Microlocal Analysis of Thermoacoustic (or Multiwave) Tomography, II

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Stability and partial data

Mini Course, Fields Institute, 2012

Stability

Stability is related to propagation of singularities. As a general principle, it is necessary (and sufficient) to be able to "detect" all singularities, i.e., the WF(f). Since $u_t = 0$ for $t = 0$, each singularity (x, ξ) splits into two parts with equal energy and they start to travel in positive (ξ) and negative $(-\xi)$ direction. We need to detect one of them, at least.

Let $T_1 \leq \infty$ be the length of the longest (maximal) geodesic through $\bar{\Omega}$. Then the "stability time" is $T_1/2$. One can show that $T_0 \leq T_1/2$. If $T_1 = \infty$, we say that the speed is **trapping** in Ω.

Theorem 1

$T > T_1/2 \implies$ *stability.*

$T < T_1/2 \implies$ *no stability, in any Sobolev norms.*

The second part follows from the fact that Λ is a smoothing FIO on an open conic subset of $T^*\Omega$ (to be discussed later). In particular, if the speed is trapping, there is no stability, whatever T.

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Comparison between the uniqueness and the stability conditions

For uniqueness:

For any $x \in \mathcal{K}$, we want to have some unit speed path from x reaching the observation part $\Gamma \subset \partial\Omega$ for time $0 \le t \le T$.

For stability:

For any $x \in \mathcal{K}$ and for any $\xi \neq 0$ we want the unit speed geodesic $\gamma_{x,\xi}$ to reach the observation part $\Gamma \subset \partial\Omega$ for time $|t| \le T$.

Examples:

- $c = 1$, $\Omega = [-1, 1]^2$. Then $T_0 = 1$, $T_1/2 = \sqrt{2}$.
- $c = 1$, $\Omega = \{|x| < 1\}$. Then $T_0 = T_1/2 = 1$.

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Why is stability related to our ability to detect all singularities? This will be made more precise below. Consider a toy problem now. Let us say that we solve $Pg = h$, h known, and P is a ΨDO of order 0 (assume a compact manifold for simplicity). If P is elliptic, then there is a parametrix Q (of order 0 as well) so that $QP = I + K$, where K is smoothing, and in particular, compact. Then

$$\|f\| \le C(\|QPf\| + \|Kf\|) \le C'(\|Pf\| + \|Kf\|).$$

Almost there but we have the K term.

If we know in addition that P is injective, there is a beautiful functional analysis argument saying that the estimate above holds without the K term but with a different constant

$$\|f\| \le C''\|Pf\|.$$

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How is this connected to detection of all singularities? To detect all singularities, as singularities of the data, means that P must be hypoelliptic. We just assumed that it was elliptic. So it was a good toy problem.

What if P cannot detect all singularities? Assume that it is of order $-\infty$ in some open cone. In other words, its essential support "has a gap". Choose f with $\mathrm{WF}(f)$ exactly in that "gap". Then $Pf \in C^\infty$, while f may be as singular as we like. The estimate

$$\|f\|_{H^{s_1}} \le C\|Pf\|_{H^{s_2}}$$

cannot hold because that estimate implies $f \in H^{s_1}$ if $Pf \in H^{s_2}$. But we just saw that we can choose f outside of any Sobolev space (with proper wave front set) and then $Pf \in C^\infty$.

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Reconstruction. Modified time reversal

Time reversal, harmonic extension

Given h (that eventually will be replaced by Λf), solve

$$\begin{cases} (\partial_t^2 - c^2\Delta)v &= 0 \quad \text{in } (0,T)\times\Omega, \\ v|_{[0,T]\times\partial\Omega} &= h, \\ v|_{t=T} &= \phi, \\ \partial_t v|_{t=T} &= 0, \end{cases} \tag{1}$$

where ϕ is the harmonic extension of $h(T,\cdot)$:

$$\Delta\phi = 0, \quad \phi|_{\partial\Omega} = h(T,\cdot).$$

Note that the initial data at $t = T$ satisfies compatibility conditions of first order (no jump at $\{T\}\times\partial\Omega$). Then we define the following pseudo-inverse

$$Ah := v(0,\cdot) \quad \text{in } \bar{\Omega}.$$

Why would we do that? We are missing the Cauchy data at $t = T$; the only thing we know there is its value on $\partial\Omega$. The time reversal methods just replace it by zero. We replace it by that data (namely, by $(\phi, 0)$), having the same trace on the boundary, that minimizes the energy.

Recall: Given $U \subset \mathbf{R}^n$, the energy in U is given by

$$E_U(t,u) = \int_U \left(|\nabla u|^2 + c^{-2}|u_t|^2\right) \mathrm{d}x.$$

We define the space $H_D(U)$ to be the completion of $C_0^\infty(U)$ under the Dirichlet norm

$$\|f\|_{H_D}^2 = \int_U |\nabla u|^2 \,\mathrm{d}x.$$

The norms in $H_D(\Omega)$ and $H^1(\Omega)$ are equivalent, so

$$H_D(\Omega) \cong H_0^1(\Omega).$$

The energy norm of a pair $[f, g]$ is given by

$$\|[f,g]\|_{\mathcal{H}(\Omega)}^2 = \|f\|_{H_D(\Omega)}^2 + \|g\|_{L^2(\Omega, c^{-2}\mathrm{d}x)}^2$$

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Consider the "error operator" K defined by $A\Lambda = I - K$. It is straightforward to see that

$$Kf = \text{first component of: } U_{\Omega,D}(-T)\Pi_\Omega U_{\mathbf{R}^n}(T)[f,0],$$

where

- $U_{\mathbf{R}^n}(t)$ is the dynamics in the whole $\mathbf{R}^n$,
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K is a contraction for $T > T_1/2$!

- We saw that $\|Kf\| \le \|f\|$. By unique continuation, $\|Kf\| < \|f\|$, $f \neq 0$.
- Assume for a moment that $T > T_1$ (twice the stability time). Then $u \in C^\infty$ in Ω because all singularities have left. Hence, K is compact.
- K^*K is also compact (and self-adjoint), with spectral radius ≤ 1. It cannot have one as an eigenvalue by the inequality above. Therefore, the largest eigenvalue is < 1.
- Then $\|Kf\|^2 = (K^*Kf, f) < \|f\|^2$. Therefore, K is a contraction.

If $T > T_1/2$, K is a sum of an operator with norm $\le 1/2 + \varepsilon$ and a compact one. Its essential spectrum is not affected by the compact part, so in $(1/2, \infty)$, its spectrum is discrete. The same kind of arguments show that K is a contraction as well.

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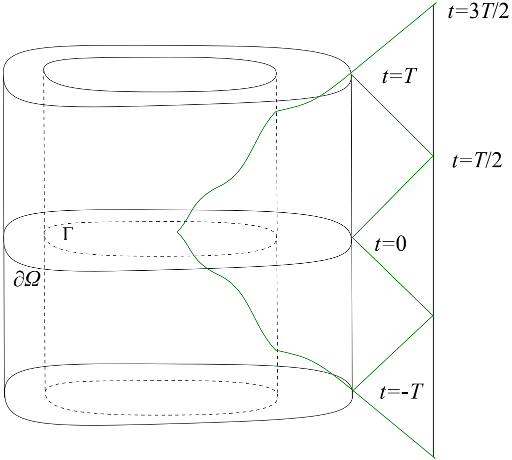
A picture explaining $\|Kf\| < \|f\|$.



Reconstruction, whole boundary

Theorem 2

Let $T > T_1/2$. Then $A\Lambda = I - K$, where $\|K\|_{\mathcal{L}(H_D(\Omega))} < 1$. In particular, $I - K$ is invertible on $H_D(\Omega)$, and the inverse thermoacoustic problem has an explicit solution of the form

$$f = \sum_{m=0}^{\infty} K^m A h, \quad h := \Lambda f.$$

If $T > T_1$, then K is compact.

We have the following estimate on $\|K\|$:

Corollary 3

$$\|Kf\|_{H_D(\Omega)} \le \left(\frac{E_\Omega(u,T)}{E_\Omega(u,0)}\right)^{1/2} \|f\|_{H_D(\Omega)}, \quad \forall f \in H_{D(\Omega)},\ f \neq 0,$$

where u is the solution with Cauchy data $(f, 0)$.

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$$\|Kf\|_{H_D(\Omega)} \le \left(\frac{E_\Omega(u,T)}{E_\Omega(u,0)} \right)^{1/2} \|f\|_{H_D(\Omega)}, \quad \forall f \in H_{D(\Omega)},\ f \neq 0,$$

where u is the solution with Cauchy data $(f, 0)$.

Summary: Dependence on T

(i) $T < T_0 \implies$ **no uniqueness**
Λf does not recover uniquely f. $\|K\| = 1$.

(ii) $T_0 < T < T_1/2 \implies$ **uniqueness, no stability**
Uniqueness but not stability (there are invisible singularities). We do not know if the Neumann series converges. $\|Kf\| < \|f\|$ but $\|K\| = 1$.

(iii) $T_1/2 < T < T_1 \implies$ **stability and explicit reconstruction**
This assumes that c is non-trapping. The Neumann series converges exponentially but maybe not as fast as in the next case (K is contraction but not compact). There is stability (we detect all singularities but some with 1/2 amplitude). $\|K\| < 1$.

(iv) $T_1 < T \implies$ **stability and explicit reconstruction**
The Neumann series converges exponentially, K is contraction and compact (all singularities have left $\bar{\Omega}$ by time $t = T$). There is stability. $\|K\| < 1$.

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Example 1: Nontrapping speed

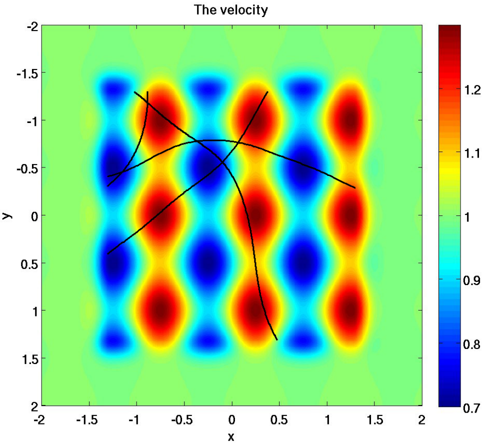

Figure: The speed, $T_0 \approx 1.15$. $\Omega = [-1.28, 1.28]^2$, computations are done in $[-2, 2]^2$

Example 1: Nontrapping speed

Figure: Original

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Figure: Neumann Series reconstruction, $T = 4T_0 = 4.6$, error $= 3.45\%$

Example 1: Nontrapping speed

Figure: Time Reversal, $T = 4T_0 = 4.6$, error $= 23\%$

Example 2: Trapping speed

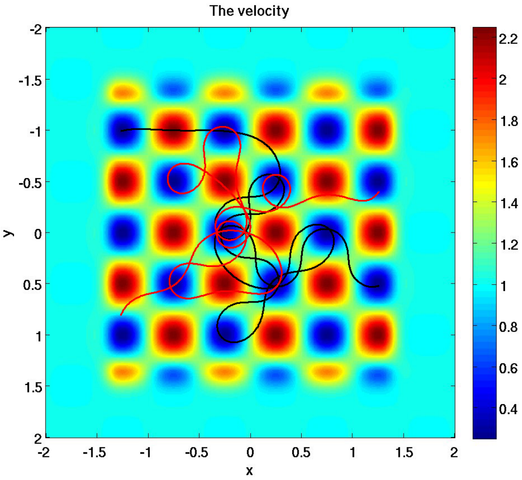

Figure: The speed, $T_0 \approx 1.18$

Example 2: Trapping speed

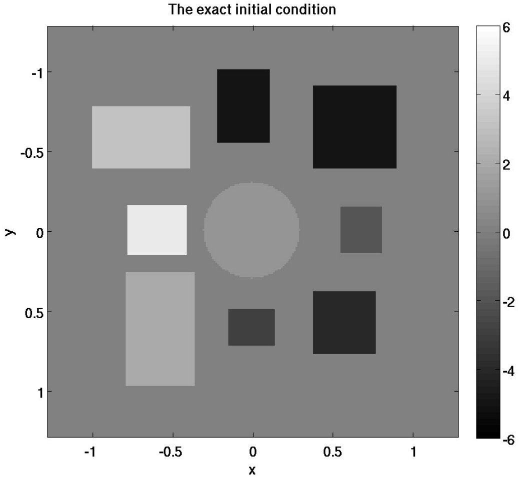

Figure: The original

Example 2: Trapping speed

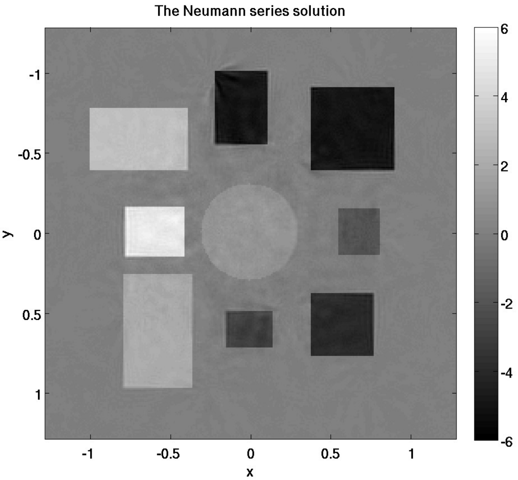

Figure: Neumann Series reconstruction, 10 steps, $T = 4T_0 = 4.7$, error $= 8.75\%$

Example 2: Trapping speed

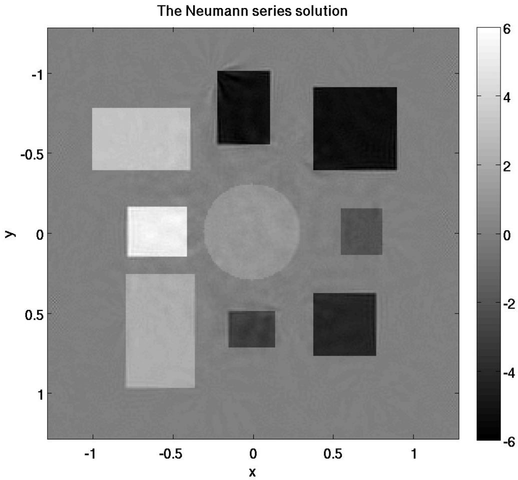

Figure: Neumann Series reconstruction, 10% noise, 15 steps, $T = 4T_0 = 4.7$, error $= 8.72\%$

Example 2: Trapping speed

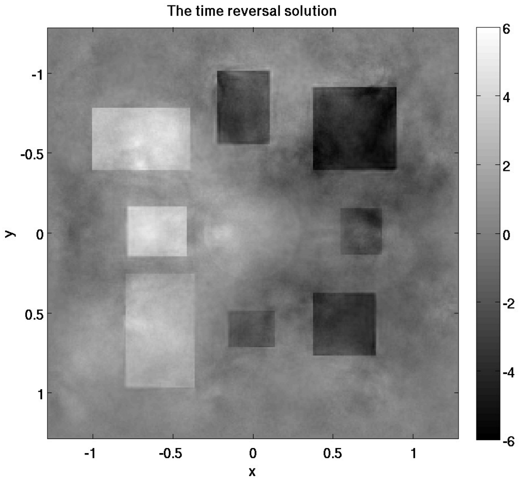

Figure: Time Reversal, $T = 4T_0 = 4.7$, error $= 55\%$

Example 2: Trapping speed

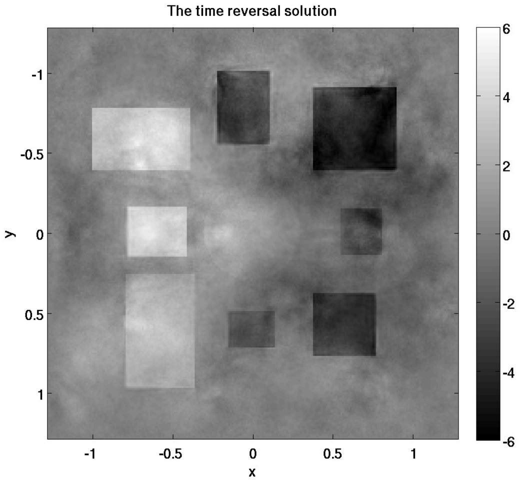

Figure: Time Reversal with 10% noise, $T = 4T_0 = 4.7$, error $= 54\%$

Example 3: The same trapping speed, Barbara

Figure: Original

Example 3: The same trapping speed, Barbara

Figure: Neumann series, $T = 4T_0 = 4.7$, error $= 7.5\%$, 10 steps

Example 3: The same trapping speed, Barbara

Figure: Time Reversal, $T = 4T_0 = 4.7$, error $= 27.7\%$

Example 3: The same trapping speed, Barbara

Figure: Time Reversal, $T = 12T_0 = 14.1$, error $= 99.67\%$

Example 4: a radial trapping speed

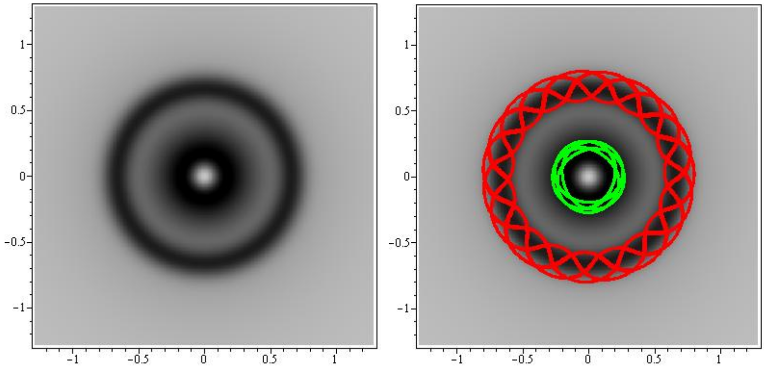

Figure: A trapping speed. Darker regions represent a slower speed. The circles of radii approximately 0.23 and 0.67 are stable periodic geodesics. Left: the speed. Right: the speed with two trapped geodesics

Example 4: a radial trapping speed

Figure: Original, lower resolution than before

Example 4: a radial trapping speed

Figure: Neumann series, 10 steps, $T = 8T_0 = 8.7$, error $= 9.7\%$

Example 4: a radial trapping speed

Figure: Time Reversal, $T = 8T_0 = 8.7$, error $= 21.7\%$

Measurements on a part of the boundary

Let $\Gamma \subset \partial\Omega$ be a relatively open subset of $\partial\Omega$.

Assume now that the observations are made on $[0, T] \times \Gamma$ only, i.e., we assume we are given

$$\Lambda f|_{[0,T]\times\Gamma}.$$

We consider f's with

$$\operatorname{supp} f \subset \mathcal{K},$$

where $\mathcal{K} \subset \Omega$ is a fixed compact.

We analyzed the uniqueness already. It holds if $T > T_0(\Gamma)$.

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Stability

Heuristic arguments for stability: To be able to recover f from Λf on $[0, T] \times \Gamma$ *in a stable way*, we need to recover all singularities. In other words, we should require that

$$\forall (x,\xi) \in \mathcal{K} \times S^{n-1}, \text{ the geodesic through it reaches } \Gamma \text{ at time } |t| < T.$$

This defines a critical time $T_1(\Gamma, \mathcal{K})$ that is a sharp time for stability. We show next that this is an "if and only if" condition (up to replacing an open set by a closed one) for stability. Actually, we show a bit more.

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Proposition 1

$\Lambda = \Lambda_+ + \Lambda_-$, where $\Lambda_\pm$ are elliptic Fourier Integral Operators of zeroth order with canonical relations given by the graphs of the maps

$$(y,\xi) \mapsto \left(\tau_\pm(y,\xi), \gamma_{y,\pm\xi}(\tau_\pm(y,\xi)), -|\xi|, \dot{\gamma}'_{y,\pm\xi}(\tau_\pm(y,\xi))\right),$$

where $|\xi|$ is the norm in the metric $c^{-2}\mathrm{d}x^2$, and the prime in $\dot{\gamma}'$ stands for the tangential projection of $\dot{\gamma}$ on $T\partial\Omega$.

Corollary 4

If the stability condition is not satisfied on $[0,T]\times\bar{\Gamma}$, then there is no stability, in any Sobolev norms.

Here, $\tau_\pm(x,\xi)$ is the time needed to reach $\partial\Omega$ starting from $(x,\pm\xi)$.

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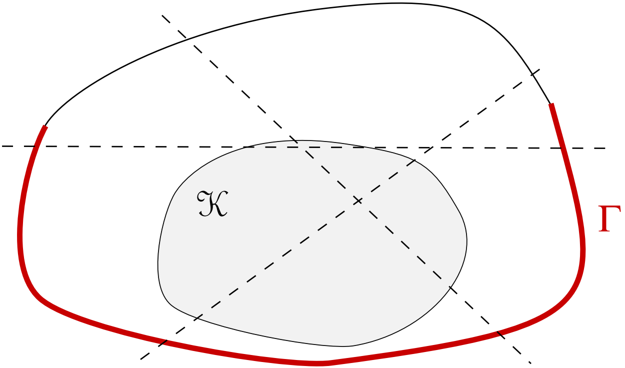
- Every geodesic through $\mathcal{K}$ intersects Γ.
- $\forall (x,\xi) \in \mathcal{K} \times S^{n-1}$, the travel time along the geodesic through it satisfies $|t| < T$.

Let us call the least such time $T_1/2$, then $T > T_1/2$ as before.

In contrast, any small open Γ suffices for uniqueness.



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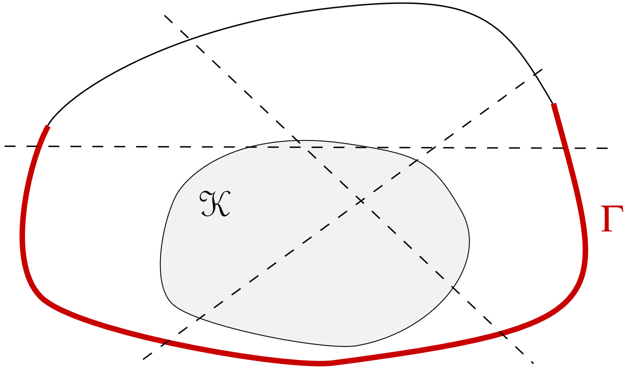

What is an FIO (with a canonical relation a graph)?

An operator that can be written in the form (locally)

$$Af = \int e^{\mathrm{i}\phi(x,\xi)} a(x,\xi)\hat{f}(\xi)\,\mathrm{d}\xi$$

with an amplitude in S^m is an example of a Fourier Integral Operator (FIO). Here ϕ is homogeneous in ξ of order 1 and $\mathrm{d}_x\phi \neq 0$ for $\xi \neq 0$. The geometric optics construction is of this type. If $\phi = x\cdot\xi$, we get a ΨDO.

To find $\mathrm{WF}(Af)$ near (x_0,ξ_0), multiply by $\chi \in C_0^\infty$, $\chi(x_0)\neq 0$, and take the Fourier transform. In other words, multiply by $\chi(x)e^{-\mathrm{i}x\cdot\eta}$, integrate in η and look for the large η behavior. This gives as an integral with a phase function

$$\Phi = \phi(x,\xi) - y\cdot\xi - x\cdot\eta.$$

Singularities can only be related to the critical points of $(x,\xi)\mapsto\Phi$.

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This shows that

$$\mathrm{WF}(Af) \subset \big\{(x,\eta);\ (\nabla_\xi \phi, \xi) \in \mathrm{WF}(f) \text{ for some } (x,\xi) \text{ and } \nabla_x \phi(x,\xi) = \eta\big\}.$$

In other words, $\mathrm{WF}(f)$ and $\mathrm{WF}(Af)$ are related by the canonical relation

$$(\nabla_\xi, \xi) \longmapsto (x, \nabla_x \phi).$$

It does not need to be defined on the whole $T^*\Omega$, not necessarily single valued. When $\phi = x\cdot\xi$, this relation is identity. When $\phi \approx x\cdot\xi$, it is close to it, and therefore it is locally a graph of a diffeomorphism. In the geometric optics construction, considering t as a parameter, we get two FIOs, and the canonical relations are just the geodesic flows on $T^*\mathbf{R}^n$ (identified with $T\mathbf{R}^n$) for $\pm t > 0$.

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Let A be the "modified time reversal" operator as before. Actually, ϕ will be 0 because of χ below. Let $\chi \in C_0^\infty([0,T]\times\partial\Omega)$ be a cutoff (supported where we have data).

Theorem 5

$A\chi\Lambda$ is a zero order classical ΨDO in some neighborhood of $\mathcal{K}$ with principal symbol

$$\frac{1}{2}\chi(\tau_+(x,\xi),\gamma_{x,\xi}(\tau_+(x,\xi))) + \frac{1}{2}\chi(\tau_-(x,\xi),\gamma_{x,\xi}(\tau_-(x,\xi))).$$

If $[0,T]\times\Gamma$ satisfies the stability condition, and $|\chi| > 1/C > 0$ there, then
(a) $A\chi\Lambda$ is elliptic,
(b) $A\chi\Lambda$ is a Fredholm operator on $H_D(\mathcal{K})$,
(c) there exists a constant $C>0$ so that

$$\|f\|_{H_D(\mathcal{K})} \le C\|\Lambda f\|_{H^1([0,T]\times\Gamma)}.$$

(b) follows by building a parametrix, and (c) follows from (b) and from the uniqueness result.

In particular, we get that for a fixed $T > T_1$, the classical Time Reversal is a parametrix (of infinite order, actually).

Proof of the main statement:
To construct a parametrix for $A\chi\Lambda f$, we apply again a geometric optic construction. It is enough to assume that $\chi\Lambda f$ has a wave front set in a conic neighborhood of some point $(t_0, y_0, \tau_0, \xi_0') \in [0, T] \times \partial\Omega$, using the notation above. For simplicity, assume that the eikonal equation is solvable for t in some neighborhood of $[0, T]$. Let $\tau_0 < 0$, for example. Then we look for a parametrix of the solution of the "back-propagated" wave equation with zero Cauchy data at $t = T$ and boundary data $\chi\Lambda_+ f$ in the form

$$v(t,x) = (2\pi)^{-n} \int e^{\mathrm{i}\phi_+(t,x,\xi)} b(x,\xi,t) \hat{f}(\xi)\, \mathrm{d}\xi.$$

Let (x_0, ξ_0) be the intersection point of the bicharacteristic issued from $(t_0, y_0, \tau_0, \xi_0')$ with $t = 0$.

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The choice of that parametrix is justified by the fact that all singularities of that solution must propagate along the geodesics close to γ_{x_0,ξ_0} in the opposite direction, as t decreases because there are no singularities for $t = T$. The critical observation is that the first transport equation for the principal term b_0 of b is a linear ODE along bicharacteristics, and starting from initial data $b_0 = \chi a_0$, where $a_0 = 1/2$, at time $t = 0$, we will get that $b_0(x,\xi)|_{t=0}$ is given by the value of $\chi/2$ at the exit point of $\gamma_{x,\xi}$ on $\partial\Omega$.

Reconstruction

One can constructively write the problem in the form

Reducing the problem to a Fredholm one

$$(I-K)f = BA\chi\Lambda f \quad \text{with the r.h.s. given,}$$

i.e., B is an explicit operator (a parametrix), where K is compact with 1 not an eigenvalue.

Constructing a parametrix without the ΨDO calculus.

Assume that the stability condition is satisfied in the interior of $\operatorname{supp}\chi$. Then

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where $I-K$ is an elliptic ΨDO with $0 \le \sigma_p(K) < 1$. Apply the formal Neumann series of $I-K$ (in Borel sense) to the l.h.s. to get

$$f \sim (I+K+K^2+\dots)A\chi\Lambda f \mod C^\infty.$$

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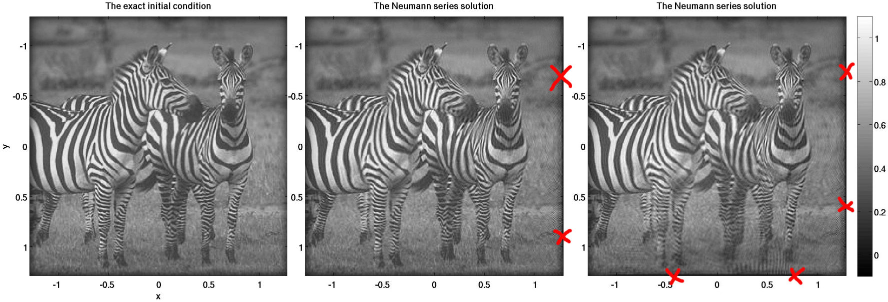
Examples: Non-trapping speed, 1 and 2 sides missing



original NS, 3 sides, error $= 7.99\%$ NS, 2 sides, error $= 12.2\%$

Figure: Partial data reconstruction, non-trapping speed, $T = 4T_0$.