INTEGRAL GEOMETRY OF TENSOR FIELDS ON A CLASS OF NON-SIMPLE RIEMANNIAN MANIFOLDS

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ABSTRACT. We study the geodesic X-ray transform I_Γ of tensor fields on a compact Riemannian manifold M with non-necessarily convex boundary and with possible conjugate points. We assume that I_Γ is known for geodesics belonging to an open set Γ with endpoints on the boundary. We prove generic s-injectivity and a stability estimate under some topological assumptions and under the condition that for any $(x,\xi)\in T^*M$, there is a geodesic in Γ through x normal to ξ without conjugate points.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let $(M,\partial M)$ be a smooth compact manifold with boundary, and let $g\in C^k(M)$ be a Riemannian metric on it. We can always assume that $(M,\partial M)$ is equipped with a real analytic atlas, while ∂M and g may or may not be analytic. We define the geodesic X-ray transform I of symmetric 2-tensor fields by

(1)
$$If(\gamma)=\int_0^{l_\gamma}\langle f(\gamma(t)),\dot\gamma^2(t)\rangle\,\mathrm{d}t,$$

where $[0,l_\gamma]\ni t\mapsto\gamma$ is any geodesic with endpoints on ∂M parameterized by its arc-length. Above, $\langle f,\theta^2\rangle$ is the action of f on the vector θ, that in local coordinates is given by $f_{ij}\theta^i\theta^j$. The purpose of this work is to study the injectivity, up to potential fields, and stability estimates for I restricted to certain subsets Γ (that we call I_Γ), and for manifolds with possible conjugate points. We require however that the geodesics in Γ do not have conjugate points. We also require that Γ is an open sets of geodesics such that the collection of their conormal bundles covers T^*M. This guarantees that I_Γ resolves the singularities. The main results are injectivity up to a potential field and stability for generic metrics, and in particular for real analytic ones.

We are motivated here by the boundary rigidity problem: to recover g, up to an isometry leaving ∂M fixed, from knowledge of the boundary distance function $\rho(x,y)$ for a subset of pairs $(x,y)\in\partial M\times\partial M$, see e.g., [Mi, Sh1, CDS, SU4, PU]. In presence of conjugate points, one should study instead the lens rigidity problem: a recovery of g from its scattering relation restricted to a subset. Then I_Γ is the linearization of those problems for an appropriate Γ. Since we want to trace the dependence of I_Γ on perturbations of the metric, it is more convenient to work with open Γ's that have dimension larger than n, if $n\geq 3$, making the linear inverse problem formally overdetermined. One can use the same method to study restrictions of I on n dimensional subvarieties but this is behind the scope of this work.

Any symmetric 2-tensor field f can be written as an orthogonal sum of a *solenoidal* part f^s and a *potential* one dv, where $v=0$ on ∂M, and d stands for the symmetric differential of the 1-form v, see Section 2. Then $I(dv)(\gamma)=0$ for any geodesic γ with endpoints on ∂M. We say that I_Γ is *s-injective*, if $I_\Gamma f=0$ implies $f=dv$ with $v=0$ on ∂M, or, equivalently, $f=f^s$. This problem has been studied before for *simple* manifolds with boundary, i.e., under the assumption that ∂M is strictly convex, and there are no conjugate points in M (then M is diffeomorphic to a ball). The book [Sh1] contains the

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main results up to 1994 on the integral geometry problem considered in this paper. Some recent results include [Sh2], [Ch], [SU3], [D], [Pe], [SSU], [ShU]. For simple 2D manifolds, following the method used in [PU] to solve the boundary rigidity problem, s-injectivity was proven in [Sh3]. In [SU4], we considered I on all geodesics and proved that the set of simple metrics on a fixed manifold for which I is s-injective is generic in $C^k(M)$, $k \gg 2$. Previous results include s-injectivity for simple manifolds with curvature satisfying some explicit upper bounds [Sh1, Sh2, Pe]. A recent result by Dairbekov [D] proves s-injectivity for non-trapping manifolds (not-necessarily convex) satisfying similar bounds, that in particular prevent the existence of conjugate points.

Fix another compact manifold M_1 with boundary such that $M_1^{\text{int}} \supset M$, where M_1^{int} stands for the interior of M_1. Such a manifold is easy to construct in local charts, then glued together.

Definition 1. *We say that the $C^k(M)$ (or analytic) metric g on M is* ***regular****, if g has a C^k (or analytic, respectively) extension on M_1, such that for any $(x,\xi) \in T^*M$ there exists $\theta \in T_xM \setminus 0$ with $\langle \xi, \theta \rangle = 0$ such that there is a geodesic segment $\gamma_{x,\theta}$ through (x,θ) such that*
(a) the endpoints of $\gamma_{x,\theta}$ are in $M_1^{\text{int}} \setminus M$.
(b) there are no conjugate points on $\gamma_{x,\theta}$.
Any geodesic satisfying (a), (b) is called a ***simple*** *geodesic.*

Note that we allow the geodesics in Γ to self-intersect.

Since we do not assume that M is convex, given (x,θ) there might be two or more geodesic segments γ_j issued from (x,θ) such that $\gamma_j \cap M$ have different numbers of connected components. Some of them might be simple, others might be not. For example for a kidney-shaped domain and a fixed (x,θ) we may have such segments so that the intersection with M has only one, or two connected components. Depending on which point in T^*M we target to recover the singularities, we may need the first, or the second extension. So simple geodesic segments through some x (that we call simple geodesics through x) are uniquely determined by an initial point x and a direction θ and its endpoints. In case of simple manifolds, the endpoints (of the only connected component in M, unless the geodesics does not intersect M) are not needed, they are a function of (x,θ). Another way to determine a simple geodesic is by parametrizing it with $(x,\eta) \in T(M_1^{\text{int}} \setminus M)$, such that $\exp_x \eta \in M_1^{\text{int}} \setminus M$ then

$$\gamma_{x,\eta} = \{\exp_x(t\eta), 0 \le t \le 1\}. \tag{2}$$

This parametrization induces a topology on the set Γ of simple geodesics.

Definition 2. *The set Γ of geodesics is called* ***complete****, if*
(a) $\forall (x,\xi) \in T^*M$ there exists a simple geodesic $\gamma \in \Gamma$ through x such that $\dot\gamma$ is normal to ξ at x.
(b) Γ is open.

In other words, a regular metric g is a metric for which a complete set of geodesics exists. Another way to express (a) is to say that

$$N^*\Gamma := \left\{N^*\gamma;\ \gamma \in \Gamma\right\} \supset T^*M, \tag{3}$$

where $N^*\gamma$ stands for the conormal bundle of γ.

We always assume that all tensor fields defined in M are extended as 0 to $M_1 \setminus M$. Notice that If does not change if we replace M by another manifold $M_{1/2}$ close enough to M such that $M \subset M_{1/2} \subset M_1$ but keep f supported in M. Therefore, assuming that M has an analytic structure as before, we can always extend M a bit to make the boundary analytic and this would keep $(M, \partial M, g)$ regular. Then s-injectivity in the extended M would imply the same in the original M, see [SU4, Prop. 4.3]. So from now on, we will assume that $(M, \partial M)$ is analytic but g does not need to be analytic. To define correctly a norm in $C^K(M)$, respectively $C^k(M_1)$, we fix a finite analytic atlas.

The motivation behind Definitions 1, 2 is the following: if g is regular, and Γ is any complete set of geodesics, we will show that $I_\Gamma f = 0$ implies that $f^s \in C^l(M)$, where $l = l(k) \to \infty$, as $k \to \infty$, in other words, the so restricted X-ray transform resolves the singularities.

The condition of g being regular is an open one for $g \in C^k(M)$, $k \geq 2$, i.e., it defines an open set. Any simple metric on M is regular but the class of regular metrics is substantially larger if dim $M \geq 3$ and allows manifolds not necessarily diffeomorphic to a ball. For regular metrics on M, we do not impose convexity assumptions on the boundary; conjugate points are allowed as far as the metric is regular; M does not need to be non-trapping. In two dimensions, a regular metric can not have conjugate points in M but the class is still larger than that of simple metrics because we do not require strong convexity of ∂M.

Example 1. To construct a manifold with a regular metric g that has conjugate points, let us start with a manifold of dimension at least three with at least one pair of conjugate points u and v on a geodesic $[a, b] \ni t \mapsto \gamma(t)$. We assume that γ is non-selfintersecting. Then we will construct M as a tubular neighborhood of γ. For any $x_0 \in \gamma$, define $S_{x_0} = \exp_{x_0}\{v;\ \langle v, \dot\gamma(x_0)\rangle = 0,\ |v| \leq \varepsilon\}$, and $M := \cup_{x_0 \in \gamma} S_{x_0}$ with $\varepsilon \ll 1$. Then there are no conjugate points along the geodesics that can be loosely described as those "almost perpendicular" to γ but not necessarily intersecting γ; and the union of their conormal bundles covers T^*M. More precisely, fix $x \in M$, then $x \in S_{x_0}$ for some $x_0 \in \gamma$. Let $0 \neq \xi \in T_x^*M$. Then there exists $0 \neq v \in T_xM$ that is both tangent to S_{x_0} and normal to ξ. The geodesic through (x, v) is then a simple one for $\varepsilon \ll 1$, and the latter can be chosen in a uniform way independent of x. To obtain a smooth boundary, one can perturb M so that the new manifold is still regular.

Example 2. This is similar to the example above but we consider a neighborhood of a periodic trajectory. Let $M = \{(x^1)^2 + (x^2)^2 \leq 1\} \times S^1$ be the interior of the torus in $\mathbf{R}^3$, with the flat metric $(dx^1)^2 + (dx^2)^2 + d\theta^2$, where θ is the natural coordinate on S^1 with period 2π. All geodesics perpendicular to $\theta =$ const. are periodic. All geodesics perpendicular to them have lengths not exceeding 2 and their conormal bundles cover the entire T^*M (to cover the boundary points, we do need to extend the geodesics in a neighborhood of M). Then M is a regular manifold that is trapping, and one can easily show that a small enough perturbation of M is also regular, and may still be trapping.

The examples above are partial cases of a more general one. Let $(M', \partial M')$ be a simple compact Riemannian manifold with boundary with dim $M' \geq 2$, and let M'' be a compact Riemannian manifold with or without boundary. Let M be a small enough perturbation of $M' \times M''$. Then M is regular.

We assume throughout this paper that M satisfies the following.

Topological Condition: Any path in M connecting two boundary points is homotopic to a polygon $c_1 \cup \gamma_1 \cup c_2 \cup \gamma_2 \cup \cdots \cup \gamma_k \cup c_{k+1}$ with the properties:

(i) c_j are paths on ∂M;

(ii) For any j, $\gamma_j = \tilde\gamma_j|_M$ for some $\tilde\gamma_j \in \Gamma$; γ_j lie in M^{int} with the exception of its endpoints and is transversal to ∂M at both ends.

Theorem 1. *Let g be an analytic, regular metric on M. Let Γ be a complete complex of geodesics. Then I_Γ is s-injective.*

The proof is based on using analytic pseudo-differential calculus, see [Sj, Tre]. This has been used before in integral geometry, see e.g., [BQ, Q], see also [SU4].

The property of γ being simple is stable under small perturbations. The parametrization by (x, η) as in (2) clearly has two more dimensions that what is needed to determine uniquely $\gamma|_M$. Indeed, a parallel transport of (x, η) along $\gamma_{x,\eta}$, close enough to x, will not change $\gamma|_M$, similarly, we can replace η by $(1 + \varepsilon)\eta$, $|\varepsilon| \ll 1$.

To formulate a stability estimate, we will parametrize the simple geodesics in a way that will remove the extra two dimensions. Let H_m be a finite collection of smooth hypersurfaces in M_1^{int}. Let $\mathcal{H}_m$ be an open subset of $\{(z,\theta)\in SM_1;\ z\in H_m, \theta\not\in T_zH_m\}$, and let $\pm l_m^{\pm}(z,\theta)\geq 0$ be two continuous functions. Let $\Gamma(\mathcal{H}_m)$ be the set of geodesics

$$\Gamma(\mathcal{H}_m)=\left\{\gamma_{z,\theta}(t);\ l_m^-(z,\theta)\leq t\leq l_m^+(z,\theta),\ (z,\theta)\in\mathcal{H}_m\right\}, \tag{4}$$

that, depending on the context, is considered either as a family of curves, or as a point set. We also assume that each $\gamma\in\Gamma(\mathcal{H}_m)$ is a simple geodesic.

If g is simple, then one can take a single $H=\partial M_1$ with $l^-=0$ and an appropriate $l^+(z,\theta)$. If g is regular only, and Γ is any complete set of geodesics, then any small enough neighborhood of a simple geodesic in Γ has the properties listed above and by a compactness argument on can choose a finite complete set of such $\Gamma(\mathcal{H}_m)$'s, that is included in the original Γ, see Lemma 1.

Given $\mathcal{H}=\{\mathcal{H}_m\}$ as above, we consider an open set $\mathcal{H}'=\{\mathcal{H}'_m\}$, such that $\mathcal{H}'_m\Subset\mathcal{H}_m$, and let $\Gamma(\mathcal{H}'_m)$ be the associated set of geodesics defined as in (4), with the same $l_m^{\pm}$. Set $\Gamma(\mathcal{H})=\cup\Gamma(\mathcal{H}_m)$, $\Gamma(\mathcal{H}')=\cup\Gamma(\mathcal{H}'_m)$.

The restriction $\gamma\in\Gamma(\mathcal{H}'_m)\subset\Gamma(\mathcal{H}_m)$ can be modeled by introducing a weight function α_m in $\mathcal{H}_m$, such that $\alpha_m=1$ on $\mathcal{H}'_m$, and $\alpha_m=0$ otherwise. More generally, we allow α_m to be smooth but still supported in $\mathcal{H}_m$. We then write $\alpha=\{\alpha_m\}$, and we say that $\alpha\in C^k(\mathcal{H})$, if $\alpha_m\in C^k(\mathcal{H}_m)$, $\forall m$.

We consider $I_{\alpha_m}=\alpha_m I$, or more precisely, in the coordinates $(z,\theta)\in\mathcal{H}_m$,

$$I_{\alpha_m}f=\alpha_m(z,\theta)\int_0^{l_m(z,\theta)}\left\langle f(\gamma_{z,\theta}),\dot\gamma_{z,\theta}^2\right\rangle \mathrm{d}t,\quad (z,\theta)\in\mathcal{H}_m. \tag{5}$$

Next, we set

$$I_\alpha=\{I_{\alpha_m}\},\quad N_{\alpha_m}=I_{\alpha_m}^*I_{\alpha_m}=I^*|\alpha_m|^2I,\quad N_\alpha=\sum N_{\alpha_m}, \tag{6}$$

where the adjoint is taken w.r.t. the measure $\mathrm{d}\mu:=|\langle\nu(z),\theta\rangle|\,\mathrm{d}S_z\,\mathrm{d}\theta$ on $\mathcal{H}_m$, $\mathrm{d}S_z\,\mathrm{d}\theta$ being the induced measure on $\mathcal{H}_m$, and $\nu(z)$ being a unit normal to H_m.

S-injectivity of N_α is equivalent to s-injectivity for I_α, which in turn is equivalent to s-injectivity of I restricted to $\operatorname{supp}\alpha$, see Lemma 2. The space $\tilde H^2$ is defined in Section 2, see (8).

Theorem 2.
(a) Let $g=g_0\in C^k$, $k\gg 1$ be regular, and let $\mathcal{H}'\Subset\mathcal{H}$ be as above with $\Gamma(\mathcal{H}')$ complete. Fix $\alpha=\{\alpha_m\}\in C^\infty$ with $\mathcal{H}'_m\subset\operatorname{supp}\alpha_m\subset\mathcal{H}_m$. Then if I_α is s-injective, we have

$$\|f^s\|_{L^2(M)}\leq C\|N_\alpha f\|_{\tilde H^2(M_1)}. \tag{7}$$

(b) Assume that $\alpha=\alpha_g$ in (a) depends on $g\in C^k$, so that $C^k(M_1)\ni g\to C^l(\mathcal{H})\ni\alpha_g$ is continuous with $l\gg 1$, $k\gg 1$. Assume that $I_{g_0,\alpha_{g_0}}$ is s-injective. Then estimate (7) remains true for g in a small enough neighborhood of g_0 in $C^k(M_1)$ with a uniform constant $C>0$.

In particular, Theorem 2 proves a locally uniform stability estimate for the class of non-trapping manifolds considered in [D].

Theorems 1, 2 allow us to formulate generic uniqueness results. One of them is formulated below. Given a family of metrics $\mathcal{G}\subset C^k(M_1)$, and $U_g\subset T(M_1^{\text{int}}\setminus M)$, depending on the metric $g\in\mathcal{G}$, we say that U_g depends continuously on g, if for any $g_0\in\mathcal{G}$, and any compact $K\subset U_{g_0}^{\text{int}}$, we have $K\subset U_g^{\text{int}}$ for g in a small enough neighborhood of g_0 in C^k. In the next theorem, we take $U_g=\Gamma_g$, that is identified with the corresponding set of (x,η) as in (2).

Theorem 3. *Let $\mathcal{G} \subset C^k(M_1)$ be an open set of regular metrics on M, and let for each $g \in \mathcal{G}$, Γ_g be a complete set of geodesics related to g and continuously depending on g. Then for $k \gg 0$, there is an open and dense subset $\mathcal{G}_s$ of $\mathcal{G}$, such that the corresponding X-ray transform I_{Γ_g} is s-injective.*

Of course, the set $\mathcal{G}_s$ includes all real analytic metrics in $\mathcal{G}$.

Corollary 1. *Let $\mathcal{R}(M)$ be the set of all regular C^k metrics on M equipped with the $C^k(M_1)$ topology. Then for $k \gg 1$, the subset of metrics for which the X-ray transform I over all simple geodesics is s-injective, is open and dense in $\mathcal{R}(M)$.*

The results above extend the generic results in [SU4], see also [SU3], in several directions: the topology of M may not be trivial, we allow conjugate points but we use only geodesics without conjugate points; the boundary does not need to be convex; and we use incomplete data, i.e., we use integrals over subsets of geodesics only.

In Section 6, we discuss versions of those results for the X-ray transform of vector fields and functions, where the proofs can be simplified. Our results remain true for tensors of any order m, the necessary modifications are addressed in the key points of our exposition. To keep the paper readable, we restrict ourselves to orders $m = 2, 1, 0$.

2. Preliminaries

We say that f is analytic in some subset U of a real analytic manifold, not necessarily open, if f can be extended analytically to some open set containing U. We will use often the word analytic instead of real analytic. Then we write $f \in \mathcal{A}(U)$. Let $g \in \mathrm{C}^k(M)$, $k \gg 2$ or $g \in \mathcal{A}(M)$ be a Riemannian metric in M. We work with symmetric 2-tensors $f = \{f_{ij}\}$ and with 1-tensors/differential forms v_j (the notation here and below is in any local coordinates). We use freely the Einstein summation convention and the convention for raising and lowering indices. We think of f_{ij} and $f^{ij} = f_{kl}g^{ki}g^{lj}$ as different representations of the same tensor. If ξ is a covector at x, then its components are denoted by ξ_j, while ξ^j is defined as $\xi^i = g^{ij}\xi_j$. Next, we denote $|\xi|^2 = \xi_i\xi^i$, similarly for vectors that we usually denote by θ. If θ_1, θ_2 are two vectors, then $\langle\theta_1, \theta_2\rangle$ is their inner product. If ξ is a covector, and θ is a vector, then $\langle\xi, \theta\rangle$ stands for $\xi(\theta)$. This notation choice is partly justified by identifying ξ with a vector, as above.

The geodesics of g can be also viewed as the x-projections of the bicharacteristics of the Hamiltonian $E_g(x, \xi) = \frac{1}{2}g^{ij}(x)\xi_i\xi_j$. The energy level $E_g = 1/2$ corresponds to parametrization with the arc-length parameter. For any geodesic γ, we have $f^{ij}(x)\xi_i\xi_j = f_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)$, where $(x, \xi) = (x(t), \xi(t))$ is the bicharacteristic with x-projection equal to γ.

2.1. Semigeodesic coordinates near a simple geodesic and boundary normal coordinates.

Let $[l^-, l^+] \ni t \mapsto \gamma_{x_0,\theta_0}(t)$ be a simple geodesic through $x_0 = \gamma_{x_0,\theta_0}(0) \in M_1$ with $\theta_0 \in S_{x_0}M_1$. The map $t\theta \mapsto \exp_{x_0}(t\theta)$ is a local diffeomorphism for θ close enough to θ_0 and $t \in [l^-, l^+]$ by our simplicity assumption but may not be a global one, since γ_{x_0,θ_0} may self-intersect. On the other hand, there can be finitely many intersections only and we can assume that each subsequent intersection happens on a different copy of M. In other words, we think of γ_0 as belonging to a new manifold that is a small enough neighborhood of γ_0, and there are no self-intersections there. The local charts of that manifold are defined through the exponential map above. Therefore, when working near γ_{x_0,θ_0} we can assume that γ_{x_0,θ_0} does not intersect itself. We will use this in the proof of Proposition 2. Then one can choose a neighborhood U of γ_0 and normal coordinates centered at x_0 there, denoted by x again, such that the radial lines $t \mapsto t\theta$, $\theta =$ const., are geodesics. If $g \in C^k$, then we lose two derivatives and the new metric is in C^{k-2}; if g is analytic near γ_0, then the coordinate change can be chosen to be analytic, as well.

If in the situation above, let $x_0 \notin M$, and moreover, assume that the part of γ_{x_0,θ_0} corresponding to $t < 0$ is still outside M. Then, one can consider (θ, t) as polar coordinates on $T_{x_0}M$. Considering them as Cartesian coordinates there, see also [SU3, sec. 9], one gets coordinates (x', x^n) near γ_{x_0,θ_0} so that the latter is given by $\{(0,\dots,0,t),\ 0\le t\le l^+\}$, $g_{in}=\delta_{in}$, and $\Gamma_{nn}^i=\Gamma_{in}^n=0$, $\forall i$. Given $x\in\mathbf{R}^n$, we write $x'=(x^1,\dots,x^{n-1})$. Moreover, the lines $x'=\text{const.}$, $|x'|\ll 1$, $x^n=t\in[0,l^+]$ are geodesics in Γ, as well. We will call those coordinates semigeodesic coordinates near γ_{x_0,θ_0}.

We will often use boundary normal (semi-geodesic) coordinates (x', x^n) near a boundary point. If $x'\in\mathbf{R}^{n-1}$ are local coordinates on ∂M, and $\nu(x')$ is the interior unit normal, for $p\in M$ close enough to ∂M, they are defined by $\exp_{(x',0)}x^n\nu=p$. Then $x^n=0$ defines ∂M, $x^n>0$ in M, $x^n=\operatorname{dist}(x,\partial M)$. The metric g in those coordinates again satisfies $g_{in}=\delta_{in}$, and $\Gamma_{nn}^i=\Gamma_{in}^n=0$, $\forall i$. We also use the convention that all Greek indices take values from 1 to $n-1$. In fact, the semigeodesic coordinates in the previous paragraph are boundary normal coordinates to a small part of the geodesic ball centered at $x_0=\gamma_{x_0,\theta_0}(0)$ with radius ε, $0<\varepsilon\ll 1$.

2.2. **Integral representation of the normal operator.** We define the L^2 space of symmetric tensors f with inner product

$$(f,h)=\int_M\langle f,\bar h\rangle(\det g)^{1/2}\,\mathrm{d}x,$$

where, in local coordinates, $\langle f,\bar h\rangle=f_{ij}\bar h^{ij}$. Similarly, we define the L^2 space of 1-tensors (vector fields, that we identify with 1-forms) and the L^2 space of functions in M. Also, we will work in Sobolev H^s spaces of 2-tensors, 1-forms and functions. In order to keep the notation simple, we will use the same notation L^2 (or H^s) for all those spaces and it will be clear from the context which one we mean.

In the fixed finite atlas on M, extended to M_1, the norms $\|f\|_{C^k}$ and the H^s norms below are correctly defined. In the proof, we will work in finitely many coordinate charts because of the compactness of M, and this justifies the equivalence of the correspondent C^k, respectively H^s norms.

We define the Hilbert space $\tilde H^2(M_1)$ used in Theorem 2 as in [SU3, SU4]. Let $x=(x',x^n)$ be local coordinates in a neighborhood U of a point on ∂M such that $x^n=0$ defines ∂M. Then we set

$$\|f\|^2_{\tilde H^1(U)}=\int_U\Big(\sum_{j=1}^{n-1}|\partial_{x^j}f|^2+|x^n\partial_{x^n}f|^2+|f|^2\Big)\,\mathrm{d}x.$$

This can be extended to a small enough neighborhood V of ∂M contained in M_1. Then we set

$$\|f\|_{\tilde H^2(M_1)}=\sum_{j=1}^{n}\|\partial_{x^j}f\|_{\tilde H^1(V)}+\|f\|_{\tilde H^1(M_1)}. \tag{8}$$

The space $\tilde H^2(M_1)$ has the property that for each $f\in H^1(M)$ (extended as zero outside M), we have $Nf\in\tilde H^2(M_1)$. This is not true if we replace $\tilde H^2(M_1)$ by $H^2(M_1)$.

Lemma 1. *Let Γ_g and $\mathcal{G}$ be as in Theorem 3. Then for $k\gg 1$, for any $g_0\in\mathcal{G}$, there exist $\mathcal{H}'=\{\mathcal{H}'_m\}\Subset\mathcal{H}=\{\mathcal{H}_m\}$ such that $\Gamma(\mathcal{H})\Subset\Gamma_{g_0}$, and $\mathcal{H}'$, $\mathcal{H}$ satisfy the assumptions of Theorem 2. Moreover, $\mathcal{H}'$ and $\mathcal{H}$ satisfy the assumptions of Theorem 2 for g in a small enough neighborhood of g_0 in C^k.*

Proof. Fix $g_0\in\mathcal{G}$ first. Given $(x_0,\xi_0)\in T^*M$, there is a simple geodesic $\gamma:[l^-,l^+]\to M_1$ in Γ_{g_0} through x_0 normal to ξ_0 at x_0. Choose a small enough hypersurface H through x_0 transversal to $\gamma\in\Gamma_{g_0}$, and local coordinates near x_0 as in Section 2.1 above, so that $x_0=0$, H is given by $x^n=0$, $\dot\gamma(0)=(0,\dots,0,1)$. Then one can set $\mathcal{H}_0=\{x;\ x^n=0;\ |x'|<\varepsilon\}\times\{\theta;\ |\theta'|<\varepsilon\}$, and $\mathcal{H}'_0$ is defined in the same way by replacing ε by $\varepsilon/2$. We define $\Gamma(\mathcal{H}_0)$ as in (4) with $l^\pm(z,\theta)=l^\pm$. Then the properties required for $\mathcal{H}_0$, including the simplicity assumption are satisfied when $0<\varepsilon\ll 1$. Choose such an ε, and replace it with a smaller one so that those properties are preserved under a small perturbation of g. Any point in SM close enough to (x_0, ξ_0) still has a geodesic in $\Gamma(\mathcal{H}'_0)$ normal to it. By a compactness argument, one can find a finite number of $\mathcal{H}'_m$ so that the corresponding $\Gamma(\mathcal{H}') = \cup\Gamma(\mathcal{H}'_m)$ is complete.

The continuity property of Γ_g w.r.t. g guarantees that the construction above is stable under a small perturbation of g. □

Similarly to [SU3], one can see that the map $I_{\alpha_m} : L^2(M) \to L^2(\mathcal{H}_m, \mathrm{d}\mu)$ defined by (5) is bounded, and therefore the *normal* operator N_{α_m} defined in (6) is a well defined bounded operator on $L^2(M)$. Applying the same argument to M_1, we see that $N_{\alpha_m} : L^2(M) \to L^2(M_1)$ is also bounded. By [SU3], at least when f is supported in the local chart near $x_0 = 0$ above, and x is close enough to x_0,

$$[N_{\alpha_m} f]^{i'j'}(x) = \int_0^\infty \int_{S_x M} |\alpha_m^\sharp(x,\theta)|^2 \theta^{i'}\theta^{j'} f_{ij}(\gamma_{x,\theta}(t))\dot\gamma^i_{x,\theta}(t)\dot\gamma^j_{x,\theta}(t)\,\mathrm{d}\theta\,\mathrm{d}t, \tag{9}$$

where $|\alpha_m^\sharp(x,\theta)|^2 = |\tilde\alpha_m(x,\theta)|^2 + |\tilde\alpha_m(x,-\theta)|^2$, and $\tilde\alpha_m$ is the extension of α_m as a constant along the geodesic through $(x,\theta) \in \mathcal{H}_m$; and equal to 0 for all other points not covered by such geodesics. Formula (9) has an invariant meaning and holds without the restriction on supp f. On the other hand, if supp f is small enough (but not necessarily near x_0), $y = \exp_x(t\theta)$ defines a local diffeomorphism $t\theta \mapsto y \in \operatorname{supp} f$. Therefore, after making the change of variables $y = \exp_x(t\theta)$, see [SU3], this becomes

$$N_{\alpha_m} f(x) = \frac{1}{\sqrt{\det g}} \int A_m(x,y) \frac{f^{ij}(y)}{\rho(x,y)^{n-1}} \frac{\partial\rho}{\partial y^i}\frac{\partial\rho}{\partial y^j}\frac{\partial\rho}{\partial x^k}\frac{\partial\rho}{\partial x^l} \det \frac{\partial^2(\rho^2/2)}{\partial x \partial y}\,\mathrm{d}y, \tag{10}$$

where

$$A_m(x,y) = \left|\alpha_m^\sharp(x, \operatorname{grad}_x \rho(x,y))\right|^2, \tag{11}$$

y are any local coordinates near supp f, and $\rho(x,y) = |\exp_x^{-1} y|$. Formula (10) can be also understood invariantly by considering $\mathrm{d}_x\rho$ and $\mathrm{d}_y\rho$ as tensors. For arbitrary $f \in L^2(M)$ we use a partition of unity in TM_1^{int} to express $N_{\alpha_m} f(x)$ as a finite sum of integrals as above, for x near any fixed x_0.

We get in particular that N_{α_m} has the pseudolocal property, i.e., its Schwartz kernel is smooth outside the diagonal. As we will show below, similarly to the analysis in [SU3, SU4], N_{α_m} is a ΨDO of order -1.

We always extend functions or tensors defined in M as 0 outside M. Then $N_\alpha f$ is well defined near M as well and remains unchanged if M is extended such that it is still in M_1, and f is kept fixed.

2.3. **Decomposition of symmetric tensors.** For more details about the decomposition below, we refer to [Sh1]. Given a symmetric 2-tensor $f = f_{ij}$, we define the 1-tensor δf called *divergence* of f by

$$[\delta f]_i = g^{jk}\nabla_k f_{ij},$$

in any local coordinates, where ∇ denotes covariant differentiation. Given an 1-tensor (a vector field or an 1-form) v, we denote by dv the 2-tensor called symmetric differential of v:

$$[dv]_{ij} = \frac{1}{2}\left(\nabla_i v_j + \nabla_j v_i\right).$$

Operators d and $-\delta$ are formally adjoint to each other in $L^2(M)$. It is easy to see that for each smooth v with $v = 0$ on ∂M, we have $I(dv)(\gamma) = 0$ for any geodesic γ with endpoints on ∂M. This follows from the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle v(\gamma(t)), \dot\gamma(t)\rangle = \langle dv(\gamma(t)), \dot\gamma^2(t)\rangle. \tag{12}$$

If $\alpha = \{\alpha_m\}$ is as in the Introduction, we get

$$I_\alpha(dv) = 0, \quad \forall v \in C_0^1(M), \tag{13}$$

and this can be extended to $v \in H_0^1(M)$ by continuity.

It is known (see [Sh1] and (15) below) that for g smooth enough, each symmetric tensor $f \in L^2(M)$ admits unique orthogonal decomposition $f = f^s + dv$ into a *solenoidal* tensor $\mathcal{S}f := f^s$ and a *potential* tensor $\mathcal{P}f := dv$, such that both terms are in $L^2(M)$, f^s is solenoidal, i.e., $\delta f^s = 0$ in M, and $v \in H_0^1(M)$ (i.e., $v = 0$ on ∂M). In order to construct this decomposition, introduce the operator $\Delta^s = \delta d$ acting on vector fields. This operator is elliptic in M, the Dirichlet problem satisfies the Lopatinskii condition, and has a trivial kernel and cokernel. Denote by Δ^s_D the Dirichlet realization of Δ^s in M. Then

$$v = \left(\Delta^s_D\right)^{-1}\delta f, \quad f^s = f - d\left(\Delta^s_D\right)^{-1}\delta f. \tag{14}$$

Therefore, we have

$$\mathcal{P} = d\left(\Delta^s_D\right)^{-1}\delta, \quad \mathcal{S} = \mathrm{Id} - \mathcal{P},$$

and for any $g \in C^1(M)$, the maps

$$(\Delta^s_D)^{-1} : H^{-1}(M) \to H_0^1(M), \quad \mathcal{P}, \mathcal{S} : L^2(M) \longrightarrow L^2(M) \tag{15}$$

are bounded and depend continuously on g, see [SU4, Lemma 1] that easily generalizes for manifolds. This admits the following easy generalization: for $s = 0, 1, \dots,$ the resolvent above also continuously maps H^{s-1} into $H^{s+1} \cap H_0^1$, similarly, $\mathcal{P}$ and $\mathcal{S}$ are bounded in H^s, if $g \in C^k$, $k \gg 1$ (depending on s). Moreover those operators depend continuously on g. Note that the 1-form v so that $\mathcal{P}f = dv$ is determined uniquely by (14).

Notice that even when f is smooth and $f = 0$ on ∂M, then f^s does not need to vanish on ∂M. In particular, f^s, extended as 0 to M_1, may not be solenoidal anymore. To stress on the dependence on the manifold, when needed, we will use the notation v_M and f^s_M as well.

Operators $\mathcal{S}$ and $\mathcal{P}$ are orthogonal projectors. The problem about the s-injectivity of I_α then can be posed as follows: if $I_\alpha f = 0$, show that $f^s = 0$, in other words, show that I_α is injective on the subspace $\mathcal{S}L^2$ of solenoidal tensors. Note that by (13) and (6),

$$N_\alpha = N_\alpha\mathcal{S} = \mathcal{S}N_\alpha, \quad \mathcal{P}N_\alpha = N_\alpha\mathcal{P} = 0. \tag{16}$$

Lemma 2. *Let $\alpha = \{\alpha_m\}$ with $\alpha_m \in C_0^\infty(\mathcal{H}_m)$ be as in the Introduction. The following statements are equivalent:*

(a) I_α is s-injective on $L^2(M)$;

(b) $N_\alpha : L^2(M) \to L^2(M)$ is s-injective;

(c) $N_\alpha : L^2(M) \to L^2(M_1)$ is s-injective;

(d) If Γ^α_m is the set of geodesics issued from $(\operatorname{supp}\alpha_m)^{\mathrm{int}}$ as in (4), and $\Gamma^\alpha = \cup\Gamma^\alpha_m$, then I_{Γ^α} is s-injective.

Proof. Let I_α be s-injective, and assume that $N_\alpha f = 0$ in M for some $f \in L^2(M)$. Then

$$0 = (N_\alpha f, f)_{L^2(M)} = \sum \|\alpha_m If\|^2_{L^2(\mathcal{H}_m, \mathrm{d}\mu)} \quad \Longrightarrow \quad f^s = 0.$$

This proves the implication $(a) \Rightarrow (b)$. Next, $(b) \Rightarrow (c)$ is immediate. Assume (c) and let $f \in L^2(M)$ be such that $I_\alpha f = 0$. Then $N_\alpha f = 0$ in M_1, therefore $f^s = 0$. Therefore, $(c) \Rightarrow (a)$. Finally, $(a) \Leftrightarrow (d)$ follows directly form the definition of I_α. ◻

Note that in (d), I_{Γ^α} is the transform I restricted to Γ^α (and weight 1), while I_α is the ray transform with weight α.

Remark. Lemma 2 above, and Lemma 4(a) in next section show that $(\operatorname{supp}\alpha_m)^{\mathrm{int}}$ in (d) can be replaced by $\operatorname{supp}\alpha_m$ if Γ^α is a complete set of geodesics.

3. MICROLOCAL PARAMETRIX OF N_α

Proposition 1. *Let $g = g_0 \in C^k(M)$ be a regular metric on M, and let $\mathcal{H}' \Subset \mathcal{H}$ be as in Theorem 2.*

(a) Let α be as in Theorem 2(a). Then for any $t = 1, 2, \dots$, there exists $k > 0$ and a bounded linear operator

$$Q : \tilde{H}^2(M_1) \longmapsto \mathcal{S}L^2(M),$$

such that

$$QN_\alpha f = f_M^s + Kf, \quad \forall f \in H^1(M), \tag{17}$$

where $K : H^1(M) \to \mathcal{S}H^{1+t}(M)$ extends to $K : L^2(M) \to \mathcal{S}H^t(M)$. If $t = \infty$, then $k = \infty$.

(b) Let $\alpha = \alpha_g$ be as in Theorem 2(b). Then, for g in some C^k neighborhood of g_0, (a) still holds and Q can be constructed so that K would depend continuously on g.

Proof. A brief sketch of our proof is the following: We construct first a parametrix that recovers microlocally $f_{M_1}^s$ from $N_\alpha f$. Next we will compose this parametrix with the operator $f_{M_1}^s \mapsto f_M^s$ as in [SU3, SU4]. Part (b) is based on a perturbation argument for the Fredholm equation (17). The need for such two step construction is due to the fact that in the definition of f^s, a solution to a certain boundary value problem is involved, therefore near ∂M, our construction is not just a parametrix of a certain elliptic ΨDO. This is the reason for losing one derivative in (7). For tensors of orders 0 and 1, there is no such loss, see [SU3] and (61), (62).

As in [SU4], we will work with ΨDOs with symbols of finite smoothness $k \gg 1$. All operations we are going to perform would require finitely many derivatives of the amplitude and finitely many seminorm estimates. In turn, this would be achieved if $g \in C^k$, $k \gg 1$ and the corresponding ΨDOs will depends continuously on g.

Recall [SU3, SU4] that for simple metrics, N is a ΨDO in M^{int} of order -1 with principal symbol that is not elliptic but $N + |D|^{-1}\mathcal{P}$ is elliptic. Here, $|D|^{-1}$ is any parametrix of $(-\Delta_g)^{1/2}$. This is a consequence of the following. We will say that N_α (and any other ΨDO acting on symmetric tensors) is *elliptic on solenoidal tensors*, if for any (x,ξ), $\xi \neq 0$, $\sigma_p(N_\alpha)^{ijkl}(x,\xi) f_{kl} = 0$ and $\xi^i f_{ij} = 0$ imply $f = 0$. Then N is elliptic on solenoidal tensors, as shown in [SU3]. That definition is motivated by the fact that the principal symbol of δ is given by $f_{ij} \mapsto \mathrm{i}\xi^i f_{ij}$, and s-injectivity is equivalent to the statement that $Nf = 0$ and $\delta f = 0$ in M imply $f = 0$. Note also that the principal symbol of d is given by $v_j \mapsto (\xi_i v_j + \xi_j v_i)/2$, and $\sigma_p(N)$ vanishes on tensors represented by the r.h.s. of the latter. We will establish similar properties of N_α below.

Let N_{α_m} be as in Section 2.2 with m fixed.

Lemma 3. *N_{α_m} is a classical ΨDO of order -1 in M_1^{int}. It is elliptic on solenoidal tensors at (x_0, ξ^0) if and only if there exists $\theta_0 \in T_{x_0}M_1 \setminus 0$ with $\langle \xi^0, \theta_0 \rangle = 0$ such that $\alpha_0(x_0, \theta_0) \neq 0$. The principal symbol $\sigma_p(N_{\alpha_m})$ vanishes on tensors of the kind $f_{ij} = (\xi_i v_j + \xi_j v_i)/2$ and is non-negative on tensors satisfying $\xi^i f_{ij} = 0$.*

Proof. We established the pseudolocal property already, and formulas (9), (10) together with the partition of unity argument following them imply that it is enough to work with x in a small neighborhood of a fixed $x_0 \in M_1^{\text{int}}$, and with f supported there as well. Then we work in local coordinates near x_0. To express N_{α_m} as a pseudo-differential operator, we proceed as in [SU3, SU4], with a starting point (10). Recall that for x

close to y we have

$$\begin{aligned}\rho^2(x,y) &= G^{(1)}_{ij}(x,y)(x-y)^i(x-y)^j,\\ \frac{\partial\rho^2(x,y)}{\partial x^j} &= 2G^{(2)}_{ij}(x,y)(x-y)^i,\\ \frac{\partial^2\rho^2(x,y)}{\partial x^j\partial y^j} &= -2G^{(3)}_{ij}(x,y),\end{aligned}$$

where $G^{(1)}_{ij}$, $G^{(2)}_{ij}$ $G^{(3)}_{ij}$ are smooth and on the diagonal. We have

$$G^{(1)}_{ij}(x,x) = G^{(2)}_{ij}(x,x) = G^{(3)}_{ij}(x,x) = g_{ij}(x).$$

Then N_{α_m} is a pseudo-differential operator with amplitude

$$\begin{aligned}M_{ijkl}(x,y,\xi) = \int e^{-\mathrm{i}\xi\cdot z}\left(G^{(1)}z\cdot z\right)^{\frac{-n+1}{2}-2}\left|\alpha^\sharp_m(x,g^{-1}G^{(2)}z)\right|^2\\ \times\left[G^{(2)}z\right]_i\left[G^{(2)}z\right]_j\left[\widetilde{G}^{(2)}z\right]_k\left[\widetilde{G}^{(2)}z\right]_l\frac{|\det G^{(3)}|}{\sqrt{\det g}}\,\mathrm{d}z,\end{aligned}\tag{18}$$

where $\widetilde{G}^{(2)}_{ij}(x,y) = G^{(2)}_{ij}(y,x)$. As in [SU4], we note that M_{ijkl} is the Fourier transform of a positively homogeneous distribution in the z variable, of order $n-1$. Therefore, M_{ijkl} itself is positively homogeneous of order -1 in ξ. Write

$$M(x,y,\xi) = \int e^{-\mathrm{i}\xi\cdot z}|z|^{-n+1}m(x,y,\theta)\,\mathrm{d}z,\quad \theta = z/|z|,\tag{19}$$

where

$$\begin{aligned}m_{ijkl}(x,y,\theta) = \left(G^{(1)}\theta\cdot\theta\right)^{\frac{-n+1}{2}-2}\left|\alpha^\sharp_m(x,g^{-1}G^{(2)}\theta)\right|^2\\ \times\left[G^{(2)}\theta\right]_i\left[G^{(2)}\theta\right]_j\left[\widetilde{G}^{(2)}\theta\right]_k\left[\widetilde{G}^{(2)}\theta\right]_l\frac{|\det G^{(3)}|}{\sqrt{\det g(x)}},\end{aligned}\tag{20}$$

and pass to polar coordinates $z = r\theta$. Since m is an even function of θ, smooth w.r.t. all variables, we get (see also [H, Theorem 7.1.24])

$$M(x,y,\xi) = \pi\int_{|\theta|=1} m(x,y,\theta)\delta(\theta\cdot\xi)\,\mathrm{d}\theta.\tag{21}$$

This proves that M is an amplitude of order -1.

To obtain the principal symbol, we set $x=y$ above (see also [SU3, sec. 5] to get

$$\sigma_p(N_{\alpha_m})(x,\xi) = M(x,x,\xi) = \pi\int_{|\theta|=1} m(x,x,\theta)\delta(\theta\cdot\xi)\,\mathrm{d}\theta,\tag{22}$$

where

$$m^{ijkl}(x,x,\theta) = \left|\alpha^\sharp_m(x,\theta)\right|^2\sqrt{\det g(x)}\left(g_{ij}(x)\theta^i\theta^j\right)^{\frac{-n+1}{2}-2}\theta^i\theta^j\theta^k\theta^l.\tag{23}$$

To prove ellipticity of $M(x,\xi)$ on solenoidal tensors at (x_0,ξ^0), notice that for any symmetric real f_{ij}, we have

$$m^{ijkl}(x_0,x_0,\theta)f_{ij}f_{kl} = \left|\alpha^\sharp_m(x_0,\theta)\right|^2\sqrt{\det g(x_0)}\left(g_{ij}(x_0)\theta^i\theta^j\right)^{\frac{-n+1}{2}-2}\left(f_{ij}\theta^i\theta^j\right)^2 \geq 0.\tag{24}$$

This, (22), and the assumption $\alpha_m(x_0,\theta_0)\neq 0$ imply that $M^{ijkl}(x_0,x_0,\xi^0)f_{ij}f_{kl}=0$ yields $f_{ij}\theta^i\theta^j=0$ for θ perpendicular to ξ^0, and close enough to θ_0. If in addition $(\xi^0)^j f_{ij}=0$, then this implies $f_{ij}\theta^i\theta^j=0$ for $\theta\in\text{neigh}(\theta_0)$, and that easily implies that it vanishes for all θ. Since f is symmetric, this means that $f=0$.

The last statement of the lemma follows directly from (22), (23), (24).

Finally, we note that (23), (24) and the proof above generalizes easily for tensors of any order. □

We continue with the proof of Proposition 1. Since (b) implies (a), we will prove (b) directly. Notice that $\mathcal{H}'$ and $\mathcal{H}$ satisfy the properties listed in the Introduction, right before Theorem 2, if $g=g_0$. On the other hand, those properties are stable under small C^k perturbation of g_0. We will work here with metrics g close enough to g_0.

By Lemma 3, since $\Gamma(\mathcal{H}')$ is complete, N_α defined by (6) is elliptic on solenoidal tensors in M. The rest of the proof is identical to that of [SU4, Proposition 4]. We will give a brief sketch of it. To use the ellipticity of N_α on solenoidal tensors, we complete N_α to an elliptic ΨDO as in [SU4]. Set

$$W=N_\alpha+|D|^{-1}\mathcal{P}_{M_1}, \tag{25}$$

where $|D|^{-1}$ is a properly supported parametrix of $(-\Delta_g)^{1/2}$ in neigh(M_1). The resolvent $(-\Delta^s_{M_1,D})^{-1}$ involved in $\mathcal{P}_{M_1}$ and $\mathcal{S}_{M_1}$ can be expressed as R_1+R_2, where R_1 is any parametrix near M_1, and $R_2: L^2_{\text{comp}}(M_1)\to C^l(M_1)$, $R_2: H^l(M_1)\to H^{l+2}(M_1)$, where $l=l(k)\gg 1$, if $k\gg 1$. Then W is an elliptic ΨDO inside M_1 of order -1 by Lemma 3.

Let P be a properly supported parametrix for W of finite order, i.e., P is a classical ΨDO in the interior of M_1 of order 1 with amplitude of finite smoothness, such that

$$PW=\text{Id}+K_1, \tag{26}$$

and $K_1: L^2_{\text{comp}}(M_1)\to H^l(M_1)$ with l as above. Then

$$P_1:=\mathcal{S}_{M_1}P$$

satisfies

$$P_1N_\alpha=\mathcal{S}_{M_1}+K_2, \tag{27}$$

where K_2 has the same property as K_1. To see this, it is enough to apply $\mathcal{S}_{M_1}$ to the left and right of (26) and to use (16).

Next step is to construct an operator that recovers f^s_M, given $f^s_{M_1}$, and to apply it to $P_1N_\alpha-K_2$. In order to do this, it is enough first to construct a map P_2 such that if $f^s_{M_1}$ and v_{M_1} are the solenoidal part and the potential, respectively, corresponding to $f\in L^2(M)$ extended as zero to $M_1\setminus M$, then $P_2: f^s_{M_1}\mapsto v_{M_1}\big|_{\partial M}$. This is done as in [SU3] and [SU4, Proposition 4]. We also have

$$P_2P_1:\tilde H^2(M_1)\to H^{1/2}(\partial M).$$

Then we showed in [SU4, Proposition 4] that one can set

$$Q=(\text{Id}+dRP_2)P_1,$$

where $R: h\mapsto u$ is the Poisson operator for the Dirichlet problem $\Delta^s u=0$ in M, $u|_{\partial M}=h$.

As explained above, we work with finite asymptotic expansions that require finite number of derivatives on the amplitudes of our ΨDOs. On the other hand, these amplitudes depend continuously on $g\in C^k$, $k\gg 1$. As a result, all operators above depend continuously on $g\in C^k$, $k\gg 1$. □

The first part of next lemma generalizes similar results in [SU3, Thm 2], [Ch, SSU] to the present situation. The second part shows that $I_\Gamma f=0$ implies that a certain $\tilde f$, with the same solenoidal projection, is flat at ∂M. This $\tilde f$ is defined by the property (29) below.

Lemma 4. *Let $g \in C^k(M)$ be a regular metric, and let Γ be a complete set of geodesics. Then*

(a) Ker $I_\Gamma \cap \mathcal{S}L^2(M)$ *is finite dimensional and included in $C^l(M)$ with $l = l(k) \to \infty$, as $k \to \infty$.*

(b) If $I_\Gamma f = 0$ with $f \in L^2(M)$, then there exists a vector field $v \in C^l(M)$, with $v|_{\partial M} = 0$ and l as above, such that for $\tilde f := f - dv$ we have

$$\partial^\alpha \tilde f|_{\partial M} = 0, \quad |\alpha| \le l, \tag{28}$$

and in boundary normal coordinates near any point on ∂M we have

$$\tilde f_{ni} = 0, \quad \forall i. \tag{29}$$

Proof. Part (a) follows directly from Proposition 1.

Without loss of generality, we may assume that M_1 is defined as $M_1 = \{x,\ \mathrm{dist}(x, M) \le \epsilon\}$, with $\epsilon > 0$ small enough. By Proposition 1, applied to M_1,

$$f^s_{M_1} \in C^l(M_1), \tag{30}$$

where $l \gg 1$, if $k \gg 1$.

Let $x = (x', x^n)$ be boundary normal coordinates in a neighborhood of some boundary point. We recall how to construct v defined in M so that (29) holds, see [SU2] for a similar argument for the non-linear boundary rigidity problem, and [E, Sh2, SU3, SU4] for the present one. The condition $(f - dv)_{in} = 0$ is equivalent to

$$\nabla_n v_i + \nabla_i v_n = 2 f_{in}, \quad v|_{x^n=0} = 0, \quad i = 1, \dots, n. \tag{31}$$

Recall that $\nabla_i v_j = \partial_i v_j - \Gamma^k_{ij} v_k$, and that in those coordinates, $\Gamma^k_{nn} = \Gamma^n_{kn} = 0$. If $i = n$, then (31) reduces to $\nabla_n v_n = \partial_n v_n = f_{nn}$, $v_n = 0$ for $x^n = 0$; we solve this by integration over $0 \le x^n \le \varepsilon \ll 1$; this gives us v_n. Next, we solve the remaining linear system of $n-1$ equations for $i = 1, \dots, n-1$ that is of the form $\nabla_n v_i = 2 f_{in} - \nabla_i v_n$, or, equivalently,

$$\partial_n v_i - 2\Gamma^\alpha_{ni} v_\alpha = 2 f_{in} - \partial_i v_n, \quad v_i|_{x^n=0} = 0, \quad i = 1, \dots, n-1, \tag{32}$$

(recall that $\alpha = 1, \dots, n-1$). Clearly, if g and f are smooth enough near ∂M, then so is v. If we set $f = f^s$ above (they both belong to Ker I_Γ), then by (a) we get the statement about the smoothness of v. Since the condition (29) has an invariant meaning, this in fact defines a construction in some one-sided neighborhood of ∂M in M. One can cut v outside that neighborhood in a smooth way to define v globally in M. We also note that this can be done for tensors of any order m, see [Sh2], then we have to solve consecutively m ODEs.

Let $\tilde f = f - dv$, where v is as above. Then $\tilde f$ satisfies (29), and let

$$\tilde f^s_{M_1} = \tilde f - d\tilde v_{M_1} \tag{33}$$

be the solenoidal projection of $\tilde f$ in M_1. Recall that $\tilde f$, according to our convention, is extended as zero in $M_1 \setminus M$ that in principle, could create jumps across ∂M. Clearly, $\tilde f^s_{M_1} = f^s_{M_1}$ because $f - \tilde f = dv$ in M with v as in the previous paragraph, and this is also true in M_1 with $\tilde f$, f and v extended as zero (and then $v = 0$ on ∂M_1). In (33), the l.h.s. is smooth in M_1 by (30), and $\tilde f$ satisfies (29) even outside M, where it is zero. Then one can get $\tilde v_{M_1}$ by solving (31) with M replaced by M_1, and f there replaced by $\tilde f^s_{M_1} \in C^l(M_1)$. Therefore, one gets that $\tilde v_{M_1}$, and therefore $\tilde f$, is smooth enough across ∂M, if $g \in C^k$, $k \gg 1$, which proves (28).

One can give the following alternative proof of (28): Let N_α be related to Γ, as in Theorem 2. One can easily check that N_α, restricted to tensors satisfying (29), is elliptic for $\xi_n \ne 0$. Since $N_\alpha \tilde f = 0$ near M, with $\tilde f$ extended as 0 outside M, as above, we get that this extension cannot have conormal singularities across ∂M. This implies (28), at least when $g \in C^\infty$. The case of g of finite smoothness can be treated by using parametrices of finite order in the conormal singularities calculus. $\square$

4. S-INJECTIVITY FOR ANALYTIC REGULAR METRICS

In this section, we prove Theorem 1. Let g be an analytic regular metrics in M, and let $M_1 \supset M$ be the manifold where g is extended analytically according to Definition 1. Recall that there is an analytic atlas in M, and ∂M can be assumed to be analytic, too. In other words, in this section, $(M, \partial M, g)$ is a real analytic manifold with boundary.

We will show first that $I_\Gamma f = 0$ implies $f^s \in \mathcal{A}(M)$. We start with interior analytic regularity. Below, $\mathrm{WF_A}(f)$ stands for the analytic wave front set of f, see [Sj, Tre].

Proposition 2. *Let $(x_0, \xi^0) \in T^*M \setminus 0$, and let γ_0 be a fixed simple geodesic through x_0 normal to ξ^0. Let $If(\gamma) = 0$ for some 2-tensor $f \in L^2(M)$ and all $\gamma \in$ neigh(γ_0). Let g be analytic in* neigh(γ_0) *and $\delta f = 0$ near x_0. Then*

$$(x_0, \xi^0) \notin \mathrm{WF_A}(f). \tag{34}$$

Proof. As explained in Section 2.1, without loss of generality, we can assume that γ_0 does not self-intersect. Let U be a tubular neighborhood of γ_0 with $x = (x', x^n)$ analytic semigeodesic coordinates in it, as in the second paragraph of Section 2.1. We can assume that $x_0 = 0$, $g_{ij}(0) = \delta_{ij}$, and $x' = 0$ on γ_0. In those coordinates, U is given by $|x'| < \varepsilon$, $l^- < x^n < l^+$, with some $0 < \varepsilon \ll 1$, and we can choose $\varepsilon \ll 1$ so that $\{x^n = l^\pm;\ |x'| \le \varepsilon\}$ lie outside M. Recall that the lines $x' =$ const. in U are geodesics.

Then $\xi^0 = ((\xi^0)', 0)$ with $\xi^0_n = 0$. We need to show that

$$(0, \xi^0) \notin \mathrm{WF_A}(f). \tag{35}$$

We choose a local chart for the geodesics close to γ_0. Set first $Z = \{x^n = 0;\ |x'| < 7\varepsilon/8\}$, and denote the x' variable on Z by z'. Then z', θ' (with $|\theta'| \ll 1$) are local coordinates in neigh(γ_0) determined by $(z', \theta') \to \gamma_{(z',0),(\theta',1)}$. Each such geodesic is assumed to be defined on $l^- \le t \le l^+$, the same interval on which γ_0 is defined.

Let $\chi_N(z')$, $N = 1, 2, \dots$, be a sequence of smooth cut-off functions equal to 1 for $|z'| \le 3\varepsilon/4$, supported in Z, and satisfying the estimates

$$\left|\partial^\alpha \chi_N\right| \le (CN)^{|\alpha|}, \quad |\alpha| \le N, \tag{36}$$

see [Tre, Lemma 1.1]. Set $\theta = (\theta', 1)$, $|\theta'| \ll 1$, and multiply

$$If\left(\gamma_{(z',0),\theta}\right) = 0$$

by $\chi_N(z')e^{\mathrm{i}\lambda z'\cdot\xi'}$, where $\lambda > 0$, ξ' is in a complex neighborhood of $(\xi^0)'$, and integrate w.r.t. z' to get

$$\iint e^{\lambda \mathrm{i} z'\cdot\xi'}\chi_N(z') f_{ij}\left(\gamma_{(z',0),\theta}(t)\right)\dot\gamma^i_{(z',0),\theta}(t)\dot\gamma^j_{(z',0),\theta}(t)\,\mathrm{d}t\,\mathrm{d}z' = 0. \tag{37}$$

For $|\theta'| \ll 1$, $(z', t) \in Z \times (l^-, l^+)$ are local coordinates near γ_0 given by $x = \gamma_{(z',0),\theta}(t)$.

If $\theta' = 0$, we have $x = (z', t)$. By a perturbation argument, for θ' fixed and small enough, (t, z') are analytic local coordinates, depending analytically on θ'. In particular, $x = (z' + t\theta', t) + O(|\theta'|)$ but this expansion is not enough for the analysis below. Performing a change of variables in (37), we get

$$\int e^{\mathrm{i}\lambda z'(x,\theta')\cdot\xi'} a_N(x, \theta') f_{ij}(x) b^i(x, \theta') b^j(x, \theta')\,\mathrm{d}x = 0 \tag{38}$$

for $|\theta'| \ll 1$, $\forall\lambda$, $\forall\xi'$, where, for $|\theta'| \ll 1$, the function $(x, \theta') \mapsto a_N$ is analytic and positive for x in a neighborhood of γ_0, vanishing for $x \notin U$, and satisfying (36). The vector field b is analytic on supp a_N, and $b(0, \theta') = \theta$, $a_N(0, \theta') = 1$.

To clarify the arguments that follow, note that if g is Euclidean in neigh(γ_0), then (38) reduces to

$$\int e^{\mathrm{i}\lambda(\xi',-\theta'\cdot\xi')\cdot x}\chi_N f_{ij}(x)\theta^i\theta^j\,\mathrm{d}x = 0,$$

where $\chi_N = \chi_N(x' - x^n\theta')$. Then $\xi = (\xi', -\theta'\cdot\xi')$ is perpendicular to $\theta = (\theta', 1)$. This implies that

$$\int e^{\mathrm{i}\lambda\xi\cdot x}\chi_N f_{ij}(x)\theta^i(\xi)\theta^j(\xi)\,\mathrm{d}x = 0 \tag{39}$$

for any function $\theta(\xi)$ defined near ξ^0, such that $\theta(\xi)\cdot\xi = 0$. This has been noticed and used before if g is close to the Euclidean metric (with $\chi_N = 1$), see e.g., [SU2]. We will assume that $\theta(\xi)$ is analytic. A simple argument (see e.g. [Sh1, SU2]) shows that a constant symmetric tensor f_{ij} is uniquely determined by the numbers $f_{ij}\theta^i\theta^j$ for finitely many θ's (actually, for $N' = (n+1)n/2$ θ's); and in any open set on the unit sphere, there are such θ's. On the other hand, f is solenoidal near x_0. To simplify the argument, assume for a moment that f vanishes on ∂M and is solenoidal everywhere. Then $\xi^i\hat{f}_{ij}(\xi) = 0$. Therefore, combining this with (39), we need to choose $N = n(n-1)/2$ vectors $\theta(\xi)$, perpendicular to ξ, that would uniquely determine the tensor $\hat{f}$ on the plane perpendicular to ξ. To this end, it is enough to know that this choice can be made for $\xi = \xi^0$, then it would be true for $\xi \in$ neigh(ξ^0). This way, $\xi^i\hat{f}_{ij}(\xi) = 0$ and the N equations (39) with the so chosen $\theta_p(\xi)$, $p = 1,\dots,N$, form a system with a tensor-valued symbol elliptic near $\xi = \xi^0$. The C^∞ ΨDO calculus easily implies the statement of the lemma in the C^∞ category, and the complex stationary phase method below, or the analytic ΨDO calculus in [Tre] with appropriate cut-offs in ξ, implies the lemma in this special case (g locally Euclidean).

We proceed with the proof in the general case. Since we will localize eventually near $x_0 = 0$, where g is close to the Euclidean metric, the special case above serves as a useful guideline. On the other hand, we work near a "long geodesic" and the lack of points conjugate to $x_0 = 0$ along it will play a decisive role in order to allow us to localize near $x = 0$.

Let $\theta(\xi)$ be a vector analytically depending on ξ near $\xi = \xi^0$, such that

$$\theta(\xi)\cdot\xi = 0, \quad \theta^n(\xi) = 1, \quad \theta(\xi^0) = e_n. \tag{40}$$

Here and below, e_j stand for the vectors $\partial/\partial x^j$. Replace $\theta = (\theta', 1)$ in (38) by $\theta(\xi)$ (the requirement $|\theta'| \ll 1$ is fulfilled for ξ close enough to ξ^0), to get

$$\int e^{\mathrm{i}\lambda\varphi(x,\xi)}\tilde{a}_N(x,\xi)f_{ij}(x)\tilde{b}^i(x,\xi)\tilde{b}^j(x,\xi)\,\mathrm{d}x = 0, \tag{41}$$

where $\tilde{a}_N$ is analytic near $\gamma_0\times\{\xi^0\}$, and satisfies (36) for ξ close enough to ξ^0 and all x. Next, φ, $\tilde{b}$ are analytic on supp $\tilde{a}_N$ for ξ close to ξ^0. In particular,

$$\tilde{b} = \dot{\gamma}_{(z',0),(\theta'(\xi),1)}(t), \quad t = t(x,\theta'(\xi)),\ z' = z'(x,\theta'(\xi)),$$

and

$$\tilde{b}(0,\xi) = \theta(\xi), \quad \tilde{a}_N(0,\xi) = 1.$$

The phase function is given by

$$\varphi(x,\xi) = z'(x,\theta'(\xi))\cdot\xi'. \tag{42}$$

To verify that φ is a non-degenerate phase in neigh $(0,\xi^0)$, i.e., that $\det\varphi_{x\xi}(0,\xi^0) \neq 0$, note first that $z' = x'$ when $x^n = 0$, therefore, $(\partial z'/\partial x')(0,\theta(\xi)) = \mathrm{Id}$. On the other hand, linearizing near $x^n = 0$, we easily get $(\partial z'/\partial x^n)(0,\theta(\xi)) = -\theta'(\xi)$. Therefore,

$$\varphi_x(0,\xi) = (\xi', -\theta'(\xi)\cdot\xi') = \xi$$

by (40). So we get $\varphi_{x\xi}(0,\xi) = \mathrm{Id}$, which proves the non-degeneracy claim above. In particular, we get that $x \mapsto \varphi_\xi(x,\xi)$ is a local diffeomorphism in neigh(0) for $\xi \in$ neigh(ξ^0), and therefore injective. We need however a semiglobal version of this along γ_0 as in the lemma below. For this reason we will make the following special choice of $\theta(\xi)$. Without loss of generality we can assume that

$$\xi^0 = e^{n-1}.$$

Set

$$\theta(\xi) = \left(\xi_1, \dots, \xi_{n-2}, -\frac{\xi_1^2 + \cdots + \xi_{n-2}^2 + \xi_n}{\xi_{n-1}}, 1\right). \tag{43}$$

If $n = 2$, this reduces to $\theta(\xi) = (-\xi_2/\xi_1, 1)$. Clearly, $\theta(\xi)$ satisfies (40). Moreover, we have

$$\frac{\partial\theta}{\partial\xi_\nu}(\xi^0) = e_\nu, \quad \nu = 1, \dots, n-2, \quad \frac{\partial\theta}{\partial\xi_{n-1}}(\xi^0) = 0, \quad \frac{\partial\theta}{\partial\xi_n}(\xi^0) = -e_{n-1}, \tag{44}$$

In particular, the differential of the map $S^{n-1} \ni \xi \mapsto \theta'(\xi)$ is invertible at $\xi = \xi^0 = e^{n-1}$.

Lemma 5. *Let $\theta(\xi)$ be as in (43), and $\varphi(x,\xi)$ be as in (42). Then there exists $\delta > 0$ such that if*

$$\varphi_\xi(x,\xi) = \varphi_\xi(y,\xi)$$

for some $x \in U$, $|y| < \delta$, $|\xi - \xi^0| < \delta$, ξ complex, then $y = x$.

Proof. We will study first the case $y = 0$, $\xi = \xi^0$, $x' = 0$. Since $\varphi_\xi(0,\xi) = 0$, we need to show that $\varphi_\xi((0,x^n),\xi^0) = 0$ for $(0,x^n) \in U$ (i.e., for $l^- < x^n < l^+$) implies $x^n = 0$.

To compute $\varphi_\xi(x,\xi^0)$, we need first to know $\partial z'(x,\theta')/\partial\theta'$ at $\theta' = 0$. Differentiate $\gamma'_{(z',0),(\theta',1)}(t) = x'$ w.r.t. θ', where $t = t(x,\theta')$, $z' = z'(x,\theta')$, to get

$$\partial_{\theta_\nu}\gamma'_{(z',0),(\theta',1)}(t) + \partial_{z'}\gamma'_{(z',0),(\theta',1)}(t)\cdot\frac{\partial z'}{\partial\theta_\nu} + \dot\gamma'_{(z',0),(\theta',1)}(t)\frac{\partial t}{\partial\theta_\nu} = 0.$$

Plug $\theta' = 0$. Since $\partial t/\partial\theta' = 0$ at $\theta' = 0$, we get

$$\frac{\partial z'}{\partial\theta_\nu} = -\partial_{\theta_\nu}\gamma'_{(z',0),(\theta',1)}(x^n)\Big|_{\theta'=0,x'=0} = -J'_\nu(x^n),$$

where the prime denotes the first $n-1$ components, as usual; $J_\nu(x^n)$ is the Jacobi field along the geodesic $x^n \mapsto \gamma_0(x^n)$ with initial conditions $J_\nu(0) = 0$, $DJ_\nu(0) = e_\nu$; and D stands for the covariant derivative along γ_0. Since $z'((0,x^n),\theta'(\xi^0)) = 0$, by (42) we then get

$$\frac{\partial\varphi}{\partial\xi_l}((0,x^n),\xi^0) = -\frac{\partial\theta^\mu}{\partial\xi_l}(\xi^0)J_\mu(x^n)\cdot(\xi^0)'.$$

By (44), (recall that $\xi^0 = e^{n-1}$),

$$\frac{\partial\varphi}{\partial\xi_l}((0,x^n),\xi^0) = \begin{cases} -J_l^{n-1}(x^n), & l = 1,\dots,n-2, \\ 0, & l = n-1, \\ J_{n-1}^{n-1}(x^n), & l = n, \end{cases} \tag{45}$$

where J_ν^{n-1} is the $(n-1)$-th component of J_ν. Now, assuming that the l.h.s. of (45) vanishes for some fixed $x^n = t_0$, we get that $J_\nu^{n-1}(t_0) = 0$, $\nu = 1,\dots,n-1$. On the other hand, J_ν are orthogonal to e_n because the initial conditions $J_\nu(0) = 0$, $DJ_\nu(0) = e_\nu$ are orthogonal to e_n, too. Since $g_{in} = \delta_{in}$, this means that $J_\nu^n = 0$. Therefore, $J_\nu(t_0)$, $\nu = 1,\dots,n-1$, form a linearly dependent system of vectors, thus some non-trivial linear combination $a^\nu J_\nu(t_0)$ vanishes. Then the solution $J_0(t)$ of the Jacobi equation along γ_0 with initial conditions $J_0(0) = 0$, $DJ_0(0) = a^\nu e_\nu$ satisfies $J(t_0) = 0$. Since $DJ_0(0) \neq 0$, J_0 is not identically zero. Therefore, we get that $x_0 = 0$ and $x = (0, t_0)$ are conjugate points. Since γ_0 is a simple geodesic x_0, we must have $t_0 = 0 = x^n$.

The same proof applies if $x' \neq 0$ by shifting the x' coordinates.

Let now y, ξ and x be as in the Lemma. The lemma is clearly true for x in the ball $B(0, \varepsilon_1) = \{|x| < \varepsilon_1\}$, where $\varepsilon_1 \ll 1$, because $\varphi(0, \xi^0)$ is non-degenerate. On the other hand, $\varphi_\xi(x, \xi) \neq \varphi_\xi(y, \xi)$ for $x \in \bar{U} \setminus B(0, \varepsilon_1)$, $y = 0$, $\xi = \xi^0$. Hence, we still have $\varphi_\xi(x, \xi) \neq \varphi_\xi(y, \xi)$ for a small perturbation of y and ξ. □

The arguments that follow are close to those in [KSU, Section 6]. We will apply the complex stationary phase method [Sj]. For x, y as in Lemma 5, and $|\eta - \xi^0| \leq \delta/\tilde{C}$, $\tilde{C} \gg 2$, $\delta \ll 1$, multiply (41) by

$$\tilde{\chi}(\xi - \eta) e^{\mathrm{i}\lambda(\mathrm{i}(\xi-\eta)^2/2 - \varphi(y,\xi))},$$

where $\tilde{\chi}$ is the characteristic function of the ball $B(0, \delta) \subset \mathbf{C}^n$, and integrate w.r.t. ξ to get

$$\iint e^{\mathrm{i}\lambda\Phi(y,x,\eta,\xi)} \tilde{\tilde{a}}_N(x, \xi, \eta) f_{ij}(x) \tilde{b}^i(x, \xi) \tilde{b}^j(x, \xi) \, \mathrm{d}x \, \mathrm{d}\xi = 0. \tag{46}$$

Here $\tilde{\tilde{a}}_N = \tilde{\chi}(\xi - \eta)\tilde{a}_N$ is another amplitude, analytic and elliptic for x close to 0, $|\xi - \eta| < \delta/\tilde{C}$, and

$$\Phi = -\varphi(y, \xi) + \varphi(x, \xi) + \frac{\mathrm{i}}{2}(\xi - \eta)^2.$$

We study the critical points of $\xi \mapsto \Phi$. If $y = x$, there is a unique (real) critical point $\xi_c = \eta$, and it satisfies $\Im \Phi_{\xi\xi} > 0$ at $\xi = \xi_c$. For $y \neq x$, there is no real critical point by Lemma 5. On the other hand, again by Lemma 5, there is no (complex) critical point if $|x - y| > \delta/C_1$ with some $C_1 > 0$, and there is a unique complex critical point ξ_c if $|x - y| < \delta/C_2$, with some $C_2 > C_1$, still non-degenerate if $\delta \ll 1$. For any $C_0 > 0$, if we integrate in (46) for $|x - y| > \delta/C_0$, and use the fact that $|\Phi_\xi|$ has a positive lower bound (for ξ real), we get

$$\left| \iint_{|x-y|>\delta/C_0} e^{\mathrm{i}\lambda\Phi(y,x,\eta,\xi)} \tilde{\tilde{a}}_N(x, \xi, \eta) f_{ij}(x) \tilde{b}^i(x, \xi) \tilde{b}^j(x, \xi) \, \mathrm{d}x \, \mathrm{d}\xi \right| \leq C_3 (C_3 N/\lambda)^N + C N e^{-\lambda/C}. \tag{47}$$

Estimate (47) is obtained by integrating N times by parts, using the identity

$$L e^{\mathrm{i}\lambda\Phi} = e^{\mathrm{i}\lambda\Phi}, \quad L := \frac{\bar{\Phi}_\xi \cdot \partial_\xi}{\mathrm{i}\lambda |\Phi_\xi|^2}$$

as well as using the estimate (36), and the fact that on the boundary of integration in ξ, the $e^{\mathrm{i}\lambda\Phi}$ is exponentially small. Choose $C_0 \gg C_2$. Note that $\Im \Phi > 0$ for $\xi \in \partial(\operatorname{supp} \tilde{\chi}(\cdot - \eta))$, and η as above, as long as $\tilde{C} \gg 1$, and by choosing $C_0 \gg 1$, we can make sure that ξ_c is as close to η, as we want.

To estimate (46) for $|x - y| < \delta/C_0$, set

$$\psi(x, y, \eta) := \Phi\big|_{\xi=\xi_c}.$$

Note that $\xi_c = -\mathrm{i}(y - x) + \eta + O(\delta)$, and $\psi(x, y, \eta) = \eta \cdot (x - y) + \frac{\mathrm{i}}{2}|x - y|^2 + O(\delta)$. We will not use this to study the properties of ψ, however. Instead, observe that at $y = x$ we have

$$\psi_y(x, x, \eta) = -\varphi_x(x, \eta), \quad \psi_x(x, x, \eta) = \varphi_x(x, \eta), \quad \psi(x, x, \eta) = 0. \tag{48}$$

We also get that

$$\Im \psi(y, x, \eta) \geq |x - y|^2/C. \tag{49}$$

The latter can be obtained by setting $h = y - x$ and expanding in powers of h. The stationary complex phase method [Sj], see Theorem 2.8 there and the remark after it, gives

$$\int_{|x-y|\le\delta/C_0} e^{\mathrm{i}\lambda\psi(x,\alpha)} f_{ij}(x)B^{ij}(x,\alpha;\lambda)\,\mathrm{d}x = O\big(\lambda^{n/2}(C_3N/\lambda)^N + Ne^{-\lambda/C}\big), \quad \forall N, \tag{50}$$

where $\alpha = (y,\eta)$, and B is a classical analytic symbol [Sj] with principal part equal to $\tilde b\otimes\tilde b$, up to an elliptic factor. The l.h.s. above is independent of N, and choosing N so that $N\le\lambda/(C_3e)\le N+1$ to conclude that the r.h.s. above is $O(e^{-\lambda/C})$.

In preparation for applying the characterization of an analytic wave front set through a generalized FBI transform [Sj], define the transform

$$\alpha\longmapsto\beta = (\alpha_x, \nabla_{\alpha_x}\varphi(\alpha)),$$

where, following [Sj], $\alpha = (\alpha_x,\alpha_\xi)$. It is a diffeomorphism from neigh$(0,\xi^0)$ to its image, and denote the inverse one by $\alpha(\beta)$. Note that this map and its inverse preserve the first (n-dimensional) component and change only the second one. This is equivalent to setting $\alpha = (y,\eta)$, $\beta = (y,\zeta)$, where $\zeta = \varphi_y(y,\eta)$. Note that $\zeta = \eta + O(\delta)$, and at $y=0$, we have $\zeta=\eta$.

Plug $\alpha = \alpha(\beta)$ in (50) to get

$$\int e^{\mathrm{i}\lambda\psi(x,\beta)} f_{ij}(x)B^{ij}(x,\beta;\lambda)\,\mathrm{d}x = O\big(e^{-\lambda/C}\big), \tag{51}$$

where ψ, B are (different) functions having the same properties as above. Then

$$\psi_y(x,x,\zeta) = -\zeta,\quad \psi_x(x,x,\zeta)=\zeta,\quad \psi(x,x,\zeta)=0. \tag{52}$$

The symbols in (51) satisfy

$$\sigma_p(B)(0,0,\zeta)\equiv\theta(\zeta)\otimes\theta(\zeta)\quad\text{up to an elliptic factor}, \tag{53}$$

and in particular, $\sigma_p(B)(0,0,\xi^0)\equiv e_n\otimes e_n$, where σ_p stands for the principal symbol.

Let $\theta_1 = e_n$, $\theta_2,\dots,\theta_N$ be $N = n(n-1)/2$ unit vectors at $x_0 = 0$, normal to $\xi^0 = e^{n-1}$ such that any constant symmetric 2-tensor f such that $f_i^{n-1} = 0$, $\forall i$ (i.e., $f_i^j\xi_j^0 = 0$) is uniquely determined by $f_{ij}\theta^i\theta^j$, $\theta=\theta_p$, $p=1,\dots,N$. Existence of such vectors is easy to establish, as mentioned above, and one can also see that such a set exists in any open set in $(\xi^0)^\perp$. We can therefore assume that θ_p belong to a small enough neighborhood of $\theta_1 = e_n$ such that the geodesics $[-l^-,l^+]\ni t\mapsto\gamma_{0,\theta_p}(t)$ through $x_0=0$ are all simple. Then we can rotate a bit the coordinate system such that $\xi^0 = e^{n-1}$ again, and $\theta_p = e_n$, and repeat the construction above. This gives us N phase functions $\psi_{(p)}$, and as many symbols $B_{(p)}$ in (51) such that (52) holds for all of them, i.e., in the coordinate system related to $\theta_1 = e_n$, we have

$$\int e^{\mathrm{i}\lambda\psi_{(p)}(x,\beta)} f_{ij}(x)B_{(p)}^{ij}(x,\beta;\lambda)\,\mathrm{d}x = O\big(e^{-\lambda/C}\big),\quad p=1,\dots,N, \tag{54}$$

and by (53),

$$\sigma_p(B_{(p)})(0,0,\xi^0)\equiv\theta_p\otimes\theta_p,\quad p=1,\dots,N,\quad\text{up to elliptic factors.} \tag{55}$$

Recall that $\delta f = 0$ near $x_0 = 0$. Let $\chi_0 = \chi_0(x)$ be a smooth cutoff close enough to $x=0$, equal to 1 in neigh(0). Integrate $\frac{1}{\lambda}\exp\big(\mathrm{i}\lambda\psi_{(1)}(x,\beta)\big)\chi_0\delta f = 0$ w.r.t. x, and by (49), after an integration by parts, we get

$$\int e^{\mathrm{i}\lambda\psi_{(1)}(x,\beta)}\chi_0(x) f_{ij}(x)C^j(x,\beta;\lambda)\,\mathrm{d}x = O\big(e^{-\lambda/C}\big),\quad i=1,\dots,n, \tag{56}$$

for $\beta_x = y$ small enough, where $\sigma_p(C^j)(0,0,\xi^0) = (\xi^0)^j$.

Now, the system of $N+n=(n+1)n/2$ equations (54), (56) can be viewed as a tensor-valued operator applied to the tensor f. Its symbol, an elliptic factor at $(0,0,\xi^0)$, has "rows" given by $\theta_p^i\theta_p^j$, $p=1,\dots,N$; and $\delta_k^i(\xi^0)^j$, $k=1,\dots,n$. It is easy to see that it is elliptic; indeed, the latter is equivalent to the statement that if for some (constant) symmetric 2-tensor f, in Euclidean geometry (because $g_{ij}(0)=\delta_{ij}$), we have $f_{ij}\theta_p^i\theta_p^j=0$, $p=1,\dots,N$; and $f_i^{n-1}=0$, $i=1,\dots,n$, then $f=0$. This however follows from the way we chose θ_p. Therefore, (35) is a consequence of (54), (56), see [Sj, Definition 6.1]. Note that in [Sj], it is required that f must be replaced by $\bar f$ in (54), (56). If f is complex-valued, we could use the fact that $I(\Re f)(\gamma)=0$, and $I(\Im f)(\gamma)=0$ for γ near γ_0 and then work with real-valued f's only.

Since the phase functions in (54) depend on p, we need to explain why the characterization of the analytic wave front sets in [Sj] can be generalized to this vector-valued case. The needed modifications are as follows. We define $h^{ij}_{(p)}(x,\beta;\lambda)=B^{ij}_{(p)}$, $p=1,\dots,N$; and $h^{ij}_{(N+k)}(x,\beta;\lambda)=C^j\delta^i_k$, $k=1,\dots,n$. Then $\{h^{ij}_{(p)}\}$, $p=1,\dots,N+n$, is an elliptic symbol near $(0,0,\xi^0)$. In the proof of [Sj, Prop. 6.2], under the conditions (49), (52), the operator Q given by

$$[Qf]_p(x,\lambda)=\iint e^{\mathrm{i}\lambda(\psi_{(p)}(x,\beta)-\overline{\psi_{(p)}(y,\beta)})}f_{ij}(y,\lambda)h^{ij}_{(p)}(x,\beta;\lambda)\,\mathrm{d}y\,\mathrm{d}\beta$$

is a ΨDO in the complex domain with an elliptic matrix-valued symbol, where we view f and Qf as vectors in $\mathbf{C}^{N+n}$. Therefore, it admits a parametrix in H_{ψ,x_0} with a suitable ψ (see [Sj]). Hence, one can find an analytic classical matrix-valued symbol $r(x,\beta,\lambda)$ defined near $(0,0,\xi^0)$, such that for any constant symmetric f we have

$$\left[Q\left(r(\cdot,\beta,\lambda)e^{\mathrm{i}\lambda\psi_{(1)}}f\right)\right]_p=e^{\mathrm{i}\lambda\psi_{(1)}}f,\quad\forall p.$$

The rest of the proof is identical to that of [Sj, Prop. 6.2] and allows us to show that (51) is preserved with a different choice of the phase functions satisfying (49), (52), and elliptic amplitudes; in particular,

$$\int e^{\mathrm{i}\lambda\psi_{(1)}(x,\beta)}\chi_2(x)f_{ij}(x)\,\mathrm{d}x=O\big(e^{-\lambda/C}\big),\quad\forall i,j$$

for $\beta\in\operatorname{neigh}(0,\xi^0)$ and for some standard cut-off χ_2 near $x=0$. This proves (35), see [Sj, Definition 6.1].

This concludes the proof of Proposition 2. Notice that the proof works in the sane way, if f is a distribution valued tensor field, supported in M. ☐

Lemma 6. *Under the assumptions of Theorem 1, let f be such that $I_\Gamma f=0$. Then $f^s\in\mathcal{A}(M)$.*

Proof. Proposition 2, combined with the completeness of Γ, imply that f^s is analytic in the interior of M. To prove analyticity up to the boundary, we do the following.

We can assume that $M_1\setminus M$ is defined by $-\varepsilon_1\le x^n\le 0$, where x^n is a boundary normal coordinate. Define the manifold $M_{1/2}\supset M$ by $x^n\ge-\varepsilon_1/2$, more precisely, $M_{1/2}=M\cup\{-\varepsilon_1/2\le x^n\le 0\}\subset M_1$.

We will show first that $f^s_{M_{1/2}}\in\mathcal{A}(M_{1/2})$. Let us first notice, that in $M_{1/2}\setminus M$, $f^s_{M_{1/2}}=-dv_{M_{1/2}}$, where $v_{M_{1/2}}$ satisfies $\Delta^s v_{M_{1/2}}=0$ in $M_{1/2}\setminus M$, $v|_{\partial M_{1/2}}=0$. Therefore, $v_{M_{1/2}}$ is analytic up to $\partial M_{1/2}$ in $M_{1/2}\setminus M$, see [MN, SU4]. Therefore, we only need to show that $f^s_{M_{1/2}}$ is analytic in some neighborhood of M. This however follows from Proposition 2, applied to $M_{1/2}$. Note that if $\varepsilon_1\ll 1$, simple geodesics through some $x\in M$ would have endpoints outside $M_{1/2}$ as well, and by a compactness argument, we need finitely many such geodesics to show that Proposition 2 implies that $f^s_{M_{1/2}}$ is analytic in, say, $M_{1/4}$, where the latter is defined similarly to $M_{1/2}$ by $x^n\ge-\varepsilon_1/4$.

To compare $f^s_{M_{1/2}}$ and $f^s=f^s_M$, see also [SU3, SU4], write $f^s_{M_{1/2}}=f-dv_{M_{1/2}}$ in $M_{1/2}$, and $f^s_M=f-dv_M$ in M. Then $dv_{M_{1/2}}=-f^s_{M_{1/2}}$ in $M_{1/2}\setminus M$, and is therefore analytic there, up to ∂M. Given $x\in\partial M$, integrate $\langle dv_{M_{1/2}},\dot\gamma^2\rangle$ along geodesics in $M_{1/2}\setminus M$, close to ones normal to the boundary, with initial point x and endpoints on $\partial M_{1/2}$. Then we get that $v_{M_{1/2}}|_{\partial M} \in \mathcal{A}(\partial M)$. Note that $v_{M_{1/2}} \in H^1$ near ∂M, and taking the trace on ∂M is well defined, and moreover, if x^n is a boundary normal coordinate, then $\text{neigh}(0) \ni x^n \mapsto v_{M_{1/2}}(\cdot, x^n)$ is continuous. Now,

$$f_M^s = f - dv_M = f_{M_{1/2}}^s + dw \quad \text{in } M, \quad \text{where } w = v_{M_{1/2}} - v_M. \tag{57}$$

The vector field w solves

$$\Delta^s w = 0, \quad w|_{\partial M} = v_{M_{1/2}}|_{\partial M} \in \mathcal{A}(\partial M).$$

Therefore, $w \in \mathcal{A}(M)$, and by (57), $f_M^s \in \mathcal{A}(M)$.

This completes the proof of Lemma 6. □

Proof of Theorem 1. Let $I_\Gamma f = 0$. We can assume first that $f = f^s$, and then $f \in \mathcal{A}(M)$ by Lemma 6. By Lemma 4, there exists $h \in \mathcal{S}^{-1}\mathcal{S} f$ such that $\partial^\alpha h = 0$ on ∂M for all α. The tensor field h satisfies (29), i.e., $h_{ni} = 0$, $\forall i$, in boundary normal coordinates, which is achieved by setting $h = f - dv_0$, where v_0 solves (31) near ∂M. Then v_0, and therefore, h is analytic for small $x^n \geq 0$, up to $x^n = 0$. Lemma 4 then implies that $h = 0$ in $\text{neigh}(\partial M)$. So we get that

$$f = dv_0, \quad 0 \leq x^n < \varepsilon_0, \quad \text{with } v_0|_{x^n=0} = 0, \tag{58}$$

where x^n is a global normal coordinate, and $0 < \varepsilon_0 \ll 1$. Note that the solution v_0 to (58) (if exists, and in this case we know it does) is unique, as can be easily seen by integrating $\langle f, \dot\gamma^2\rangle$ along paths close to normal ones to ∂M and using (12).

We show next that v_0 admits an analytic continuation from a neighborhood of any $x_1 \in \partial M$ along any path in M.

Fix $x \in M$. Let $c(t)$, $0 \leq t \leq 1$ be a path in M such that $c(0) = x_0 \in \partial M$ and $c(1) = x$. Given $\varepsilon > 0$, one can find a polygon $x_0 x_1 \dots x_k x$ consisting of geodesic segments of length not exceeding ε, that is close enough and therefore homotopic to c. One can also assume that the first one is transversal to ∂M, and if $x \in \partial M$, the last one is transversal to ∂M as well; and all other points of the polygon are in M^{int}. We choose $\varepsilon \ll 1$ so that there are no conjugate points on each geodesic segment above. We also assume that $\varepsilon \leq \varepsilon_0$. Then $f = dv$ near x_0x_1 with $v = v_0$ by (58). As in the second paragraph of Section 2.1, one can choose semigeodesic coordinates (x', x^n) near x_1x_2, and a small enough hypersurface H_1 through x_1 given locally by $x^n = 0$. As in Lemma 4, one can find an analytic 1-form v_1 defined near x_1x_2, so that $(f - dv_1)_{in} = 0$, $v_1|_{x^n=0} = v_0(x', 0)$. Close enough to x_1, we have $v_1 = v_0$ because v_0 is also a solution, and the solution is unique, see also (32). Since v_1 is analytic, we get that it is an analytic extension of v_0 along x_1x_2. Since f and v_1 are both analytic in $\text{neigh}(x_1x_2)$, and $f = dv_1$ near x_1, this is also true in $\text{neigh}(x_1x_2)$. So we extended v_0 along $x_0x_1x_2$, let us call this extension v. Then we do the same thing near x_2x_3, etc., until we reach $\text{neigh}(x)$, and then $f = dv$ there.

This defines v in $\text{neigh}(x)$, where $x \in M$ was chosen arbitrary. It remains to show that this definition is independent of the choice of the path. Choose another path that connects some $y_1 \in \partial M$ and x. Combine them both to get a path that connects $x_1 \in \partial M$ and $y_1 \in \partial M$. It suffices to prove that the analytic continuation of v_0 from x_1 to y_1 equals v_0 again. Let $c_1 \cup \gamma_1 \cup c_2 \cup \gamma_2 \cup \dots \cup \gamma_k \cup c_{k+1}$ be the polygon homotopic to the path above. Analytic continuation along c_1 coincides with v_0 again by (58). Next, let p_1, p_2 be the initial and the endpoint of γ_1, respectively, where p_1 is also the endpoint of c_1. We continue analytically v_0 from $\text{neigh}(p_1)$ to $\text{neigh}(p_2)$ along γ_1, let us call this continuation v. By what we showed above, $f = dv$ near γ_1. Since $If(\gamma_1) = 0$, and $v(p_1) = 0$, we get by (12), that $\langle v(p_2), \dot\gamma_1(l)\rangle = 0$ as well, where l is such $\gamma_1(l) = p_2$. Using the assumption that γ_1 is transversal to ∂M at both ends, one can perturb the tangent vector $\dot\gamma_1(l)$ and this will define a new geodesic through p_2 that hits ∂M transversely again near p_1, where $v = v_0 = 0$. Since Γ is open, integral of f over this geodesic vanishes again, therefore $\langle v(p_2), \xi_2\rangle = 0$ for ξ_2 in an open set. Hence $v(p_2) = 0$. Choose $q_2 \in \partial M$ close enough to p_2, and η_2 close enough to ξ_2 (in a fixed chart). Then the geodesic through (q_2, η_2) will hit ∂M transversally close to

p_1, and we can repeat the same arguments. We therefore showed that $v = 0$ on ∂M near p_2. On the other hand, v_0 has the same property. Since $f = dv = dv_0$ there, by the remark after (58), we get that $v = v_0$ near p_2. We repeat this along all the legs of the polygon until we get that the analytic continuation v of v_0 along the polygon, from x_1 to y_1, equals v_0 again.

As a consequence of this, we get that $f = dv$ in M with $v = 0$ on ∂M. Since $f = f^s$, this implies $f = 0$.

This completes the proof of Theorem 1. □

5. Proof of Theorems 2 and 3

Proof of Theorem 2. Theorem 2(b), that also implies (a), is a consequence of Proposition 1, as shown in [SU4], see the proof of Theorem 2 and Proposition 4 there. Part (a) only follows more directly from [Ta1, Prop. V.3.1] and its generalization, see [SU3, Thm 2]. □

Proof of Theorem 3. First, note that for any analytic metric in $\mathcal{G}$, I_{Γ_g} is s-injective by Theorem 1. We build $\mathcal{G}_s$ as a small enough neighborhood of the analytic metrics in $\mathcal{G}$. Then $\mathcal{G}_s$ is dense in $\mathcal{G}$ (in the $C^k(M_1)$ topology) since it includes the analytic metrics. To complete the definition of $\mathcal{G}_s$, fix an analytic $g_0 \in \mathcal{G}$. By Lemma 1, one can find $\mathcal{H}' \Subset \mathcal{H}$ related to $g = g_0$ and Γ_g, satisfying the assumptions of Theorem 2, and they have the properties required for g close enough to g_0.

Let α be as in Theorem 2 with $\alpha = 1$ on $\mathcal{H}'$. Then, by Theorem 2, $I_{\alpha,g}$ is s-injective for g close enough to g_0 in $C^k(M_1)$. By Lemma 2, for any such g, I_{Γ^α} is s-injective, where $\Gamma^\alpha = \Gamma(\mathcal{H}^\alpha)$, $\mathcal{H}^\alpha = \operatorname{supp}\alpha$. If g is close enough to g_0, $\Gamma^\alpha \subset \Gamma_g$ because when $g = g_0$, $\Gamma^\alpha \subset \Gamma(\mathcal{H}) \Subset \Gamma_{g_0}$, and Γ_g depends continuously on g in the sense described before the formulation of Theorem 3. Those arguments show that there is a neighborhood of each analytic $g_0 \in \mathcal{G}$ with an s-injective I_{Γ_g}. Therefore, one can choose an open dense subset $\mathcal{G}_s$ of $\mathcal{G}$ with the same property. □

Proof of Corollary 1. It is enough to notice that the set of all simple geodesics related to g depends continuously on g in the sense of Theorem 3. Then the proof follows from the paragraph above. □

6. The geodesic X-ray transform of functions and 1-forms/vector fields

If f is a vector field on M, that we identify with an 1-form, then its X-ray transform is defined quite similarly to (1) by

$$I_\Gamma f(\gamma) = \int_0^{l_\gamma} \langle f(\gamma(t)), \dot\gamma(t)\rangle \, \mathrm{d}t, \quad \gamma \in \Gamma. \tag{59}$$

If f is a function on M, then we set

$$I_\Gamma f(\gamma) = \int_0^{l_\gamma} f(\gamma(t)) \, \mathrm{d}t, \quad \gamma \in \Gamma. \tag{60}$$

The latter case is a partial case of the X-ray transform of 2-tensors; indeed, if $f = \alpha g$, where f is a 2-tensor, α is a function, and g is the metric, then $I_\Gamma f = I_\Gamma \alpha$, where in the l.h.s., I_Γ is as in (1), and on the right, I_Γ is as in (60). The proofs for the X-ray transform of functions are simpler, however, and in particular, there is no loss of derivatives in the estimate (7), as in [SU3]. This is also true for the X-ray transform of vector fields and the proofs are more transparent than those for tensors of order 2 (or higher). Without going into details (see [SU3] for the case of simple manifolds), we note that the main theorems in the Introduction remain true. In case of 1-forms, estimate (7) can be improved to

$$\|f^s\|_{L^2(M)}/C \le \|N_\alpha f\|_{H^1(M_1)} \le C\|f^s\|_{L^2(M)}, \tag{61}$$

while in case of functions, we have

$$\|f\|_{L^2(M)}/C \le \|N_\alpha f\|_{H^1(M_1)} \le C\|f\|_{L^2(M)}. \tag{62}$$

If $(M, \partial M)$ is simple, then the full X-ray transform of functions and 1-forms (over all geodesics) is injective, respectively s-injective, see [Mu2, MuR, BG, AR].

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