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Uniqueness of the Multi-Dimensional Inverse Scattering Problem for Time Dependent Potentials*

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1. Introduction

Consider the wave equation

$$u_{tt}-\triangle u+q(t,x)\,u=0 \tag{1.1}$$

$t\in\mathbb{R}$, $x\in\mathbb{R}^n$, $n\geqq 3$ odd. The potential $q(t, x)$ depends on t and satisfies the following conditions:

(i) $q\in C^\infty(\mathbb{R}_t\times\mathbb{R}^n_x)$,

(ii) there exist some positive constants ρ, C, N such that $q(t, x)=0$ for $|x|>\rho$ and $|q(t, x)|\leqq C(1+|t|)^N$.

The aim of this paper is to prove that the potential $q(t, x)$ is uniquely determined by the scattering data.

The inverse scattering problem for stationary (time independent) potentials $q(x)$ has been attacked by many authors. The reader should consult [2], [10], [18] and the references given there for the history and the recent progress in the analysis of this problem. Most of the works in this direction deal with the Schrödinger equation. Nevertheless it is clear that the results obtained in these papers are applicable to the wave equation $u_{tt}-\triangle u+q(x)\ u=0$. It is known that for stationary potentials the uniqueness holds. The proof is based on the Born approximation of the scattering amplitude $A(k, \omega', \omega)$ as $k\to\infty$ (see [7], [16], [22]).

The situation considerably changes when we deal with time dependent potentials. The techniques used in the papers cited above are not available in this case. One of the reasons is that we cannot use the tools from the spectral theory. The local energy decay for (1.1) was examined by Tamura [28], while the existence of the scattering operator was proved in [1], [8], [19], [20] for certain classes of potentials $q(t, x)$. In [8] Ferreira and Perla Menzala proved that if the scattering operators related to $q_i(t, x)$, $i=1, 2$ coincide and if $q_1\geqq q_2$ then $q_1=q_2$, provided that q_i are non-negative, "small" in a suitable sense and $q_t=O(|t|^{-\alpha})$ as $|t|\to\infty$, $0<\alpha\leqq 1$. Recently the author [23–26] extended

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this result to a large class of potentials without assuming that the potentials are small. Nevertheless the problem of uniqueness was not completely solved because of the restriction $q_1 \geqq q_2$.

The main purpose of this work is to prove the uniqueness of the inverse scattering problem for (1.1) assuming only (i) and (ii). It is important to note that (i) and (ii) do not guarantee the existence of the scattering operator S. Following Cooper and Strauss [5], [6] for this reason we define a *generalized scattering kernel* $K^{\#}(s', \omega'; s, \omega)$, $(s', s) \in \mathbb{R}^2$, $(\omega', \omega) \in S^{n-1} \times S^{n-1}$. This kernel describes the scattering of the plane waves $\delta(t+s-x\cdot\omega)$ by the potential. Here the unprimed variables denote the parameters of the incoming wave and the primed variables those of the outgoing wave. The kernel $K^{\#}$ will be shown to be smooth in ω', s, ω with values in the space of distributions $\mathcal{D}'(\mathbb{R}_{s'})$ and C^∞ off the diagonal $(s', \omega') = (s, \omega)$. If the scattering operator S exists, $K^{\#}$ is the Scwartz kernel of the operator $S-I$ in the Lax-Phillips translation representation [14]. Our principal result is the following.

Theorem 1.1. *Let* $q_1(t, x)$ *and* $q_2(t, x)$ *be two potentials satisfying* (i) *and* (ii) *and let* $K_i^{\#}(s', \omega'; s, \omega)$, $i=1, 2$ *be the corresponding generalized scattering kernels. Suppose there exist* $\varepsilon > 0$ *and* $\omega_0 \in S^{n-1}$ *such that* $K_1^{\#}(s', \omega'; s, \omega) = K_2^{\#}(s', \omega'; s, \omega)$ *for* $|\omega - \omega_0| < \varepsilon$, $|\omega' - \omega_0| < \varepsilon$, $|s' - s| < \varepsilon$. *Then* $q_1(t, x) = q_2(t, x)$ *for all* t, x.

The structure of the paper is as follows. Section 2 contains some basic facts which are necessary for our exposition. The generalized scattering kernel $K^{\#}$ is introduced in Sect. 3. Next we examine the singularities of the solution $u(t, x; s, \omega)$ of (1.1) which coincides with the plane wave $\delta(t+s-x\cdot\omega)$ for t large negative. For this reason in Theorem 4.1 we construct a parametrix for $u(t, x; s, \omega)$ in the form of a progressive wave expansion. In particular, we prove that $u(t, x; s, \omega) = \delta(t+s-x\cdot\omega) + u_{sc}(t, x; s, \omega)$ where u_{sc} is a locally bounded function. In Sect. 5 we derive a suitable representation of $K^{\#}$ involving the solution $u(t, x; s, \omega)$. As a consequence, for $\omega' \neq \omega$ $K^{\#}$ has the form

$$K^{\#}(s', \omega'; s, \omega) = -2^{-1}(2\pi)^{1-n}\, \partial_s^{(n-3)/2}\, \partial_{s'}^{(n-1)/2}\, M(s', \omega'; s, \omega),$$

where

$$M(s', \omega'; s, \omega) = \frac{1}{|\omega' - \omega|} \int_{x\cdot(\omega'-\omega) = s'-s} q(x\cdot\omega - s, x)\, dS_x + \int q(x\cdot\omega' - s', x)\, u_{sc}(x\cdot\omega' - s', x; s, \omega)\, dx \tag{1.2}$$

(see Corollary 5.4). In Sect. 6 we prove Theorem 1.1. The main idea is the following. Let us examine the limit of $|\omega' - \omega|\, M$ as $(s', \omega') \to (s, \omega)$. Since u_{sc} is locally bounded, the second term in the right hand side of (1.2) gives no contribution to this limit. On the other hand, the limit of the first integral in (1.2) depends on the choice of the sequence $(s', \omega') \to (s, \omega)$. We set

$$\omega'(\mu) = \omega \cos\mu + a \sin\mu,$$
$$s'(\mu) = s + \alpha \sin\mu, \qquad (\mu, \alpha) \in \mathbb{R}^2,$$

where $a \in S^{n-1}$, $a \cdot \omega = 0$. Note that the primes here do not denote derivatives. Letting $\mu \to 0$, we obtain that

$$\lim_{\mu \to 0} |\omega' - \omega| M = \int_{a \cdot x = \alpha} q(x \cdot \omega - s, x)\, dS_x.$$

This relation allows us to determine the integrals of q over some characteristic rays. In this way the problem of uniqueness is reduced to the proof of the fact that these integrals determine q uniquely. Finally, in Sect. 7 we discuss briefly the connection between our time dependent approach and the Born approximation of the scattering amplitude for stationary potentials.

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2. Preliminary Results

We first recall some facts about the free wave equation $\square u = 0$, $\square = \partial_t^2 - \triangle_x$ (see [14]). Denote by $\mathscr{H} = H_D(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ the completion of the set $C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n) \ni \mathbf{f} = (f_1, f_2)$ with respect to the energy norm

$$\|\mathbf{f}\| = \{\tfrac{1}{2} \int (|\nabla f_1|^2 + |f_2|^2)\, dx\}^{1/2}.$$

$\mathscr{H}$ is a Hilbert space with energy scalar product. It is well-known that for all $R > 0$, $g \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int_{|x| \leq R} |g(x)|^2\, dx \leqq \frac{R^2}{2(n-2)} \int |\nabla g(x)|^2\, dx. \tag{2.1}$$

Estimate (2.1) shows that $H_D(\mathbb{R}^n) \subset L^2_{\mathrm{loc}}(\mathbb{R}^n)$. Denote by $U_0(t) = \exp(tA)$ the unitary group in $\mathscr{H}$ with generator $A\mathbf{f} = (f_2, \triangle f_1)$, $D(A) = \{\mathbf{f} = (f_1, f_2) \in \mathscr{H};\ (f_2, \triangle f_1) \in \mathscr{H}\}$. Then the solution of the Cauchy problem

$$\square u = 0,$$
$$u(0, x) = f_1(x), \qquad u_t(0, x) = f_2(x), \qquad \mathbf{f} = (f_1, f_2) \in \mathscr{H}$$

is given by $\mathbf{u}(t) = U_0(t)\mathbf{f}$. Here and in what follows we denote the pair $(u(t, \cdot), u_t(t, \cdot))$ by $\mathbf{u}(t)$. The Huygens' principle says that supp $U_0(t)\mathbf{f} \subset \{x \in \mathbb{R}^n;\ |x - y| = |t|$ for some $y \in \operatorname{supp} \mathbf{f}\}$. There exists a unitary map $\mathscr{R}_n\colon \mathscr{H} \to L^2(\mathbb{R} \times S^{n-1})$ called the free translation representation, such that $(\mathscr{R}_n U_0(t) \mathscr{R}_n^{-1} k)(s, \omega) = k(s - t, \omega)$. For $\mathbf{f} \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$ the map $\mathscr{R}_n$ is given by

$$k(s, \omega) = (\mathscr{R}_n \mathbf{f})(s, \omega) = c_n(-\partial_s^{(n+1)/2} R f_1 + \partial_s^{(n-1)/2} R f_2), \tag{2.2}$$

where Rg is the Radon transform of g and for simplicity of notations we set $c_n = 2^{-1}(2\pi)^{(1-n)/2}$, $c_n^- = 2^{-1}(-2\pi)^{(1-n)/2}$. The inverse map $\mathscr{R}_n^{-1}$ is given by

$$(\mathscr{R}_n^{-1} k)(x) = 2c_n^- \int_{|\omega| = 1} (-\partial_s^{(n-3)/2} k(x \cdot \omega, \omega), \quad \partial_s^{(n-1)/2} k(x \cdot \omega, \omega))\, d\omega. \tag{2.3}$$

Next we turn our attention to the perturbed Eq. (1.1). Following [12, Theorems 4.5 and 6.1] (see also [1], [24], [26]) it is easy to show that the solution of the Cauchy problem

$$(\square + q(t,x))\,u = 0,$$
$$u(s,x) = f_1(x), \quad u_t(s,x) = f_2(x), \quad \mathbf{f} = (f_1, f_2) \in \mathscr{H}$$

is given by $\mathbf{u}(t) = U(t,s)\mathbf{f}$. Here $U(t,s)$ is a strongly continuous two-parameter family of operators in $\mathscr{H}$ satisfying the conditions

a) $U(t,s)\,U(s,r) = U(t,r)$ for all t, s, r; $U(s,s) = I$;

b) $\|U(t,s)\| \leqq \exp\left\{\gamma |t-s| \sup_{\substack{s \leqq \tau \leqq t \\ x \in \mathbb{R}^n}} |q(\tau, x)|\right\}$;

c) for $\mathbf{f} \in D(A)$ we have $U(t,s)\,\mathbf{f} \in D(A)$ and

$$(d/dt)\,U(t,s)\,\mathbf{f} = (A - Q(t))\,U(t,s)\,\mathbf{f},$$
$$(d/ds)\,U(t,s)\,\mathbf{f} = -U(t,s)\,(A - Q(s))\,\mathbf{f},$$

where $Q(t)\mathbf{f} = (0, q(t,x)\,f_1(x))$. Note that $Q(t)$ is a bounded operator in $\mathscr{H}$ in view of (2.1).

The operatur $U(t,s)$ admits the representation

$$U(t,s) = U_0(t-s) + \sum_{k=1}^{\infty} V_k(t,s), \tag{2.4}$$

where

$$V_k(t,s)\,\mathbf{f} = (-1)^k \int_s^t d\sigma_1 \int_s^{\sigma_1} d\sigma_2 \ldots \int_s^{\sigma_{k-1}} d\sigma_k \cdot U_0(t-\sigma_1)\,Q(\sigma_1) \ldots U_0(\sigma_{k-1}-\sigma_k)\,Q(\sigma_k)\,U_0(\sigma_k - s)\,\mathbf{f}, \tag{2.5}$$

$k \geqq 1$ (see [12, Theorem 4.5], [24], [26]). The convergence of (2.4) follows from the estimate

$$\|V_k(t,s)\| \leqq \frac{|t-s|^k}{k!}\left(\sup_{s \leqq \tau \leqq t} \|Q(\tau)\|\right)^k.$$

Expansion (2.4) shows that for (1.1) is valid the principle of causality. More precisely, causality means that

$$\operatorname{supp} U(t,s)\,\mathbf{f} \subset \{x \in \mathbb{R}^n;\quad |x-y| \leqq |t-s| \quad \text{for some } y \in \operatorname{supp} \mathbf{f}\}.$$

This property enables us to extend $U(t,s)$ (and $U_0(t)$) on the space $\mathscr{H}_{\mathrm{loc}} = \{\mathbf{f};\ \varphi\mathbf{f} \in \mathscr{H}$ for each $\varphi \in C_0^\infty(\mathbb{R}^n)\}$ by using a partition of unity.

Below we recall some results of Cooper and Strauss [4], [5] which are necessary for our exposition.

Definition 2.1. The continuous function $t \to \mathbf{v}(t) \in \mathscr{H}_{\mathrm{loc}}$ is called *outgoing* if $\lim_{t \to -\infty} (\mathbf{v}(t), U_0(t)\,\mathbf{g}) = 0$ for each $\mathbf{g} \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$.

Here $(\cdot,\cdot)$ is the scalar product in $\mathscr{H}$.

Theorem 2.2 [4], [5]. *Let* $p(t,x)\in L^1_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{R}^n))$, $p(t,x)=0$ *for* $|x|>\rho$. *Then*

a) *there exists a unique outgoing solution* $\mathbf{u}(t)=(u,u_t)$ *of the equation* $u_{tt}-\triangle u=p(t,x)$;

b) *there exists a unique function* $u^{\#}\in L^2_{\mathrm{loc}}(\mathbb{R}\times S^{n-1})$, *called the asymptotic wave profile of* u, *such that for each pair* $R_1<R_2$ *we have*

$$\int\limits_{R_1+t\leqq|x|\leqq R_2+t} |u_t(t,x)-|x|^{-(n-1)/2}u^{\#}\left(|x|-t,\frac{x}{|x|}\right)|^2\,dx\to 0, \qquad as\ t\to\infty; \tag{2.6}$$

c) *the map* $p\to u^{\#}$ *is continuous.*

Note that condition (2.6) can be written in the following equivalent form

$$u^{\#}(s,\omega)=\lim_{t\to\infty} t^{(n-1)/2}u_t(t,(t+s)\omega), \tag{2.7}$$

where the convergence takes place in $L^2_{\mathrm{loc}}(\mathbb{R}\times S^{n-1})$.

There is another situation when the asymptotic wave profile exists. Let $\mathbf{u}(t)=U_0(t)\,\mathbf{f}$, $\mathbf{f}\in\mathscr{H}$. Setting

$$u^{\#}=(-1)^{(n-1)/2}\mathscr{R}_n\mathbf{f}, \tag{2.8}$$

one can prove that (2.6) and (2.7) hold for $t\to\pm\infty$. Moreover, then the integration in (2.6) can be taken over $\mathbb{R}^n$ and similarly (2.7) holds in $L^2(\mathbb{R}\times S^{n-1})$ (see [14], [5]).

3. The Generalized Scattering Kernel

We are going to introduce the generalized scattering kernel $K^{\#}$. Our definition (see also [26], [27]) is close to that given in [5], [6] in the case of scattering by a moving obstacle. Let $h_j(\tau)=\tau^j/j!$ for $\tau\geqq 0$ and $h_j(\tau)=0$ otherwise. Then $h_j'=h_{j-1}$, h_0 is the Heaviside function. Denote by $\Gamma(t,x;s,\omega)$ the solution of the Cauchy problem

$$(\Box+q(t,x))\,\Gamma=0,$$
$$\Gamma|_{t<-s-\rho}=h_1(t+s-x\cdot\omega), \qquad (s,\omega)\in\mathbb{R}\times S^{n-1}.$$

In other words, $(\Gamma,\Gamma_t)=U(t,-s-\rho)\,(h_1(-\rho-x\cdot\omega),h_0(-\rho-x\cdot\omega))$. The function $\Gamma_{sc}(t,x;s,\omega)=\Gamma(t,x;s,\omega)-\mathrm{h}_1(t+s-x\cdot\omega)$ is the unique outgoing solution of the equation $\Box\Gamma_{sc}=-q(t,x)\,\Gamma(t,x;s,\omega)\in C(\mathbb{R}_s\times S^{n-1}_\omega; L^1_{\mathrm{loc}}(\mathbb{R}_t; L^2(\mathbb{R}^n_x)))$. By Theorem 2.2 the function Γ_{sc} has an asymptotic wave profile $\Gamma^{\#}_{sc}(s',\omega';s,\omega)\in C(\mathbb{R}_s\times S^{n-1}_\omega; L^2_{\mathrm{loc}}(\mathbb{R}_{s'}\times S^{n-1}_{\omega'}))$ satisfying (2.6) and (2.7). We define

$$K^{\#}(s',\omega';s,\omega)=2c_n^{-}\,\partial_s^{(n+1)/2}\,\Gamma^{\#}_{sc}(s',\omega';s,\omega). \tag{3.1}$$

To describe the role which plays $K^{\#}$ in the scattering theory for Eq. (1.1), consider the following situation. Let $\mathbf{f}\in\mathscr{H}$ be such that $\mathscr{R}_n\mathbf{f}\in C_0^\infty(\mathbb{R}\times S^{n-1})$,

$(\mathscr{R}_n \mathbf{f})(s, \omega)=0$ for $|s|>R$ and set $\mathbf{v}_0(t)=U_0(t)\mathbf{f}$. Then $\mathbf{v}_0(t)$ vanishes in the backward cone $|x|<-t-R$ [14], hence $\mathbf{v}_0(t)$ is a solution of (1.1) for $t<-R-\rho$. Denote by $\mathbf{v}(t)$ that solution of (1.1) which coincides with $\mathbf{v}_0(t)$ for $t<-R-\rho$. Then the function $\mathbf{v}(t)-\mathbf{v}_0(t)$ is the outgoing solution of the equation $\square(v-v_0)=-qv$. By Theorem 2.2 $v-v_0$ has an asymptotic wave profile. Since v_0 is a free solution, there exists $v_0^{\#}$ given by (2.8). Consequently, we can define $v^{\#}=(v-v_0)^{\#}+v_0^{\#}$. Definition (3.1) of the generalized scattering kernel $K^{\#}$ is motivated by the following.

Proposition 3.1. *Let v_0 and v be as above. Then*

$$v^{\#}(s', \omega')=v_0^{\#}(s', \omega')+\int_{|\omega|=1}\int K^{\#}(s', \omega'; s, \omega)\, v_0^{\#}(s, \omega)\, ds\, d\omega,$$

where the integral is to be considered in the distribution sense.

Proof (following [5]). First we will show that

$$v(t, x)=v_0(t, x)-2c_n \int_{|\omega|=1}\int \Gamma_{sc}(t, x; s, \omega)\, \partial_s^{(n+1)/2}\, v_0^{\#}(s, \omega)\, ds\, d\omega. \tag{3.2}$$

Denote by $v_1(t, x)$ the right hand side of (3.2). Since Γ_{sc} vanishes for $s<-t-\rho$ and $v_0^{\#}$ vanishes for $|s|>R$, it follows that $v_1=v_0$ for $t<-R-\rho$. On the other hand, (2.3) and (2.8) yield

$$v_0(t, x)=-2c_n \int_{|\omega|=1} \partial_s^{(n-3)/2}\, v_0^{\#}(x\cdot\omega-t, \omega)\, d\omega.$$

We have

$$\begin{aligned} v_1(t, x)&=v_0(t, x)+2c_n \int_{|\omega|=1}\int h_1(t+s-x\cdot\omega)\, \partial_s^{(n+1)/2}\, v_0^{\#}(s, \omega)\, ds\, d\omega \\ &\quad -2c_n \int_{|\omega|=1}\int \Gamma(t, x; s, \omega)\, \partial_s^{(n+1)/2}\, v_0^{\#}(s, \omega)\, ds\, d\omega \\ &=-2c_n \int_{|\omega|=1}\int \Gamma(t, x; s, \omega)\, \partial_s^{(n+1)/2}\, v_0^{\#}(s, \omega)\, ds\, d\omega. \end{aligned}$$

Hence $v_1(t, x)$ is a solution of (1.1) with Cauchy data $v_1=v_0$ for $t<-R-\rho$. This fact proves (3.2). Taking the asymptotic wave profiles of each side of (3.2) we complete the proof of the proposition.

In the remainder of this section we will study the case when the scattering operator exists. The wave operators associated with (1.1) are

$$\Omega_+=\mathrm{s}-\lim_{t\to\infty} U(0, -t)\, U_0(-t),$$

$$W\mathbf{f}=\lim_{t\to\infty} U_0(-t)\, U(t, 0)\,\mathbf{f}, \qquad \mathbf{f}\in\overline{\operatorname{Ran}\Omega_+}.$$

We say that the scattering operator S exists, if the limits Ω_+ and W exist defining bounded operators. Then we set $S=W\Omega_+$. We refer to [1], [8], [19], [20] for sufficient conditions guaranteeing the existence of S.

Proposition 3.2. *Suppose the scattering operator S exists. Then the distribution $K^{\#}$ is the Schwartz kernel of the operator $\mathscr{R}_n(S-I)\,\mathscr{R}_n^{-1}$.*

Proof. Let $k\in C_0^\infty(\mathbb{R}\times S^{n-1})$ and set $\mathbf{f}=\mathscr{R}_n^{-1}k$, $\mathbf{v}_0(t)=U_0(t)\,\mathbf{f}$. Let $\mathbf{v}(t)$ be related to $\mathbf{v}_0(t)$ as in Proposition 3.1. Obviously $\mathbf{v}(t)=U(t,0)\,\Omega_+\mathbf{f}$. The definition of S implies that

$$\|U_0(t)\,S\mathbf{f}-\mathbf{v}(t)\|\to 0,\qquad \text{as } t\to\infty.$$

Thus the asymptotic wave profiles of $U_0(t)\,S\mathbf{f}$ and $\mathbf{v}(t)$ must coincide. Combining (2.8) and Proposition 3.1 we find

$$(\mathscr{R}_n S\mathbf{f})(s',\omega')=(\mathscr{R}_n\mathbf{f})(s',\omega')+\int\limits_{|\omega|=1}\int K^{\#}(s',\omega';s,\omega)\,(\mathscr{R}_n\mathbf{f})(s,\omega)\,ds\,d\omega,$$

which yields the required result.

4. Construction of Parametrix for u_{sc}

An important role in our analysis plays the function u_{sc} defined by $u_{sc}(t,x;s,\omega)=\partial_s^2\,\Gamma_{sc}(t,x;s,\omega)$. Clearly $u_{sc}(t,x;s,\omega)=u(t,x;s,\omega)-\delta(t+s-x\cdot\omega)$, where $u(t,x;s,\omega)$ solves the Cauchy problem

$$\begin{aligned}&(\Box+q(t,x))\,u=0,\\ &u|_{t<-s-\rho}=\delta(t+s-x\cdot\omega).\end{aligned}\tag{4.1}$$

Note that by causality u, u_{sc} and Γ_{sc} vanish for $t+s<x\cdot\omega$. A priori u_{sc} is a distribution. Below we prove that in fact u_{sc} is a function which is continuous for $t+s>x\cdot\omega$ and it has a jump at $t+s=x\cdot\omega$. The next theorem could be considered as a generalization of the corresponding result in [17] (see also [3]) established for stationary potentials.

Recall the functions $h_j(\tau)$ introduced in Sect. 2. We are looking for an approximative solution u_N of the problem (4.1) in the form of the progressive wave expansion

$$u_N(t,x;s,\omega)=\delta(t+s-x\cdot\omega)+\sum_{j=0}^{N}A_j(t,x,\omega)\,h_j(t+s-x\cdot\omega).$$

For convenience denote $h_{-1}=\delta$ and $h_j=h_j(t+s-x\cdot\omega)$. We have

$$\begin{aligned}(\Box+q)\,u_N&=-\sum_{j=0}^{N}(\nabla_x-\omega\partial_t)\cdot(\nabla_x+\omega\partial_t)\,(A_j h_j)+q\delta+q\sum_{j=0}^{N}A_j h_j\\ &=2\sum_{j=0}^{N}[(\partial_t+\omega\cdot\nabla_x)\,A_j]\,h_{j-1}+\sum_{j=0}^{N}[(\Box+q)\,A_j]\,h_j+q\delta.\end{aligned}$$

Equating singular coefficients we obtain the following transport equations for A_j

$$2(\partial_t+\omega\cdot\nabla_x)A_0=-q,$$
$$2(\partial_t+\omega\cdot\nabla_x)A_j=-(\square+q)A_{j-1},\quad j=1,2,\dots,N.$$

These equations can be solved setting

$$A_0(t,x,\omega)=-\frac{1}{2}\int_{-\infty}^{0} q(t+\sigma,x+\sigma\omega)\,d\sigma,$$

$$A_j(t,x,\omega)=-\frac{1}{2}\int_{-\infty}^{0}(\square+q)A_{j-1}(t+\sigma,x+\sigma\omega,\omega)\,d\sigma,\quad j=1,\dots,N.$$

Clearly,

$$A_j(t,x,\omega)=0\quad\text{for } x\cdot\omega<-\rho\quad\text{and for } |x-(x\cdot\omega)\,\omega|>\rho. \tag{4.2}$$

Thus we obtain $(\square+q)u_N=[(\square+q)A_N]\,h_N$. This equality shows that the remainder $R_N=u-u_N$ solves the Cauchy problem

$$(\square+q)R_N=-[(\square+q)A_N]\,h_N,\qquad R_N|_{t<-s-\rho}=0. \tag{4.3}$$

Observe that the right hand side of (4.3) is supported in the compact set $\{x\in\mathbb{R}^n;\ -\rho\leqq x\cdot\omega\leqq t+s,\ |x-(x\cdot\omega)\,\omega|\leqq\rho\}$ in view of (4.2). Hence it belongs to $C(\mathbb{R}_t\times\mathbb{R}_s\times S^{n-1}_\omega;\ H^N(\mathbb{R}^n_x))$, where H^N is the Sobolev space. Since $\square+q(t,x)$ is a strictly hyperbolic operator, the above Cauchy problem has a unique solution $R_N\in C(\mathbb{R}_t;\ H^{N+1}(\mathbb{R}^n_x))\cap C^1(\mathbb{R}_t;\ H^N(\mathbb{R}^n_x))$ (see [11, Theorem 23.2.3]). Moreover, R_N depends continuously on s, ω. Thus we have proved

Theorem 4.1. *For each integer N we have*

$$u(t,x;s,\omega)=\delta(t+s-x\cdot\omega)+\sum_{j=0}^{N}A_j(t,x,\omega)\,h_j(t+s-x\cdot\omega)+R_N(t,x,s,\omega),$$

where $R_N\in C(\mathbb{R}_t\times\mathbb{R}_s\times S^{n-1}_\omega;\ H^{N+1}(\mathbb{R}^n_x))$.

Following similar arguments, it is easy to prove that R_N depends smoothly on t, s, ω with derivatives $\partial_t^\alpha\,\partial_s^\beta\,\partial_\omega^\gamma\,R_N$ belonging to $H^m(\mathbb{R}^n_x)$ with $m=N+1-\alpha-\beta-|\gamma|$.

Corollary 4.2. $u_{sc}(t,x;s,\omega)=A_0(t,x,\omega)\,h_0(t+s-x\cdot\omega)+R_0(t,x,s,\omega)$, *where R_0 is continuous in t, x, s, ω.* Moreover, $\operatorname{supp}u_{sc}\subset\{x;\ |x|\leqq t+s+2\rho,\ x\cdot\omega\leqq t+s\}$.

Proof. Fix $N>n/2-1$. Then $R_0=\sum_{j=1}^{N}A_jh_j+R_N$. Since the sum is continuous function, it is sufficient to examine R_N. Clearly R_N depends continuously on x provided t, s, ω are fixed. Furthermore,

$$|R_N(t,x,s,\omega)-R_N(t_0,x_0,s_0,\omega_0)|\leqq|R_N(t_0,x,s_0,\omega_0)-R_N(t_0,x_0,s_0,\omega_0)|$$
$$+|R_N(t,x,s,\omega)-R_N(t_0,x,s_0,\omega_0)|.$$

The first term tends to zero, as $x \to x_0$, while the second one can be estimated by

$$\begin{aligned} &C\|\hat{R}_N(t,\cdot,s,\omega)-\hat{R}_N(t_0,\cdot,s_0,\omega_0)\|_{L^1(\mathbb{R}^n)} \\ &\leqq C[\int(1+|\xi|^2)^{-N-1}d\xi]^{1/2}\|R_N(t,\cdot,s,\omega)-R_N(t_0,\cdot,s_0,\omega_0)\|_{H^{N+1}(\mathbb{R}^n)} \end{aligned}$$

which goes to zero, as $t \to t_0$, $s \to s_0$, $\omega \to \omega_0$ in view of Theorem 4.1. Here $\hat{R}_N$ denotes the Fourier transform of R_N with respect to x.

To prove the last assertion of the corollary, it is sufficient to consider the support of Γ_{sc}. Duhamel's principle implies that

$$(\Gamma_{sc}, \partial_t \Gamma_{sc}) = -\int_{-s-\rho}^{t} U(t,\sigma)\, Q(\sigma)\, \mathbf{h}(\sigma)\, d\sigma,$$

where $\mathbf{h}(\sigma)=(h_1(\sigma+s-x\cdot\omega),\ h_0(\sigma+s-x\cdot\omega))$. According to (ii), $Q(\sigma)\ \mathbf{h}(\sigma)$ vanishes for $|x|>\rho$. By causality $\Gamma_{sc}=0$ for $|x|>t+s+2\rho$. This completes the proof of the corollary.

5. Representation of the Generalized Scattering Kernel

In this section we obtain a representation formula for $K^{\#}$ involving the solution $u(t,x;s,\omega)$ of the Cauchy problem (4.1).

Lemma 5.1. *Let p and $u^{\#}$ be the same as in Theorem 2.2. Then*

a) *if $p \in C^\infty$, we have*

$$u^{\#}(s,\omega) = c_n^- \, \partial_s^{(n-1)/2} \int p(x\cdot\omega - s, x)\, dx,$$

b) *for arbitrary $p \in L^1_{\text{loc}}(\mathbb{R}; L^2(\mathbb{R}^n))$ the above formula is valid in the following generalized sense:*

$$\int u^{\#}(s,\omega)\,\psi(s)\,ds = c_n \iint p(t,x)\, \partial_s^{(n-1)/2}\psi(x\cdot\omega - t)\, dt\, dx \tag{5.1}$$

for each $\psi(s) \in C_0^\infty(\mathbb{R})$.

Proof. The outgoing solution $\mathbf{u}(t)$ of the equation $\Box u = p(t,x)$ has the form [4], [5]

$$\mathbf{u}(t) = \int_{-\infty}^{t} U_0(t-\tau)\, \mathbf{p}(\tau)\, d\tau,$$

where $\mathbf{p}(\tau)=(0, p(\tau,\cdot))$. Fix $R_1 < R_2$ and set

$$\mathbf{f} = \int_{-R_2-\rho}^{-R_1+\rho} U_0(-\tau)\, \mathbf{p}(\tau)\, d\tau$$

and denote $\mathbf{v}(t)=U_0(t)\,\mathbf{f}$. Huygens' principle implies that $\mathbf{v}(t)=\mathbf{u}(t)$ for $R_1+t \leqq |x| \leqq R_2+t$. Hence $v^{\#}(s,\omega)=u^{\#}(s,\omega)$ for $R_1 \leqq s \leqq R_2$. On the other hand, by (2.2) we get

$$v^{\#}(s,\omega)=(-1)^{(n-1)/2}(\mathscr{R}_n\mathbf{f})(s,\omega)=c_n^-\,\partial_s^{(n-1)/2}\int_{-R_2-\rho}^{-R_1+\rho}\int_{x\cdot\omega=s+\tau} p(x\cdot\omega-s,x)\,dS_x\,d\tau.$$

For $R_1 \leqq s \leqq R_2$ we obtain

$$u^{\#}(s,\omega)=v^{\#}(s,\omega)=c_n^-\,\partial_s^{(n-1)/2}\int p(x\cdot\omega-s,x)\,dx.$$

Since the numbers $R_1<R_2$ are arbitrary, we complete the proof of (a). To prove (b), choose a sequence $p_k\in C^\infty$, $p_k=0$ for $|x|>\rho+1$, $p_k\to p$ in $L^1_{\mathrm{loc}}(\mathbb{R};\,L^2(\mathbb{R}^n))$, as $k\to\infty$. Then $u_k^{\#}\to u^{\#}$ in $L^2_{\mathrm{loc}}(\mathbb{R}\times S^{n-1})$ and by (a) we have

$$\int u_k^{\#}(s,\omega)\,\psi(s)\,ds=c_n\iint p_k(t,x)\,\partial_s^{(n-1)/2}\,\psi(x\cdot\omega-t)\,dt\,dx.$$

Letting $k\to\infty$, we see that the left hand side of the above equality tends weakly to the left hand side of (5.1), while the second term above tends to the second term in (5.1) uniformly in $\omega\in S^{n-1}$. This completes the proof of the lemma.

Remark. A straightforward proof of (a) in the case $n=3$ can be found in [4].

Now we are ready to derive the desired representation formula for $K^{\#}$. Recall the equality $\Box\,\Gamma_{sc}=-q\Gamma$. By Lemma 5.1 for each $\varphi(s')\in C_0^\infty(\mathbb{R})$ we have

$$\begin{aligned}\langle K^{\#}(s',\omega';s,\omega),\varphi(s')\rangle&=2c_n^-\,\partial_s^{(n+1)/2}\int\Gamma_{sc}^{\#}(s',\omega';s,\omega)\,\varphi(s')\,ds'\\&=-2c_n^2\,\partial_s^{(n+1)/2}\iint q(t,x)\,\Gamma(t,x;s,\omega)\,(-\partial_{s'})^{(n-1)/2}\,\varphi(x\cdot\omega'-t)\,dt\,dx.\end{aligned}\tag{5.2}$$

Here and in what follows we denote by $\langle f,\varphi\rangle$ the action of the distribution f on the test function φ. Applying the results of the preceding section it is easy to show that (5.2) depends smoothly on ω', s, ω. Thus we have proved

Theorem 5.2. *$K^{\#}$ admits the following representation*

$$\begin{aligned}K^{\#}(s',\omega';s,\omega)=&-2^{-1}(2\pi)^{1-n}\iint q(t,x)\,\partial_s^{(n-3)/2}\,u(t,x;s,\omega)\\&\cdot\,\partial_{s'}^{(n-1)/2}\,\delta(t+s'-x\cdot\omega')\,dt\,dx,\end{aligned}\tag{5.3}$$

where $u(t,x;s,\omega)$ is the solution of the Cauchy problem (4.1). *Moreover, $K^{\#}(s',\omega';s,\omega)\in C^\infty(S^{n-1}_{\omega'}\times\mathbb{R}_s\times S^{n-1}_\omega;\,\mathscr{D}'(\mathbb{R}_{s'}))$.*

Remark 1. Integral (5.3) is to be considered in the sense of (5.2)

Remark 2. A similar formula was found by Morawetz [17] for stationary potentials. Representations of this kind for scattering by obstacles were established in [6], [15].

Corollary 5.3. $K^{\#}(s',\omega';s,\omega)=0$ *for* $s'>s+\varrho|\omega'-\omega|$.

Proof. Assume $s'>s+\rho|\omega'-\omega|$, $|x|\leqq\rho$, $t+s'-x\cdot\omega'=0$. Then $s'-s>x\cdot(\omega'-\omega)$, hence $t+s-x\cdot\omega<0$ and by causality the solution $u(t,x;s,\omega)$ in integral (5.3) vanishes.

Formula (5.3) can be written in the following useful form.

Corollary 5.4. *For $\omega' \neq \omega$ we have*

$$K^{\#}(s', \omega'; s, \omega) = -2c_n^2\, \partial_s^{(n-3)/2}\, \partial_{s'}^{(n-1)/2} \left[\frac{1}{|\omega' - \omega|} \int_{x\cdot(\omega'-\omega)=s'-s} q(x\cdot\omega - s, x)\, dS_x \right.$$
$$\left. + \int q(x\cdot\omega' - s', x)\, u_{sc}(x\cdot\omega' - s', x; s, \omega)\, dx \right]. \tag{5.4}$$

Proof. Inserting the equality $\Gamma = \Gamma_{sc} + h_1(t + s - x\cdot\omega)$ in (5.2) we find

$$\langle K^{\#}(s', \omega'; s, \omega), \varphi(s')\rangle = I_1 + I_2, \tag{5.5}$$

where

$$\begin{aligned} I_1 &= -2c_n^2\, \partial_s^{(n-3)/2} \iint q(x\cdot\omega' - s', x)\, u_{sc}(x\cdot\omega' - s', x; s, \omega) \\ &\quad \cdot (-\partial_{s'})^{(n-1)/2} \varphi(s')\, dx\, ds', \\ I_2 &= -2c_n^2\, \partial_s^{(n+1)/2} \iint q(t, x)\, h_1(t + s - x\cdot\omega) \\ &\quad \cdot (-\partial_{s'})^{(n-1)/2}\, \varphi(x\cdot\omega' - t)\, dx\, dt \\ &= -2c_n^2\, \partial_s^{(n-3)/2} \int q(x\cdot\omega - s, x)\, (-\partial_{s'})^{(n-1)/2}\, \varphi(x\cdot(\omega' - \omega) + s)\, dx. \end{aligned} \tag{5.6}$$

Let us represent $\mathbb{R}^n$ as a union of the planes $\{x;\, x\cdot(\omega' - \omega) = s' - s\}$, $s' \in \mathbb{R}$. Then I_2 becomes

$$\begin{aligned} I_2 &= -2c_n^2\, \partial_s^{(n-3)/2} \frac{1}{|\omega' - \omega|} \int \int_{x\cdot(\omega'-\omega)=s'-s} q(x\cdot\omega - s, x) \\ &\quad \cdot (-\partial_{s'})^{(n-1)/2}\, \varphi(s')\, dS_x\, ds' \end{aligned} \tag{5.7}$$

Combining (5.5), (5.6) and (5.7), we complete the proof of (5.4).

Remark. Applying the results of Sect. 4 to (5.3) or (5.4), it is not hard to see that $K^{\#}$ is C^∞ off the diagonal $(s', \omega') = (s, \omega)$.

6. Proof of Theorem 1.1

Denote by $M(s', \omega'; s, \omega)$ the expression in the square brackets in (5.4). As we have just shown, M is C^∞ for $(s', \omega') \neq (s, \omega)$. We are going to examine the limit of $|\omega' - \omega|\, M(s', \omega'; s, \omega)$ as $(s', \omega') \to (s, \omega)$. Since M is singular at this point, the limit will depend on the choice of the sequence (s', ω'). Fix $(s, \omega) \in \mathbb{R} \times S^{n-1}$ and let $a \in S^{n-1}$, $a\cdot\omega = 0$. Set

$$\begin{aligned} \omega'(\mu) &= \omega \cos\mu + a \sin\mu, \\ s'(\mu) &= s + \alpha \sin\mu, \qquad (\alpha, \mu) \in \mathbb{R}^2. \end{aligned}$$

By Corollary 4.2 the function u_{sc} is locally bounded. Hence the second integral in (5.4) remains bounded when s, ω are fixed and s', ω' run over bounded sets. Consequently,

$$\lim_{\substack{\mu\to 0\\ \mu\neq 0}} |\omega'(\mu)-\omega|\, M(s'(\mu),\omega'(\mu);s,\omega) = \lim_{\mu\to 0} \int_{x\cdot(a-\omega\tan\mu/2)=\alpha} q(x\cdot\omega - s, x)\, dS_x$$

$$= \int_{x\cdot a=\alpha} q(x\cdot\omega - s, x)\, dS_x. \tag{6.1}$$

Now, let q_1 and q_2 be the potentials of Theorem 1.1. Denote by M_i the function M related to q_i, $i=1, 2$. We first claim that

$$M_1(s',\omega';s,\omega) = M_2(s',\omega';s,\omega)$$
$$\text{for } |s'-s|<\varepsilon_1, |\omega'-\omega_0|<\varepsilon_1, |\omega-\omega_0|<\varepsilon_1, \omega'\neq\omega \tag{6.2}$$

with some positive $\varepsilon_1 \leqq \varepsilon$. Indeed, fix ω', ω so that $\omega'\neq\omega$, $|\omega'-\omega_0|<v$, $|\omega-\omega_0|<v$ with $0<v\leqq\varepsilon$. Then $|\omega'-\omega|<2v$. The assumptions of Theorem 1.1 imply

$$\partial_s^{(n-3)/2}\,\partial_{s'}^{(n-1)/2}(M_1-M_2)=0 \quad \text{for } |s'-s|<\varepsilon, \tag{6.3}$$

$$M_1=M_2=0 \quad \text{for } s'>s+2v\rho. \tag{6.4}$$

Choosing v so that $2\rho v<\varepsilon/2$, it is easy to derive from (6.3) and (6.4) that $M_1=M_2$ for $|s'-s|<\varepsilon$, $|\omega'-\omega_0|<v$, $|\omega-\omega_0|<v$, $\omega'\neq\omega$. Of course, the same is true for $\omega'=\omega$. This proves (6.2). For convenience we denote $q=q_1-q_2$. Combining (6.1) and (6.2) we find

$$\int_{x\cdot a=\alpha} q(x\cdot\omega-s,x)\,dS_x=0 \tag{6.5}$$

for all $(s,\alpha)\in\mathbb{R}^2$, $(\omega, a)\in S^{n-1}\times S^{n-1}$ such that $|\omega-\omega_0|<\varepsilon_1$, $a\cdot\omega=0$. Fix (s, ω) so that $|\omega-\omega_0|<\varepsilon_1$. Given $a\in S^{n-1}$ orthogonal to ω, denote by $\Sigma_{\omega,a}=\{x\in\mathbb{R}^n; x\cdot\omega=x\cdot a=0\}$ the hyperplane in $\mathbb{R}^n$ orthogonal to ω and a. Setting $x=\alpha a+\beta\omega+y$, $y\in\Sigma_{\omega,a}$, we write (6.5) in the form

$$\int_{\Sigma_{\omega,a}}\int_{-\rho}^{\rho} q(\beta-s,\alpha a+\beta\omega+y)\,d\beta\,dS_y=0 \quad \text{for } \alpha a\perp\omega, \tag{6.6}$$

where dS_y is the surface measure on $\Sigma_{\omega,a}$. Consider the function

$$Q_{s,\omega}(z)=\int_{-\rho}^{\rho} q(\beta-s,z+\beta\omega)\,d\beta, \qquad z\in\Sigma_\omega=\{x;x\cdot\omega=0\}.$$

Letting a run over the set $\{a \in S^{n-1};\ a \cdot \omega = 0\}$, we deduce from (6.6) that the integrals of the function $Q_{s,\omega}(z)$ over each hyperplane in Σ_ω vanish. Hence $Q_{s,\omega} = 0$, i.e.

$$\int_{-\rho}^{\rho} q(\beta - s, \alpha a + \beta \omega + y)\, d\beta = 0 \tag{6.7}$$

for $|\omega - \omega_0| < \varepsilon_1$, all s, $\alpha a \perp \omega$, $y \in \Sigma_{\omega, a}$.

We wish to show that (6.7) implies $q = 0$. To do this we realize below the following idea (see also [9, Proposition 7.5]). By (ii), $q \in \mathscr{S}'(\mathbb{R}^{n+1})$, hence the Fourier transform $\hat{q}(\tau, \xi)$ is well-defined. Let us write formally

$$\hat{q}(\tau, \xi) = \iint e^{-it\tau - ix \cdot \xi} q(t, x)\, dt\, dx. \tag{6.8}$$

Introduce new variables β, s, α, y, such that

$$\begin{aligned} &t = \beta - s, \\ &x = \beta \omega + \alpha a + y, \qquad (\beta, s, \alpha) \in \mathbb{R}^3, \qquad y \in \Sigma_{\omega, a}. \end{aligned} \tag{6.9}$$

Clearly, $dt\, dx = d\beta\, ds\, d\alpha\, dS_y$ and (6.8) becomes (formally)

$$\hat{q}(\tau, \xi) = \int_{\Sigma_{\omega, a}} \iiint e^{-i\varphi} q(\beta - s, \beta\omega + \alpha a + y)\, d\beta\, ds\, d\alpha\, dS_y,$$

where the phase function $\varphi = \beta(\tau + \xi \cdot \omega) - s\tau + \alpha a \cdot \xi + y \cdot \xi$ does not depend on β if $\tau + \xi \cdot \omega = 0$. In this case $\hat{q} = 0$ in view of (6.7). Thus we obtain that $\hat{q}$ vanishes in the interior of the set $U = \bigcup_{|\omega - \omega_0| < \varepsilon_1} \{(\tau, \xi);\ \tau + \xi \cdot \omega = 0\}$. Then exploiting the fact that $\hat{q}$ is analytic with respect to ξ, we can show that $q = 0$. If $q(t, x)$ is compactly supported in t these arguments prove Theorem 1.1.

To justify these considerations in the general case, fix $a \in S^{n-1}$, so that $a \cdot \omega_0 = 0$ and for simplicity denote $\Sigma = \Sigma_{\omega_0, a}$. Put

$$\begin{aligned} \omega(p) &= \omega_0 \cos p + a \sin p, \\ a(p) &= -\omega_0 \sin p + a \cos p. \end{aligned}$$

Clearly $|\omega(p)| = |a(p)| = 1$, $\omega(p) \perp a(p)$ and $|p| < \varepsilon_1$ implies $|\omega(p) - \omega_0| < \varepsilon_1$. Relation (6.7) yields

$$\int_{-\rho}^{\rho} q(\beta - s, \alpha a(p) + \beta \omega(p) + y)\, d\beta = 0 \tag{6.10}$$

for all $(s, \alpha) \in \mathbb{R}^2$, $y \in \Sigma$, $|p| < \varepsilon_1$. Fix $\tau_0 \in \mathbb{R}$ and choose $\xi_0 \in \mathbb{R}^n$ such that $\tau_0 + \xi_0 \cdot \omega_0 = 0$, $\xi_0 \cdot a > 0$. Then

$$\xi_0 = \alpha_0 a - \tau_0 \omega_0 + y_0, \qquad y_0 \in \Sigma, \qquad \alpha_0 > 0.$$

By the implicit function theorem there exist functions $\alpha=\alpha(\tau, \xi)$, $y=y(\tau, \xi)$, $p=p(\tau, \xi)$ defined in some neighborhood $|\tau-\tau_0|<v$, $|\xi-\xi_0|<v$ of (τ_0, ξ_0) such that $\alpha=\alpha_0$, $y=y_0$, $p=0$ for $(\tau, \xi)=(\tau_0, \xi_0)$ and

$$\xi=\alpha a(p)-\tau\omega(p)+y, \qquad y\in\Sigma \tag{6.11}$$

(then $\tau+\xi\cdot\omega=0$). Moreover, the Jacobian $|\partial(\tau, \xi)/\partial(\tau, \alpha, y, p)|$ is equal to α. Let $\varphi(\tau, \xi)\in C_0^\infty(\mathbb{R}\times\mathbb{R}^n)$ be supported in the set $|\tau-\tau_0|<v$, $|\xi-\xi_0|<v$. Then

$$\begin{aligned}\langle\hat{q}(\tau, \xi), \varphi(\tau, \xi)\rangle=\iint q(t, x)\int\int_{\Sigma}\iint e^{-it\tau-ix\cdot\xi(\tau, \alpha, y, p)}\\ \cdot\ \varphi(\tau, \xi(\tau, \alpha, y, p))\,\alpha\, d\tau\, d\alpha\, dS_y\, dp\, dt\, dx,\end{aligned} \tag{6.12}$$

where $\xi(\tau, \alpha, y, p)$ is determined by (6.11). Here the integration in τ, α, p is taken over small neighborhoods of τ_0, α_0, 0 and $y\in\Sigma$ belongs to some neighborhood of y_0. Choosing a new constant $v>0$, we can assume that the support of φ is so small that $(\tau, \xi(\tau, \alpha, y, p))\in\operatorname{supp}\varphi$ implies $|p|<\varepsilon_1$.

We will change the order of integration $dp\ dt\ dx\to dt\ dx\ dp$ in (6.12). To justify this, it is sufficient to prove that the function

$$f(t, x, p)=q(t, x)\int_{\Sigma}\iint e^{-it\tau-ix\cdot\xi(\tau, \alpha, y, p)}\,\varphi(\tau, \xi(\tau, \alpha, y, p))\,\alpha\, d\tau\, d\alpha\, dS_y$$

satisfies the estimate

$$|f(t, x, p)|\leqq C(1+t^2)^{-1} \qquad \text{for all } t, x, p. \tag{6.13}$$

Indeed, according to (ii) we have

$$\begin{aligned}&|(1+t^2)f(t, x, p)|\\ &\leqq C(1+t^2)^{1+N}\sup_{|x|\leqq\rho}\left|\mathscr{F}_{\tau\to t}\int_{\Sigma}\int e^{-ix\cdot\xi(\tau, \alpha, y, p)}\,\varphi(\tau, \xi(\tau, \alpha, y, p))\,\alpha\, d\alpha\, dS_y\right|\\ &\leqq C_1\sup_{|x|\leqq\rho}\sup_{\tau}\int_{\Sigma}\int|(1-\partial_\tau^2)^{1+N}e^{-ix\cdot\xi(\tau, \alpha, y, p)}\,\varphi(\tau, \xi(\tau, \alpha, y, p))|\,\alpha\, d\alpha\, dS_y.\end{aligned}$$

Here $\mathscr{F}_{\tau\to t}$ denotes the Fourier transform with respect to τ. The inequalities above lead immediately to (6.13). Changing the order of integration we write (6.12) in the form

$$\begin{aligned}\int_{|p|<\varepsilon_1}\Bigg[\iint q(t, x)\Bigg(\int_{\Sigma}\iint e^{-it\tau-ix\cdot\xi(\tau, \alpha, y, p)}\\ \cdot\ \varphi(\tau, \xi(\tau, \alpha, y, p))\,\alpha\, d\tau\, d\alpha\, dS_y\Bigg)dt\, dx\Bigg]dp.\end{aligned} \tag{6.14}$$

A second change of variables (see (6.9))

$$\begin{aligned}&t=\beta-s,\\ &x=\beta\omega(p)+\alpha' a(p)+y', \qquad (\beta, s, \alpha')\in\mathbb{R}^3, \qquad y'\in\Sigma,\end{aligned}$$

yields

$$\langle \hat{q}(\tau,\xi), \varphi(\tau,\xi)\rangle = \int_{|p|<\varepsilon_1} \{\int \ldots \int [\int q(\beta - s, \beta\omega(p) + \alpha' a(p) + y')\, d\beta]$$
$$\cdot\, e^{-i(-s\tau+\alpha'\alpha+y'\cdot y)}\, \varphi(\tau, \xi(\tau,\alpha,y,p))\, \alpha\, d\tau\, d\alpha\, dS_y\, ds\, d\alpha'\, dS_{y'}\}\, dp.$$

Here we have used the fact that the phase function $-s\tau+\alpha'\alpha+y'\cdot y$ does not depend on β. By (6.10) the integral in the square brackets above vanishes. Thus we have proved that for each $\tau_0 \in \mathbb{R}$ there exists $\xi_0 \in \mathbb{R}^n$ and $v>0$ such that for each $\varphi(\tau,\xi) \in C_0^\infty(\mathbb{R}^{n+1})$ supported in $\{(\tau,\xi); |\tau-\tau_0|<v, |\xi-\xi_0|<v\}$ we have $\langle \hat{q}(\tau,\xi), \varphi(\tau,\xi)\rangle = 0$. Let $\varphi(\tau,\xi) = \psi(\tau)\,\eta(\xi)$, where $\psi \in C_0^\infty(\mathbb{R})$, $\psi(\tau)=0$ for $|\tau-\tau_0|>v$, $\eta \in C_0^\infty(\mathbb{R}^n)$, $\eta(\xi)=0$ for $|\xi-\xi_0|>v$. Then

$$\begin{aligned}\langle \hat{q}(\tau,\xi), \psi(\tau)\,\eta(\xi)\rangle &= \iint q(t,x) \iint e^{-it\tau-ix\cdot\xi}\, \psi(\tau)\,\eta(\xi)\, d\tau\, d\xi\, dt\, dx \\ &= \iint q(t,x) \int e^{-ix\cdot\xi}\, \hat{\psi}(t)\,\eta(\xi)\, d\xi\, dt\, dx \\ &= \int \mathscr{F}_{x\to\xi}(\int q(t,x)\, \hat{\psi}(t)\, dt)\, \eta(\xi)\, d\xi.\end{aligned}$$

Hence the Fourier transform

$$F(\xi) = \mathscr{F}_{x\to\xi}(\int q(t,x)\, \hat{\psi}(t)\, dt)$$

vanishes for $|\xi-\xi_0|<v$. Since the last integral is compactly supported function, it follows that $F(\xi)$ is real analytic. Hence $F(\xi)=0$ for all ξ. Consequently,

$$(\mathscr{F}_{t\to\tau}\, q)(\tau,x) = 0 \qquad \text{for } |\tau-\tau_0|<v, \quad \text{all } x.$$

Since τ_0 is arbitrary, we obtain $q=0$. This completes the proof of Theorem 1.1.

7. The Stationary Case

In this section we assume $q(x)$ stationary (time-independent). In this case we obtain a simple recovering procedure for $q(x)$. At the end of the section we make a connection with the proof of the uniqueness of the inverse scattering problem for stationary potentials based on high-frequency asymptotics of the scattering amplitude.

The introduction of $K^\#$ in this case is simplified as follows. Let $\tilde{\Gamma}(t,x,\omega)$ be the solution of the Cauchy problem

$$(\square + q(x))\,\tilde{\Gamma} = 0,$$
$$\tilde{\Gamma}|_{t<-\rho} = h_1(t - x\cdot\omega)$$

and set $\tilde{\Gamma}_{sc}(t,x,\omega) = \tilde{\Gamma}(t,x,\omega) - h_1(t-x\cdot\omega)$. The generalized scattering kernel we define by (see (3.1))

$$\tilde{K}^\#(s,\omega',\omega) = (2\pi)^{(1-n)/2}\, \partial_s^{(n-1/2}\, \tilde{\Gamma}_{sc}^\#(s,\omega',\omega),$$

where for each $\omega \in S^{n-1}$, $\tilde{\Gamma}_{sc}^\#(s,\omega',\omega)$ is the asymptotic wave profile of $\tilde{\Gamma}_{sc}$. It is not hard to see that the kernel $K^\#(s',\omega'; s,\omega)$ introduced in Sect. 3 depends merely on $s'-s$ in the stationary case and we have $\tilde{K}^{\#}(s'-s, \omega', \omega)=K^{\#}(s', \omega'; s, \omega)$. Theorem 1.1 then reduces to the following

Corollary 7.1. *Assume $q_i(x)\in C_0^\infty(\mathbb{R}^n)$, $n\geqq 3$ odd and let $\tilde{K}_i^{\#}(s, \omega', \omega)$ be the generalized scattering kernels related to q_i, $i=1, 2$. Suppose there exist $\varepsilon>0$ and $\omega_0\in S^{n-1}$ such that*

$$\tilde{K}_1^{\#}(s, \omega', \omega)=\tilde{K}_2^{\#}(s, \omega', \omega) \quad \text{for } |s|<\varepsilon,\ |\omega-\omega_0|<\varepsilon,\ |\omega'-\omega_0|<\varepsilon.$$

Then $q_1=q_2$.

Suppose further, we know the kernel $\tilde{K}^{\#}(s, \omega', \omega)$ for *all* incident directions ω, $|\omega'-\omega|<\varepsilon$, $|s|<\varepsilon$ with some $\varepsilon>0$. Then we shall obtain explicit formulae for recovering $q(x)$. By Theorem 5.2 we have

$$\tilde{K}^{\#}(s, \omega', \omega)=(-1)^{(n-1)/2}2^{-1}(2\pi)^{1-n}\,\partial_s^{n-2}\,\tilde{M}(s, \omega', \omega), \tag{7.1}$$

where

$$\tilde{M}(s, \omega', \omega)=\int q(x)\,u(x\cdot\omega'-s, x, \omega)\,dx. \tag{7.2}$$

$\tilde{M}$ and $\tilde{K}^{\#}$ depend smoothly on ω', ω with values in $\mathscr{D}'(\mathbb{R}_s)$ and for $\omega'\neq\omega$ or $s\neq 0$ they are smooth functions. For $\omega\in S^{n-1}$, $0<|\omega'-\omega|<\varepsilon$, $|s|<\varepsilon$ let us define

$$\tilde{M}_1(s, \omega', \omega)=2(-1)^{(n-1)/2}(2\pi)^{n-1}\int_s^\varepsilon ds_1\int_{s_1}^\varepsilon ds_2\ldots\int_{s_{n-3}}^\varepsilon ds_{n-2}\,\tilde{K}^{\#}(s_{n-2}, \omega', \omega). \tag{7.3}$$

By the arguments following (6.1) we see that for $|\omega'-\omega|<\nu=\min(\varepsilon, \varepsilon/4\rho)$ $\tilde{M}_1$ coincides with $\tilde{M}$. Put

$$\begin{aligned} \omega'(\mu)&=\omega\cos\mu+a\sin\mu,\\ s(\mu)&=\alpha\sin\mu,\qquad a\in S^{n-1},\ a\cdot\omega=0,\ (\alpha,\mu)\in\mathbb{R}^2. \end{aligned} \tag{7.4}$$

By (6.1) we can find

$$J(a, \alpha)=\lim_{\substack{\mu\to 0\\ \mu\neq 0}}|\omega'(\mu)\text{-}\omega|\,\tilde{M}_1(s(\mu), \omega'(\mu), \omega)=\int_{a\cdot x=\alpha} q(x)\,dS_x. \tag{7.5}$$

When ω runs over S^{n-1} we can determine $J(a, \alpha)$ for all $a\in S^{n-1}$, $\alpha\in\mathbb{R}$. Thus we obtain the Radon transform of $q(x)$. The potential q can then be recovered by [9]

$$q(x)=2^{-1}(2\pi)^{1-n}(-1)^{(n-1)/2}\,\triangle_x^{(n-1)/2}\int_{|a|=1} J(a, a\cdot x)\,da.$$

Here $J(a, \alpha)$ is related to $\tilde{K}^{\#}(s, \omega', \omega)$ via (7.3) and (7.5).

In the remainder of this section we shall discuss briefly the connection between our time dependent approach and the stationary scattering theory for time independent potentials. In particular, we shall show that Corollary 7.1 and the recovering procedure described above are closely related to the Born approximation of the scattering amplitude at high frequencies. Assume for simplicity $n=3$ and $q(x)\geqq 0$. Then the scattering operator associated with the perturbed wave equation

$$(\square + q(x))\, u = 0 \tag{7.6}$$

is known to exists [13], [14], [21] and by Proposition 3.2 the distribution $\tilde{K}^{\#}(s'-s, \omega', \omega)$ is the Schwartz kernel of $\mathcal{R}_n(S-I)\,\mathcal{R}_n^{-1}$. Thus $\tilde{K}^{\#}$ (and $\tilde{M}$) is tempered with respect to s and the *scattering matrix* related to (7.6) is given by

$$S(k, \omega', \omega) = \delta(\omega'-\omega) + \int e^{-iks}\, \tilde{K}^{\#}(s, \omega', \omega)\, ds.$$

We define the *scattering amplitude* by

$$A(k, \omega', \omega) = -\frac{1}{4\pi} \int e^{-iks}\, \tilde{M}(s, \omega', \omega)\, ds. \tag{7.7}$$

By (7.1) A and $\tilde{K}^{\#}$ are related by

$$-\frac{k}{2\pi i} A(k, \omega', \omega) = \int e^{-iks}\, \tilde{K}^{\#}(s, \omega', \omega)\, ds.$$

Substituting (7.2) into (7.7) we see that

$$A(k, \omega', \omega) = -\frac{1}{4\pi} \int e^{-ik\omega' \cdot x}\, q(x)\, v(k, x, \omega)\, dx, \tag{7.8}$$

where $v(k, x, \omega) = \int \exp(ikt)\, u(t, x, \omega)\, dt$. Duhamel's principle applied to $u(t, x, \omega)$ implies that

$$u(t, x, \omega) = \delta(t - x \cdot w) - \int E_+(t-\tau, x) * [q(x)\, u(\tau, x, \omega)]\, d\tau, \tag{7.9}$$

where $E_+(t, x) = (4\pi|x|)^{-1}\delta(t-|x|)$ and the star denotes convolution with respect to x. Taking the inverse Fourier transform with respect to t in (7.9) we obtain that $v(k, x, \omega)$ solves the Lippmann-Schwinger equation

$$v(k, x, \omega) = e^{ik\omega \cdot x} - \int G_+(k, |x-y|)\, q(y)\, v(k, y, \omega)\, dy,$$

where $G_+(k, r) = (4\pi r)^{-1} \exp(ikr)$ is the outgoing Green function. Thus relation (7.8) shows that $A(k, \omega', \omega)$ defined a priori by (7.7) is the classical scattering amplitude (see [2]).

According to (7.7), the behaviour of $\tilde{M}$ near the singularity $s=0$, $\omega'=\omega$ affects the high-frequency ($k \to \infty$) asymptotic of A. To understand this relation, recall the Born approximation of the scattering amplitude [22]

$$A(k, \omega', \omega) = -\frac{1}{4\pi} \int e^{-ik(\omega'-\omega)\cdot x} q(x)\, dx + o(1), \qquad k \to \infty. \tag{7.10}$$

Asymptotic (7.10) is used to reconstruct the Fourier transform of q. To do this, given $\xi \in \mathbb{R}^3 \setminus 0$ *choose* $\omega \in S^2$ so that $\xi \cdot \omega = 0$. Set

$$\omega'(\mu) = \omega \cos \mu + (\xi/|\xi|) \sin \mu,$$
$$k(\mu) = |\xi|/\sin \mu$$

(compare with (7.4)). From (7.10) we find (see also [22])

$$\hat{q}(\xi) = -4\pi \lim_{\mu \to 0} A(k(\mu), \omega'(\mu), \omega). \tag{7.11}$$

Let us take the inverse Fourier transform of (7.11) with respect to $|\xi|$. Then the left hand side of (7.11) becomes the Radon transform of q, while the right hand side can be expressed in terms of $\tilde{M}$ in view of (7.7) and the result is (7.5). Hence, in the stationary case our approach is closely related to the asymptotic (7.10).

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