basis at x. By the inverse function theorem, $U_x = (u_x^1, \ldots, u_x^n)$ is a diffeomorphism on a neighborhood V_x of x. Choose a finite cover $K \subset V_{x_1} \cup \cdots \cup V_{x_m}$ and set

$$F_0 = (u_{x_1}^1, \ldots, u_{x_1}^n, \ldots, u_{x_m}^1, \ldots, u_{x_m}^n).$$

Then F_0 is an immersion on K and is injective on each V_{x_j}.

By the Lebesgue number lemma [3, Chapter XI, Theorem 4.5], F_0 separates all pairs in K with $0 < |x - y| < \lambda$ for some $\lambda > 0$. The remaining pairs form the compact set

$$\{(x, y) \in K \times K : |x - y| \geq \lambda\}.$$

For each such pair, choose a Helmholtz solution separating its points. By continuity and compactness, finitely many of these solutions $h^1, \ldots, h^q$ suffice. Thus $F = (F_0, h^1, \ldots, h^q)$ is an injective immersion on K. Since K is compact, F is a homeomorphism onto its image [3, Chapter XI, Theorem 2.1], hence a smooth embedding. Restricting to $U = B_1$ proves the proposition. □

2.2. **The Helmholtz gauge.** The next lemma extends the gauge construction in [7, Lemma 3.2] to the Helmholtz equation.

Lemma 2.2. *Let M, g_0, k, s be as in Theorem 1, and let F be the embedding in Proposition 2.1. Let g be a smooth metric sufficiently close to g_0 in $H^s(M)$. Then there exist a smooth diffeomorphism $\Phi\colon M \to M$ fixing ∂M pointwise and $Z \in C^\infty(M; \mathbb{R}^N)$ with the following properties. Set*

$$\overline{g} = \Phi_* g, \qquad B = a^{-2}(A_{\overline{g}} - A_0). \tag{2.2}$$

Then

$$(\Delta_{\overline{g}} + k^2)(F + Z) = 0, \qquad Z|_{\partial M} = 0, \qquad \partial_j F \cdot Z = 0, \quad j = 1, \ldots, n, \tag{2.3}$$

and

$$\begin{aligned} \|\Phi - \mathrm{Id}\|_{H^{s+1}(M)} + \|B\|_{H^s(M)} + \|Z\|_{H^{s+1}(M)} &\le C\|g - g_0\|_{H^s(M)}, \\ \|Z\|_{H^1(M)} &\le C\|B\|_{L^2(M)}. \end{aligned} \tag{2.4}$$

The constant C and the required smallness of $\|g - g_0\|_{H^s(M)}$ depend only on M, g_0, k, s, F.

Proof. Since k^2 is not a Dirichlet eigenvalue of $-\Delta_{g_0}$, it is not a Dirichlet eigenvalue of $-\Delta_g$ for g sufficiently close to g_0. We can therefore solve

$$(\Delta_g + k^2)F_g^\alpha = 0 \quad \text{in } M, \qquad F_g^\alpha|_{\partial M} = F^\alpha|_{\partial M}, \qquad \alpha = 1, \dots, N.$$

By elliptic regularity up to the boundary, $F_g \in C^\infty(M; \mathbb{R}^N)$. Since F solves the background equation, we have

$$(\Delta_{g_0} + k^2)(F_g - F) = (\Delta_{g_0} - \Delta_g)F_g, \qquad (F_g - F)|_{\partial M} = 0.$$

By standard elliptic estimates and Sobolev product estimates,

$$\begin{aligned} \|F_g - F\|_{H^{s+1}} &\le C\|(\Delta_{g_0} - \Delta_g)F_g\|_{H^{s-1}} \\ &\le C\|g - g_0\|_{H^s}\big(1 + \|F_g - F\|_{H^{s+1}}\big), \end{aligned}$$

where we used that F is fixed and smooth. All norms here and below are over M. For g sufficiently close to g_0, we can absorb the last term to get

$$\|F_g - F\|_{H^{s+1}} \le C\|g - g_0\|_{H^s}.$$

Let π be the Euclidean nearest-point projection onto $F(U)$, defined and smooth near $F(M)$. By the preceding estimate and Sobolev embedding, $F_g(M)$ lies in its domain for g sufficiently close to g_0. Set

$$\Phi = F^{-1} \circ \pi \circ F_g.$$

As in the proof of [7, Lemma 3.2], Φ is a smooth diffeomorphism of M onto itself fixing ∂M pointwise. Set

$$Z = F_g \circ \Phi^{-1} - F.$$

The Sobolev estimates in that proof also establish the first estimate of the lemma. By the definition of π, Z is normal to $F(U)$, and $Z = 0$ on ∂M because $F_g = F$ there. Invariance of the Helmholtz equation under diffeomorphisms implies $(\Delta_{\overline{g}} + k^2)(F + Z) = 0$.

It remains to estimate Z in H^1. Since $(\Delta_{g_0} + k^2)F = 0$, we have

$$\big(L_{A_{\overline{g}}} + k^2|\overline{g}|^{1/2}\big)Z = -\operatorname{div}(a^2 B\nabla F) - k^2\big(|\overline{g}|^{1/2} - |g_0|^{1/2}\big)F, \qquad Z|_{\partial M} = 0. \tag{2.5}$$

Since $A_{\overline{g}} = a^2(\mathrm{Id} + B)$ and $n \ge 3$,

$$|\overline{g}|^{1/2} = |g_0|^{1/2}\,\det(\mathrm{Id} + B)^{1/(n-2)}.$$

For B small in L^∞, this implies

$$\big\||\overline{g}|^{1/2} - |g_0|^{1/2}\big\|_{L^2} \le C\|B\|_{L^2}.$$

By the assumption on k^2, the operator $L_{A_0} + k^2|g_0|^{1/2}$ is invertible from $H_0^1(M)$ to $H^{-1}(M)$. The H^s bound for B and Sobolev embedding show that the corresponding operator for $\overline{g}$ is close to it in the operator norm from $H_0^1(M)$ to $H^{-1}(M)$. Its inverse is therefore uniformly bounded. Applying it to the equation for Z, we get

$$\|Z\|_{H^1} \le C\left(\|a^2 B\nabla F\|_{L^2} + k^2\big\|(|\overline{g}|^{1/2} - |g_0|^{1/2})F\big\|_{L^2}\right) \le C\|B\|_{L^2}.$$

□

2.3. **Boundary jets.**

Lemma 2.3. *Let g and $\overline{g}$ be as in Lemma 2.2, and assume that $\Lambda_g(k) = \Lambda_{g_0}(k)$. Then $\overline{g} - g_0$ is flat at ∂M.*

Proof. We work near an arbitrary boundary point, and write $\equiv$ for equality of boundary jets. We will use the boundary jet recovery Theorem 8.4 in [2] for general elliptic second order operators which allows a conformal gauge because it treats both the metric and the potential as unknowns; while in our case, the potential is fixed to be $-k^2$. Applying it to the pairs $(g, -k^2)$ and $(g_0, -k^2)$, and composing the conformal gauges by multiplying their factors, we obtain a local diffeomorphism ψ fixing the boundary pointwise and a smooth $f > 0$, with $f = 1$ and $\mathrm{d}f = 0$ on the boundary, such that, with $\widehat{g} = \psi^* g$,

$$\widehat{g} \equiv f^{4/(n-2)} g_0, \qquad -k^2 \equiv f^{-4/(n-2)} \left(-k^2 - \frac{\Delta_{g_0} f}{f} \right).$$

Then

$$\Delta_{g_0} f \equiv k^2 \left(f^{(n+2)/(n-2)} - f \right).$$

This is an elliptic semilinear PDE (which we are not trying to solve). Successive normal differentiation, using $f = 1$ and $\mathrm{d}f = 0$ on the boundary, shows that $f - 1$ is flat, hence so is $\widehat{g} - g_0$.

Set $W = F_g \circ \psi - F$. Then

$$(\Delta_{\widehat{g}} + k^2) W = (\Delta_{g_0} - \Delta_{\widehat{g}}) F,$$
$$W|_{\partial M} = \partial_{\nu_{\widehat{g}}} W|_{\partial M} = 0.$$

The Neumann condition follows from the DN equality and $\widehat{g} = g_0$ on the boundary. Since the right-hand side is flat, normal differentiation shows that W is flat as well. The projection argument in [7, Lemma 3.3] now shows that $\Phi \circ \psi - \mathrm{Id}$ is flat, too. Since $(\Phi \circ \psi)_* \widehat{g} = \overline{g}$, the lemma follows. □

Lemma 2.3 allows us to extend $\overline{g}$ by g_0 outside M, and B by zero; then $B \in C_0^\infty(\mathbb{R}^n)$. By (2.4),

$$\|B\|_{H^s(\mathbb{R}^n)} \leq C \|g - g_0\|_{H^s(M)}. \tag{2.6}$$

The estimate in [7, Lemma 3.4] also holds for the Helmholtz gauge.

Lemma 2.4. *For B as above,*

$$\|\operatorname{div} B\|_{L^2(\mathbb{R}^n)} \leq C \|B\|_{L^2(\mathbb{R}^n)}, \tag{2.7}$$

where C depends only on M, g_0, k, s, F.

Proof. By Lemma 2.2,

$$L_{A_{\overline{g}}} Z = -\partial_i (a^2 B^{ij} \partial_j F) - k^2 |\overline{g}|^{1/2} Z - k^2 \big(|\overline{g}|^{1/2} - |g_0|^{1/2} \big) F.$$

Take scalar products with $\partial_\ell F$. The term $-k^2 |\overline{g}|^{1/2} Z$ makes no contribution, since $Z \cdot \partial_\ell F = 0$. The tangential calculation in [7, Lemma 3.4] then gives a system for $\partial_i (a^2 B^{ij})$ with the same coefficient matrix $\partial_\ell F \cdot \partial_j F - Z \cdot \partial_j \partial_\ell F$. The only additional term on the right is

$$-k^2 \big(|\overline{g}|^{1/2} - |g_0|^{1/2} \big) \, \partial_\ell F \cdot F.$$

Since F is fixed and smooth, the estimate for the volume-density difference in the proof of Lemma 2.2 bounds its $L^2(M)$ norm by $C\|B\|_{L^2(M)}$. Since F is an embedding and Z is small in L^∞, the matrix is uniformly invertible, as in [7, Lemma 3.4]. Therefore,

$$\|\operatorname{div}(a^2B)\|_{L^2} \le C\left(\|B\|_{L^2} + \|Z\|_{H^1} + k^2\big\||\overline{g}|^{1/2} - |g_0|^{1/2}\big\|_{L^2}\right) \le C\|B\|_{L^2},$$

where we used Lemma 2.2 again, and all norms are in M. Finally, $\operatorname{div} B = a^{-2}\operatorname{div}(a^2B) - 2B\nabla\log a$. Since B extends smoothly by zero by Lemma 2.3, the estimate holds on $\mathbb{R}^n$ as well. □

2.4. **The Liouville transform.** Extend $\overline{g}$ by the fixed extended background g_0 and B by zero outside M, as in Lemma 2.3. Then $A_{\overline{g}} = a^2(\mathrm{Id} + B)$ on $\mathbb{R}^n$. Set

$$b_2 = \|B\|_{L^2}, \qquad b_s = \|B\|_{H^s}, \qquad b_\infty = \|B\|_{L^\infty}. \tag{2.8}$$

All norms below are over $\mathbb{R}^n$ unless another domain is indicated. Sobolev embedding and interpolation imply

$$b_\infty + \|\operatorname{div} B\|_{L^\infty} \le Cb_s \tag{2.9}$$

and

$$b_\infty \le C\|B\|_{H^{s-1}} \le Cb_2^{1/s}b_s^{1-1/s}. \tag{2.10}$$

As in [7, Proposition 2.5], the continuation argument will require τb_∞ to be small; the interpolation estimate will allow us to choose the frequency cutoff within this range.

The substitution $u = v/a$ in $(\Delta_{\overline{g}} + k^2)u = 0$ leads to the following version of [7, Lemma 3.5]. Set

$$q = \frac{\Delta a}{a} - k^2e^{2c}, \qquad q + k^2 \in C_0^\infty(\mathbb{R}^n),$$

where Δ is the Euclidean Laplacian.

Lemma 2.5. *Set*

$$q_B = a^{-1}\operatorname{div}(B\nabla a) - k^2e^{2c}\left(\det(\mathrm{Id} + B)^{1/(n-2)} - 1\right).$$

Then, for $v \in C^\infty(\mathbb{R}^n)$,

$$a^{-1}\big(L_{A_{\overline{g}}} + k^2|\overline{g}|^{1/2}\big)(v/a) = (\Delta - q)v + \operatorname{div}(B\nabla v) - q_Bv. \tag{2.11}$$

Furthermore,

$$\|\operatorname{div} B\|_{L^2} + \|q_B\|_{L^2} \le Cb_2, \qquad \|\operatorname{div} B\|_{L^\infty} + \|q_B\|_{L^\infty} \le Cb_s.$$

The constant C depends only on M, g_0, k, s, F and the fixed extension of c.

Proof. By the calculation in [7, Lemma 3.5],

$$a^{-1}L_{A_0}(v/a) = \Delta v - \frac{\Delta a}{a}v,$$

and the conductivity perturbation contributes $\operatorname{div}(B\nabla v) - a^{-1}\operatorname{div}(B\nabla a)v$. The additional Helmholtz term is determined by

$$a^{-2}|\overline{g}|^{1/2} = e^{2c}\det(\mathrm{Id} + B)^{1/(n-2)},$$

which proves the identity in the lemma.

For B small in L^∞,

$$\big\|\det(\mathrm{Id} + B)^{1/(n-2)} - 1\big\|_{L^p} \le C\|B\|_{L^p}, \qquad p = 2, \infty.$$

The differential term in q_B is $\nabla\log a \cdot \operatorname{div} B + a^{-1}B^{ij}\partial_i\partial_j a$. The estimates therefore follow from Lemma 2.4 and Sobolev embedding. □

The potential q equals $-k^2$ outside a compact set. To construct the background complex geometric optics solutions, we will cut it off outside a neighborhood of the region under consideration.

2.5. **Background complex geometric optics solutions.**

Lemma 2.6. *Fix a closed ball $\mathcal{B}$ containing M in its interior and an integer $\ell \geq 0$. There are constants τ_*, $C > 0$ such that, if $\zeta \in \mathbb{C}^n$, $\zeta \cdot \zeta = 0$, and $|\zeta| \geq \tau_*$, then there is a smooth function on $\mathbb{R}^n$ of the form*

$$u_\zeta^0 = a^{-1} e^{x\cdot\zeta}(1 + r_\zeta^0) \tag{2.12}$$

satisfying

$$(\Delta_{g_0} + k^2)u_\zeta^0 = 0 \quad \text{in a neighborhood of } \mathcal{B}$$

and

$$\|r_\zeta^0\|_{C^\ell(\mathcal{B})} \leq \frac{C}{|\zeta|}. \tag{2.13}$$

The constants depend only on the fixed extended background g_0, k, $\mathcal{B}$, ℓ, and the fixed cutoff in the proof.

Proof. By the Liouville identity (2.11) with $B = 0$ and $\zeta \cdot \zeta = 0$, the ansatz (2.12) satisfies

$$a^{-1}|g_0|^{1/2} e^{-x\cdot\zeta}(\Delta_{g_0} + k^2)u_\zeta^0 = P_\zeta r_\zeta^0 - q(1 + r_\zeta^0), \qquad P_\zeta = \Delta + 2\zeta \cdot \nabla.$$

Choose $\chi \in C_0^\infty(\mathbb{R}^n)$ equal to 1 near $\mathcal{B}$, and set $\tilde{q} = \chi q$. It suffices to solve

$$P_\zeta r_\zeta^0 = \tilde{q}(1 + r_\zeta^0) \quad \text{on } \mathbb{R}^n.$$

The Sylvester–Uhlmann construction [13], with the C^ℓ remainder estimate in [7, Lemma 3.6], yields a smooth solution satisfying (2.13) for $|\zeta|$ sufficiently large. Since $\tilde{q} = q$ near $\mathcal{B}$, the first identity proves the assertion. □

For $\tau \geq 1$, set

$$X_\tau(v) = \|v\|_{L^2} + \tau^{-1}\|\nabla v\|_{L^2} + \tau^{-2}\|\nabla^2 v\|_{L^2}.$$

Corollary 2.7. *There are constants $\tau_0 \geq 1$ and $C > 0$, depending only on $\mathcal{B}, n$ and $\|q\|_{L^\infty(\mathcal{B})}$, such that, for $v \in H^2(\mathbb{R}^n)$ supported in $\mathcal{B}$ and $\zeta \in \mathbb{C}^n$ with $\zeta \cdot \zeta = 0$ and $|\zeta| = \sqrt{2}\tau$, we have*

$$X_\tau(v) \leq \frac{C}{\tau}\|(P_\zeta - q)v\|_{L^2}, \qquad \tau \geq \tau_0. \tag{2.14}$$

Proof. The proof of [7, Corollary 3.7] requires only a bound for q on the support of v. Indeed, for $\tau \geq 1$,

$$X_\tau(v) \leq \frac{C}{\tau}\|P_\zeta v\|_{L^2} \leq \frac{C}{\tau}\|(P_\zeta - q)v\|_{L^2} + \frac{C\|q\|_{L^\infty(\mathcal{B})}}{\tau} X_\tau(v).$$

Choose τ_0 so that $C\|q\|_{L^\infty(\mathcal{B})}/\tau_0 \leq 1/2$ and absorb the last term to obtain (2.14). □

The continuation argument of [7, Lemma 3.8] applies to the Helmholtz equation as follows.

Lemma 2.8. *Let $g, \overline{g}$ be as in Lemma 2.3. After increasing τ_0 from Corollary 2.7, there are constants $c_0, C > 0$ such that the following holds. Let $\zeta \cdot \zeta = 0$, $|\zeta| = \sqrt{2}\tau$, and assume*

$$b_s \leq c_0, \qquad \tau \geq \tau_0, \qquad \tau b_\infty \leq c_0.$$

Let u_ζ^0 be the background solution of Lemma 2.6, with the $C^2(\mathcal{B})$ bound in (2.13)*. The solution u_ζ of $(\Delta_{\overline{g}} + k^2)u_\zeta = 0$ in M with boundary value u_ζ^0 extends by u_ζ^0 outside M to a smooth function on $\mathbb{R}^n$ solving the same equation on a neighborhood of $\mathcal{B}$. It has the form*

$$u_\zeta = a^{-1}e^{x\cdot\zeta}(1 + r_\zeta^0 + r_\zeta), \qquad r_\zeta \in C_0^\infty(\mathbb{R}^n), \qquad \operatorname{supp} r_\zeta \subset M,$$

with

$$X_\tau(r_\zeta) \le C\tau b_2.$$

The constants depend only on $M, g_0, k, s, F, \mathcal{B}$, the fixed extension of c, and the cutoff in Lemma 2.6.

Proof. The argument of [7, Lemma 3.8] applies with the potentials q and q_B defined above, using Lemmas 2.5 and 2.6 and Corollary 2.7. □

2.6. **Estimates on the Fourier transform.** The following polarization lemma from the previous paper applies without change.

Lemma 2.9 ([7, Lemma 3.9])**.** *For every $\omega \in \mathbb{S}^{n-1}$, one can choose $L = L(n)$ pairs $(z_\ell, w_\ell) \in \mathbb{C}^n \times \mathbb{C}^n$ such that*

$$z_\ell \cdot z_\ell = w_\ell \cdot w_\ell = 0, \qquad z_\ell + w_\ell = -\mathrm{i}\omega, \qquad |z_\ell| = |w_\ell| = \sqrt{2},$$

and, for every complex symmetric matrix S,

$$|S| \le C\left(\sum_{\ell=1}^{L} |S(z_\ell, w_\ell)| + |S\omega|\right). \tag{2.15}$$

Here $S(z,w) = S^{ij}z_i w_j$, $|S|$ is the Frobenius norm, and C depends only on n. The vectors may be chosen piecewise smoothly in ω.

For $S = \widehat{B}(\xi)$ and $\omega = \xi/|\xi|$, the additional term in (2.15) is controlled by Lemma 2.4. Indeed, $\widehat{\operatorname{div} B}(\xi) = \mathrm{i}\widehat{B}(\xi)\xi$, so Plancherel's identity and (2.7) imply

$$\left\| \mathbf{1}_{\{|\xi|>T\}} \widehat{B}(\xi)\frac{\xi}{|\xi|} \right\|_{L^2} \le \frac{C}{T}\|\operatorname{div} B\|_{L^2} \le \frac{C}{T}b_2, \qquad T > 0.$$

We next estimate the errors in the background complex geometric optics amplitudes.

Lemma 2.10 ([7, Lemma 3.10])**.** *Let $e(x,\xi) = (e^{ij}(x,\xi))$ be measurable in ξ and C^{n+1} in x, with support in a fixed compact set in x, and suppose that*

$$|\partial_x^\alpha e(x,\xi)| \le C|\xi|^{-1}, \qquad |\alpha| \le n+1, \quad |\xi| \ge \tau_0. \tag{2.16}$$

For a square-integrable matrix field B, define

$$E_T B(\xi) = \mathbf{1}_{\{|\xi|>T\}} \int_{\mathbb{R}^n} e^{-\mathrm{i}x\cdot\xi} e^{ij}(x,\xi)B^{ij}(x)\,\mathrm{d}x, \qquad T \ge \tau_0.$$

Then

$$\|E_T B\|_{L^2} \le \frac{C}{T}\|B\|_{L^2},$$

where C depends only on n, the fixed compact support, and the constant in (2.16)*.*

We next adapt the Fourier estimate of [7, Lemma 3.11] to the Helmholtz equation.

Lemma 2.11. *Let $g, \overline{g}$ be as in Lemma 2.3, with B as in (2.2). After increasing τ_0 and decreasing c_0 from Lemma 2.8, if necessary, we have*

$$\left\|\mathbf{1}_{\{|\xi|>T\}}\widehat{B}\right\|_{L^2} \le C\big(R^{n/2+1}b_2^2 + T^{-1}b_2 + R^{-s}b_s\big) \tag{2.17}$$

whenever

$$b_s \le c_0, \qquad R \ge T \ge \tau_0, \qquad Rb_\infty \le c_0.$$

The constant C has the same dependence as in Lemma 2.8 and is independent of B, T, R.

Proof. Write $\xi = \rho\omega$ and take $\zeta = \rho z_\ell$, $\eta = \rho w_\ell$ with the vectors of Lemma 2.9. Choose the background solutions with the $C^{n+2}(\mathcal{B})$ bound in Lemma 2.6, increasing τ_0 if necessary. For $T < \rho \le R$, the assumptions of Lemma 2.8 hold. Applying (2.1) to $\overline{g}$, by invariance of the DN map, we obtain

$$\int_M a^2 B^{ij}(\partial_i u_\zeta^0)\,(\partial_j u_\eta^0)\,\mathrm{d}x = -\int_M a^2 B^{ij}(\partial_i(u_\zeta - u_\zeta^0))\,(\partial_j u_\eta^0)\,\mathrm{d}x + k^2\int_M \big(|\overline{g}|^{1/2} - |g_0|^{1/2}\big)u_\zeta u_\eta^0\,\mathrm{d}x.$$

The gradient terms are treated as in [7, Lemma 3.11]. We estimate the additional term, divided by ρ^2, after writing $u_\zeta = u_\zeta^0 + (u_\zeta - u_\zeta^0)$. By the estimate for the volume-density difference in the proof of Lemma 2.2,

$$\big\||\overline{g}|^{1/2} - |g_0|^{1/2}\big\|_{L^2} \le Cb_2.$$

This density difference is supported in M by Lemma 2.3.

For the two background solutions, use $\zeta + \eta = -\mathrm{i}\xi$ and choose $\chi \in C_0^\infty(\mathcal{B}^{\mathrm{int}})$ equal to 1 near M. The amplitude multiplying the density difference is

$$\frac{k^2\chi(x)}{\rho^2 a^2}(1 + r_\zeta^0)(1 + r_\eta^0) = O_{C^{n+1}}(k^2\rho^{-2}), \qquad \rho \ge \tau_0.$$

Since $\rho^{-2} \le T^{-1}\rho^{-1}$ for $\rho > T$, the scalar version of Lemma 2.10, applied to T times this amplitude with the cutoff $\rho > T$, implies

$$\left\|\mathbf{1}_{\{|\xi|>T\}}\frac{k^2}{\rho^2}\int_M \big(|\overline{g}|^{1/2} - |g_0|^{1/2}\big)u_\zeta^0 u_\eta^0\,\mathrm{d}x\right\|_{L^2_\xi} \le Ck^2T^{-2}b_2.$$

For the part containing the correction, cancellation of the exponentials and Lemma 2.8 imply

$$\frac{k^2}{\rho^2}\left|\int_M \big(|\overline{g}|^{1/2} - |g_0|^{1/2}\big)(u_\zeta - u_\zeta^0)u_\eta^0\,\mathrm{d}x\right| \le \frac{Ck^2}{\rho^2}\big\||\overline{g}|^{1/2} - |g_0|^{1/2}\big\|_{L^2}\|r_\zeta\|_{L^2} \le Ck^2\rho^{-1}b_2^2.$$

Its L^2 norm on $T < |\xi| \le R$ is therefore at most $Ck^2R^{n/2+1}b_2^2$, since $\rho \ge 1$. As $T \ge 1$ and k is fixed, both contributions have the bounds required in (2.17), with constants depending also on k. The polarization and Sobolev tail arguments of [7, Lemma 3.11] complete the proof. □

2.7. **Compactness.** If the local rigidity statement of Theorem 1 fails, we have a sequence of nonzero perturbations B_j with $\|B_j\|_{H^s} \to 0$. We now prove that their L^2-normalizations have a convergent subsequence.

Lemma 2.12. *Let $B_j \ne 0$ be conductivity perturbations satisfying the hypotheses of Lemma 2.11, with constants C, c_0, τ_0 independent of j, and write $b_{2,j}, b_{s,j}, b_{\infty,j}$ for their norms as in (2.8). If $b_{s,j} \to 0$, then a subsequence of $H_j = B_j/b_{2,j}$ converges in $L^2(\mathbb{R}^n)$ to a tensor H satisfying*

$$\operatorname{supp} H \subset M, \qquad \|H\|_{L^2} = 1.$$

Proof. The compactness argument of [7, Lemma 3.12] applies, using Lemma 2.11 and the interpolation estimate (2.10). □

2.8. **The linearized problem.** We identify below covariant and contravariant tensors using e and denote the Euclidean tensor by e in either case. For a smooth positive function a, and a vector field X, define the normalized infinitesimal gauge by

$$\mathcal{G}_a X = DX + DX^\top - \big(\operatorname{div} X + 2X \cdot \nabla \log a\big)\mathsf{e}. \tag{2.18}$$

As shown in the proof of Lemma 2.13, $a^2\mathcal{G}_a X$ is the variation of $A_0 = a^2\mathrm{Id}$ under pushforward by the flow of X.

Lemma 2.13. *Let $X \in H^1_0(M;\mathbb{R}^n)$, set $H = \mathcal{G}_a X$, and let $u, v \in C^\infty(M)$ satisfy $(\Delta_{g_0} + k^2)u = (\Delta_{g_0} + k^2)v = 0$. Then*

$$\int_M \left(a^2 H^{ij} (\partial_i u)\, (\partial_j v) - \frac{k^2}{n-2} |g_0|^{1/2} (\operatorname{tr} H) uv \right) \mathrm{d}x = 0, \tag{2.19}$$

and

$$\big(L_{A_0} + k^2 |g_0|^{1/2}\big)(-X \cdot \nabla u) = -\operatorname{div}(a^2 H \nabla u) - \frac{k^2}{n-2} |g_0|^{1/2} (\operatorname{tr} H) u \tag{2.20}$$

in the distributional sense. The same identities hold on $\mathbb{R}^n$ for compactly supported $X \in H^1(\mathbb{R}^n;\mathbb{R}^n)$ and $u, v \in C^\infty(\mathbb{R}^n)$ satisfying the background Helmholtz equation on a neighborhood of $\operatorname{supp} X$.

Proof. First let X be smooth and compactly supported, let φ_t be its flow, and set $g_t = (\varphi_t)_* g_0$. By [7, Lemma 3.13], the conductivity variation is $\frac{\mathrm{d}}{\mathrm{d}t} A_{g_t}\big|_{t=0} = a^2 H$. The volume density transforms by

$$|g_t|^{1/2} = \left(\frac{|g_0|^{1/2}}{\det D\varphi_t} \right) \circ \varphi_t^{-1},$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t} |g_t|^{1/2} \bigg|_{t=0} = -\operatorname{div}\big(|g_0|^{1/2} X\big) = \frac{|g_0|^{1/2}}{n-2} \operatorname{tr} H.$$

The last equality follows from (2.18) and $|g_0|^{1/2} = a^{2n/(n-2)}$. Differentiating

$$\big(L_{A_{g_t}} + k^2 |g_t|^{1/2}\big)(u \circ \varphi_t^{-1}) = 0 \quad \text{near } \operatorname{supp} X$$

at $t = 0$ proves (2.20). Testing against v and integrating by parts proves (2.19), since $(L_{A_0} + k^2|g_0|^{1/2})v = 0$ near the support of $X \cdot \nabla u$. Both identities extend by H^1 density to the stated classes of vector fields, as in [7, Lemma 3.13]. □

We can call $\mathcal{G}_a X$ with $X \in H^1_0(M;\mathbb{R}^n)$ *potential tensor fields*, by analogy with the X-ray geodesic transform of symmetric two-tensor fields. Then (2.19) is the analog of the fact that the geodesic X-ray transform of two-tensor fields vanishes on potential ones.

The next two lemmas prepare the proof that the kernel of the linearization consists exactly of potential tensors $H = \mathcal{G}_a X$, with $X \in H^1_0(M;\mathbb{R}^n)$. Background complex geometric optics solutions yield a condition on $\widehat{H}$, which the next lemma converts into the decomposition $H = DX + DX^\top + \beta\mathsf{e}$. By (2.18), this can be rewritten as $H = \mathcal{G}_a X + \sigma\mathsf{e}$. Lemma 2.15 eliminates the scalar remainder σ, and we then show that X vanishes outside M. In the completion of the proof, the limiting gauge condition forces $X = 0$, contradicting the unit norm of the limit obtained by compactness.

Lemma 2.14 ([7, Lemma 3.14])**.** *Let $H \in L^2(\mathbb{R}^n)$ be a real symmetric tensor with compact support, and assume that for almost every $\xi \neq 0$ the restriction of $\widehat{H}(\xi)$ to $\xi^\perp$ is a scalar multiple of the Euclidean form. Then there are compactly supported $X \in H^1(\mathbb{R}^n;\mathbb{R}^n)$ and $\beta \in L^2(\mathbb{R}^n)$ such that*

$$H = DX + DX^\top + \beta\mathsf{e}.$$

The next lemma is the Helmholtz version of [7, Lemma 3.15]. It proves injectivity of the linearization for conformal perturbations.

Lemma 2.15. *Let $a > 0$ be smooth on $\mathbb{R}^n$ and equal to one outside a ball, and set $g_0 = a^{4/(n-2)}\mathsf{e}$. Suppose that $\sigma \in L^2(\mathbb{R}^n)$ has support in the interior of a closed ball $\mathcal{B}$ and*

$$\int_{\mathbb{R}^n} \sigma \left(a^2 \nabla u \cdot \nabla v - \frac{nk^2}{n-2} |g_0|^{1/2} uv \right) \mathrm{d}x = 0 \tag{2.21}$$

for every pair $u, v \in C^\infty(\mathbb{R}^n)$ satisfying $(\Delta_{g_0} + k^2)u = (\Delta_{g_0} + k^2)v = 0$ on a neighborhood of $\mathcal{B}$. Then $\sigma = 0$.

Proof. The product identity is now

$$L_{A_0}(uv) = 2a^2 \nabla u \cdot \nabla v - 2k^2 |g_0|^{1/2} uv.$$

Substituting this into (2.21) and integrating by parts yields

$$\left\langle a^{-2} \left(L_{A_0}\sigma - \frac{4k^2}{n-2} |g_0|^{1/2} \sigma \right), (au)(av) \right\rangle = 0.$$

The distribution in this pairing belongs to H^{-2} and has compact support. Using the background construction of Lemma 2.6 with two derivatives on $\mathcal{B}$, the complex geometric optics limiting argument of [7, Lemma 3.15] shows that its Fourier transform vanishes. Hence

$$L_{A_0}\sigma - \frac{4k^2}{n-2} |g_0|^{1/2} \sigma = 0.$$

Elliptic regularity makes σ smooth. Since σ has compact support, unique continuation (or the negativity of the associated quadratic form) implies $\sigma = 0$. □

The next proposition identifies the kernel of the linearized Helmholtz DN map, expressed in conductivity variables, with the potential tensors. It proves injectivity of the full linearization modulo infinitesimal boundary-fixing diffeomorphisms. It is the Helmholtz version of [7, Proposition 3.16]; its L^2 formulation is needed for the limit in Lemma 2.12.

Proposition 2.16. *A real symmetric tensor $H \in L^2(M)$ satisfies*

$$\int_M \left(a^2 H^{ij} (\partial_i u)(\partial_j v) - \frac{k^2}{n-2} |g_0|^{1/2} (\operatorname{tr} H) uv \right) \mathrm{d}x = 0 \tag{2.22}$$

for all $u, v \in C^\infty(M)$ satisfying $(\Delta_{g_0} + k^2)u = (\Delta_{g_0} + k^2)v = 0$ if and only if

$$H = \mathcal{G}_a X \qquad \text{for some } X \in H^1_0(M; \mathbb{R}^n). \tag{2.23}$$

Proof. The reverse implication is Lemma 2.13. For the forward implication, extend H by zero to $\mathbb{R}^n$. Fix $\xi \neq 0$ and orthonormal $p, q \in \xi^\perp$, and test (2.22) with the background complex geometric optics solutions with phases ζ_t, η_t used in [7, Proposition 3.16]. Since $\zeta_t + \eta_t = -\mathrm{i}\xi$, the additional zeroth-order term, after division by t^2, is $O(k^2 t^{-2} \|H\|_{L^2})$ by (2.13). Thus the same limit yields

$$\widehat{H}(\xi)(p + \mathrm{i}q, p + \mathrm{i}q) = 0.$$

As in that proposition, the restriction of $\widehat{H}(\xi)$ to $\xi^\perp$ is scalar. Lemma 2.14 supplies compactly supported X, β, chosen using only H, and

$$H - \mathcal{G}_a X = \sigma \mathsf{e}, \qquad \sigma = \beta + \operatorname{div} X + 2X \cdot \nabla \log a.$$

The scalar σ belongs to L^2 and has compact support. For background Helmholtz solutions near a ball containing $M \cup \operatorname{supp} X \cup \operatorname{supp} \sigma$ in its interior, subtracting (2.19) from (2.22) gives (2.21). Lemma 2.15 therefore implies $\sigma = 0$.

To prove that X vanishes outside M, we vary the background there as in [7, Proposition 3.16]. For real $\varphi \in C_0^\infty(\mathbb{R}^n \setminus M)$, set $\tilde{a} = ae^{\varphi}$ and $\tilde{g}_0 = \tilde{a}^{4/(n-2)}\mathsf{e}$. Both the conductivity and volume density agree with those of g_0 on M. Hence (2.22) holds for this background as well. Keeping the same X, β and repeating the scalar argument with $\tilde{a}$ gives $H = \mathcal{G}_{\tilde{a}} X$, and therefore

$$0 = \mathcal{G}_{\tilde{a}} X - \mathcal{G}_a X = -2(X \cdot \nabla\varphi)\mathsf{e}. \tag{2.24}$$

Since (2.24) holds for every such φ, the support argument of [7, Proposition 3.16] implies that $X = 0$ outside M, including its bounded complementary components. This proves (2.23). □

2.9. **Completion of the proof.**

Proof of Theorem 1. Suppose the conclusion fails. As in the proof of [7, Theorem 2], apply Lemmas 2.2–2.3 to a sequence of metrics $g_j \to g_0$ in $H^s(M)$ with $\Lambda_{g_j}(k) = \Lambda_{g_0}(k)$ which are not related to g_0 by boundary-fixing diffeomorphisms. Write $\overline{g}_j = (\Phi_j)_* g_j$, with perturbations B_j as in (2.2). Then $b_{s,j} \to 0$ by (2.6), and $B_j \neq 0$, since $B_j = 0$ would imply $\overline{g}_j = g_0$. Lemma 2.12 yields, after passing to a subsequence,

$$H_j = B_j / b_{2,j} \longrightarrow H \quad \text{in } L^2, \qquad \|H\|_{L^2} = 1.$$

Set $A_j = A_{\overline{g}_j} = a^2(\mathrm{Id} + B_j)$. By (2.9), $A_j \to A_0$ and $|\overline{g}_j|^{1/2} \to |g_0|^{1/2}$ in $L^\infty(M)$. Since

$$|\overline{g}_j|^{1/2} = |g_0|^{1/2} \det(\mathrm{Id} + B_j)^{1/(n-2)},$$

Taylor's formula implies

$$\left\| \frac{|\overline{g}_j|^{1/2} - |g_0|^{1/2}}{b_{2,j}} - \frac{|g_0|^{1/2}}{n-2} \operatorname{tr} H_j \right\|_{L^2(M)} \leq C b_{\infty,j} \longrightarrow 0. \tag{2.25}$$

Thus the normalized volume-density differences converge to $|g_0|^{1/2} \operatorname{tr} H/(n-2)$ in $L^2(M)$.

Fix $u, v \in C^\infty(M)$ satisfying $(\Delta_{g_0} + k^2)u = (\Delta_{g_0} + k^2)v = 0$, and let u_j solve $(\Delta_{\overline{g}_j} + k^2)u_j = 0$ with boundary value u. Then

$$\big(L_{A_j} + k^2 |\overline{g}_j|^{1/2}\big)(u_j - u) = -\operatorname{div}(a^2 B_j \nabla u) - k^2\big(|\overline{g}_j|^{1/2} - |g_0|^{1/2}\big)u, \qquad (u_j - u)|_{\partial M} = 0.$$

By the assumption on k^2, the Dirichlet inverses are uniformly bounded from $H^{-1}(M)$ to $H_0^1(M)$, as in Lemma 2.2. Therefore

$$\|u_j - u\|_{H^1(M)} \leq C b_{2,j}, \tag{2.26}$$

where C may depend on u. Dividing the Alessandrini identity (2.1) by $b_{2,j}$, replacing u_j by u using (2.26), and applying (2.25), we obtain

$$\left| \int_M \left(a^2 H_j^{ij} (\partial_i u)\, (\partial_j v) - \frac{k^2}{n-2} |g_0|^{1/2} (\operatorname{tr} H_j) uv \right) \mathrm{d}x \right| \leq C(b_{2,j} + b_{\infty,j}) \longrightarrow 0. \tag{2.27}$$

Passing to the limit in (2.27) proves (2.22) for H. Proposition 2.16 therefore implies $H = \mathcal{G}_a X$ for some $X \in H_0^1(M; \mathbb{R}^n)$.

Let Z_j be the normal fields from Lemma 2.2 and set $V_j = Z_j / b_{2,j}$. By (2.4), this sequence is bounded in $H_0^1(M; \mathbb{R}^N)$, and (2.3), (2.5) imply

$$\big(L_{A_j} + k^2 |\overline{g}_j|^{1/2}\big) V_j = -\operatorname{div}(a^2 H_j \nabla F) - k^2 \frac{|\overline{g}_j|^{1/2} - |g_0|^{1/2}}{b_{2,j}} F, \qquad DF^\top V_j = 0. \tag{2.28}$$

After passing to a further subsequence, $V_j \rightharpoonup V$ in H_0^1. Using the L^∞ convergence of the coefficients, $H_j \to H$ in L^2, and (2.25), we pass to the weak formulation of (2.28) and obtain

$$\begin{aligned} \big(L_{A_0} + k^2|g_0|^{1/2}\big)V &= -\operatorname{div}(a^2 H \nabla F) - \frac{k^2}{n-2}|g_0|^{1/2}(\operatorname{tr} H)F, \\ DF^\top V &= 0. \end{aligned} \tag{2.29}$$

Each component of F solves the background Helmholtz equation. Lemma 2.13, applied to $H = \mathcal{G}_a X$, shows that $-DFX \in H_0^1(M;\mathbb{R}^N)$ solves the same Dirichlet problem as V in (2.29). Since k^2 is not a Dirichlet eigenvalue of $-\Delta_{g_0}$, we have $V = -DFX$. The final argument is the same as in the proof of [7, Theorem 2]: normality implies $DF^\top DF\, X = 0$, and the full rank of DF from Proposition 2.1 forces $X = 0$. Hence $H = 0$, contradicting $\|H\|_{L^2} = 1$. □

3. Proof of Corollary 1.1

By the strict domain monotonicity of the Dirichlet eigenvalues, we can choose a slightly larger concentric ball $B' \Supset \overline{B}$, if needed, so that k^2 is not a Dirichlet eigenvalue of $-\Delta_{g_0}$ in B'. The same holds for g when ε_0 is sufficiently small.

The next argument is standard in inverse scattering. Equality of the scattering matrices implies equality of the DN maps on $\partial B'$; see [14, Section 1, p. 237]. Indeed, Rellich's lemma shows that the scattering solutions with the same incident plane wave coincide outside B. Their Cauchy data therefore agree on $\partial B'$. By duality and unique continuation, the linear span of their Dirichlet traces is dense in $H^{1/2}(\partial B')$, which proves the assertion. See also [12, Chapter IV, Theorem 5.1].

Theorem 1, applied in B', yields $g = \Phi^* g_0$ with $\Phi|_{\partial B'} = \mathrm{Id}$. Since Φ is an isometry fixing the boundary pointwise and preserving the interior, $D\Phi$ fixes tangent vectors and the inward unit normal on $\partial B'$. The inward normal geodesics for both metrics are radial segments in $B' \setminus \overline{B}$, so uniqueness for the geodesic equation implies $\Phi(r,\omega) = (r,\omega)$ there, in polar coordinates about the common center of the balls. Extending Φ by the identity outside B' proves the corollary.

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Université Paris-Saclay, Laboratoire de mathématiques d'Orsay, 91405, Orsay, France
Email address: sebastian.munoz-thon@universite-paris-saclay.fr

Department of Mathematics, Purdue University, West Lafayette, IN 47907
Email address: stefanov@math.purdue.edu