Stochastic Processes and their Applications 175 (2024) 104412

Contents lists available at ScienceDirect

Stochastic Processes and their Applications

journal homepage: www.elsevier.com/locate/spa

Euler scheme for SDEs driven by fractional Brownian motions: Malliavin differentiability and uniform upper-bound estimates

Jorge A. León [a], Yanghui Liu [b,*], Samy Tindel [c]

[a] Departamento de Control Automático, Cinvestav-IPN, Mexico
[b] Baruch College, CUNY, NY, United States of America
[c] Department of Mathematics, Purdue University, West Lafayette, United States of America

ARTICLE INFO

Keywords:
Rough paths
Discrete sewing lemma
Fractional Brownian motion
Stochastic differential equations
Euler scheme
Asymptotic error distributions

ABSTRACT

The Malliavin differentiability of a SDE plays a crucial role in the study of density smoothness and ergodicity among others. For Gaussian driven SDEs the differentiability issue is solved essentially in Cass et al., (2013). In this paper, we consider the Malliavin differentiability for the Euler scheme of such SDEs. We will focus on SDEs driven by fractional Brownian motions (fBm), which is a very natural class of Gaussian processes. We derive a uniform (in the step size n) path-wise upper-bound estimate for the Euler scheme for stochastic differential equations driven by fBm with Hurst parameter $H > 1/3$ and its Malliavin derivatives.

1. Introduction

In this paper we are interested in the following stochastic differential equation driven by a d-dimensional fractional Brownian motion (fBm in the sequel) x with Hurst parameter $\frac{1}{3} < H < \frac{1}{2}$:

$$\begin{aligned} dy_t &= V_0(y_t)dt + V(y_t)dx_t\,, \quad t\in[0,T], \\ y_0 &= a \in \mathbb{R}^m. \end{aligned} \tag{1.1}$$

Throughout the paper we assume that the collection of vector fields $V_0 = (V_0^i, 1 \le i \le m) \in C_b^3(\mathbb{R}^m, \mathbb{R}^m)$ and $V = (V_j^i, 1 \le i \le m, 1 \le j \le d)$ all sit in the class $C_b^3(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m))$. Here C_b^3 denotes the space of functions whose derivatives up to the third order exist and are continuous and bounded. The existence and uniqueness of path-wise solution of Eq. (1.1) is guaranteed by the theory of rough paths; see e.g. [10]. In addition, the unique solution y in the sense of [10] has γ-Hölder continuity for all $0 < \gamma < H$.

The aim of this paper is to consider the numerical approximation of Eq. (1.1). It is well-known (see the introduction in [8] for more details about this issue) that the classical Euler scheme is divergent under this setting. The simplest possible solution to this problem is to use a second-order Euler (that is a Milstein type) scheme, which however involves iterated integrals of the fBm x and is not implementable directly. Several contributions are made to tackle the implementation issue [8,9,12,18]; see also [13,14].

In this paper we will focus our attention on the (implementable) Euler scheme introduced in [12,18]. Take the uniform partition $\pi : 0 = t_0 < t_1 < \cdots < t_n = T$ on $[0,T]$, where for $k = 0, \ldots, n$ we have $t_k = k\Delta$ with $\Delta = \frac{T}{n}$. The Euler scheme is recursively defined as follows:

$$y^n_{t_{k+1}} = y^n_{t_k} + V_0(y^n_{t_k})\Delta + V(y^n_{t_k})\delta x_{t_k t_{k+1}} + \frac{1}{2}\sum_{j=1}^{d} \partial V_j V_j(y^n_{t_k})\, \Delta^{2H} \tag{1.2}$$

* Corresponding author.
E-mail addresses: jleon@ctrl.cinvestav.mx (J.A. León), yanghui.liu@baruch.cuny.edu (Y. Liu), stindel@purdue.edu (S. Tindel).

https://doi.org/10.1016/j.spa.2024.104412
Received 16 May 2023; Received in revised form 5 June 2024; Accepted 6 June 2024
Available online 8 June 2024

and $y_0^n = y_0$, where we have used the notation

$$\partial V_i V_j = \left(\sum_{\ell=1}^{m} \partial_\ell V_i^k V_j^\ell \,;\, k = 1, \ldots, m \right) \tag{1.3}$$

and ∂_i stands for the partial derivative in the $y^{(i)}$ direction: $\partial_i = \frac{\partial}{\partial y^{(i)}}$. The exact rate of convergence of y^n to y is shown to be of order $1/n^{2H-1/2}$ in [18].

In this paper, we are interested in proving that the approximation y^n defined by (1.2) is Malliavin differentiable under sufficient smoothness assumption on the coefficients. More importantly, we will establish pathwise upper bounds estimates of the Malliavin derivative which will be uniform in n. Our motivation for this endeavor is twofold:

(i) The integrability of Malliavin derivatives for rough differential equations has been an important open problem a decade ago. This is mostly due to the prominent role played by Malliavin calculus techniques in obtaining results about the density of random variables like y_t in (1.1). The integrability issue for the Malliavin derivatives Dy_t has been solved completely in [5]. Subsequent applications to the smoothness of densities of y_t are contained in [2,4,11]. The corresponding question for numerical approximations of y is thus in order. We propose to start a detailed answer to this natural problem in the current paper.

(ii) Upper bounds on Malliavin derivatives open the way to important results for numerical schemes. Among others, one can quote weak convergence as well as convergence of densities. In our companion paper [17] we prove the weak convergence of y^n defined by (1.2) towards the solution to (1.1). The uniform bounds on Malliavin derivatives obtained in the current contribution are a crucial ingredient in [17].

With those motivations in mind, our main result can be informally spelled out as follows. Please refer to Theorem 4.10, Remark 4.11 and Theorem 4.15 for a more precise statement.

Theorem 1.1. *Let y and y^n be the solution of (1.1) and the corresponding Euler scheme (1.2), respectively. Take an integer $L \geq 1$. Let $\bar{D}^L y_t^n$ be the Lth Malliavin derivative of y_t^n in the Cameron–Martin space $\bar{\mathcal{H}}$ corresponding to the fBm x. Suppose that $V \in C_b^{L+2}$. Then for each $n \in \mathbb{N}$ there is a functional $\mathcal{G}_L^n$ of the fBm x which is almost surely finite and such that the following pathwise bound holds true:*

$$\|\bar{D}^L y_t^n\|_{\bar{\mathcal{H}}^{\otimes L}} \leq \mathcal{G}_L^n, \quad \text{for all } t = t_k \text{ and } k = 1, \ldots, n. \tag{1.4}$$

The explicit expression of $\mathcal{G}_L^n$ is given in Theorem 4.10. Furthermore, we have the uniform integrability of $\mathcal{G}_L^n$ for $n \in \mathbb{N}$:

$$\sup_{n\in\mathbb{N}} \mathbb{E}[|\mathcal{G}_L^n|^p] < \infty.$$

Remark 1.2. In Theorem 4.10 we will see that, roughly speaking, $\mathcal{G}_L^n$ is the product of the values of a control function ω over the sequence of intervals: $[s_0, s_1]$, $[s_1, s_2]$, ... Here $S := \{s_0, s_1, \ldots\} \subset [\![0, T]\!]$ is a discrete version of the so-called greedy sequence introduced in [5] (the reader is referred to (4.36) for its precise definition). As far as the control function ω is concerned, it will be expressed as the addition of pth moments of a (discrete-version) of the p-variation of the fBm x, plus a quadratic functional of x (see (4.35) below). While $\mathcal{G}_L^n$ still depends on n we have shown (see [17]) that the moments of $\mathcal{G}_L^n$ are uniformly bounded in the variable n, due to the proper choice of the sequence S.

As mentioned above, Theorem 1.1 is a crucial step in the analysis of weak convergence for the Euler type scheme (1.2). In addition, the proof of our main estimate (1.4) relies on techniques which are interesting in their own right. Specifically, we will first resort to rough paths type estimates (recalled in Section 2), and our Malliavin calculus setting will follow Inahama's approach [15] for all computations in the Cameron–Martin space. On top of those classical ingredients, our main technical tool will be a representation of higher order Malliavin derivatives of y^n in terms of a tree expansion (see Lemmas 3.8 and 3.18 below). This kind of expression has to be contrasted with the standard form of higher order Malliavin derivatives, based on sums over partitions of the set $\{1, \ldots, n\}$ (see [20, Proposition 5]). We note that the advantage of our directed-tree notations is that it allows us to distinguish all terms in the chain differentiations. We will benefit from this feature while proving an identities (3.26)–(3.27) in Lemma 3.18. We also believe that the tree-based computations presented here can be usefully applied to other numerical schemes. We plan on developing this line of research in subsequent publications.

The paper is organized as follows. In Section 2 we recall some basic material on rough paths and Malliavin calculus. We also review results on the Euler scheme which will be used throughout the paper. In Section 3 we derive a representation of Malliavin derivatives of the Euler scheme via tree notations. Finally, in Section 4 we prove the uniform upper-bound estimate for the Euler scheme and its Malliavin derivatives.

Notation 1.3. *In what follows, we take $n \in \mathbb{N}$ and $\Delta = T/n$, and consider the uniform partition: $0 = t_0 < t_1 < \cdots < t_n = T$ on $[0, T]$, where $t_k = k\Delta$. We denote by $[\![s, t]\!]$ the discrete interval: $[\![s, t]\!] = \{t_k \in [s, t] : k = 0, \ldots, n\}$. For $u \in [t_k, t_{k+1})$, we denote $\eta(u) = t_k$. For an interval $[s, t] \subset [0, T]$ we define the simplex $\mathcal{S}_2([s, t]) = \{(u, v) : s \leq u \leq v \leq t\}$. For a vector $a = (a^1, \ldots, a^m) \in \mathbb{R}^d$ we define the norm $|a| = \max_{j=1,\ldots,m} |a_j|$. Throughout the paper, we use C and K to represent constants that are independent of n and whose values may change from line to line.*

2. Preliminary results

In this section we recall some basic notions of rough paths theory and their application to fractional Brownian motion, which allows a proper definition of Eq. (1.1). We also give the necessary elements of Malliavin calculus in order to estimate densities of random variables.

2.1. Elements of rough paths and fractional Brownian motion

This subsection is devoted to introduce some basic concepts of rough paths theory. We are going to restrict our analysis to a generic Hölder regularity of the driving path of order $\frac{1}{3} < \gamma \le \frac{1}{2}$, in order to keep expansions to a reasonable size. We also fix a finite time horizon $T > 0$. The following notation will prevail until the end of the paper: for a Banach space $\mathcal{V}$ (which can be either finite or infinite dimensional) and two functions $f \in C([0,T],\mathcal{V})$ and $g \in C(\mathcal{S}_2([0,T]),\mathcal{V})$ we set

$$\delta f_{st} = f_t - f_s, \quad \text{and} \quad \delta g_{sut} = g_{st} - g_{su} - g_{ut}, \qquad 0 \le s \le u \le t \le T. \tag{2.1}$$

Let us introduce the analytic requirements in terms of Hölder regularity which will be used in the sequel. Namely consider two paths $x \in C([0,T],\mathbb{R}^d)$ and $x^2 \in C(\mathcal{S}_2([0,T]),(\mathbb{R}^d)^{\otimes 2})$. Then we denote

$$\|x\|_{[s,t],\gamma} := \sup_{(u,v)\in\mathcal{S}_2([s,t]):u\neq v} \frac{|\delta x_{uv}|}{|v-u|^{\gamma}}, \qquad \|x^2\|_{[s,t],2\gamma} := \sup_{(u,v)\in\mathcal{S}_2([s,t]):u\neq v} \frac{|x^2_{uv}|}{|v-u|^{2\gamma}}. \tag{2.2}$$

When the semi-norms in (2.2) are finite we say that x and x^2 are respectively in $C^{\gamma}([s,t],\mathbb{R}^d)$ and $C^{2\gamma}(\mathcal{S}_2([s,t]),(\mathbb{R}^d)^{\otimes 2})$. For convenience, we denote $\|x\|_{\gamma} := \|x\|_{[0,T],\gamma}$ and $\|x^2\|_{2\gamma} := \|x^2\|_{[0,T],2\gamma}$. With this preliminary notation in hand, we can now turn to the definition of rough path.

Definition 2.1. Let $x \in C([0,T],\mathbb{R}^d)$, $x^2 \in C(\mathcal{S}_2([0,T]),(\mathbb{R}^d)^{\otimes 2})$, and $\frac{1}{3} < \gamma \le \frac{1}{2}$. For $(s,t) \in \mathcal{S}_2([0,T])$ we denote $x^1_{st} = \delta x_{st}$. We call $\mathbf{x} := S_2(x) := (x^1, x^2)$ a (second-order) γ-rough path if $\|x^1\|_{\gamma} < \infty$ and $\|x^2\|_{2\gamma} < \infty$, and if the following algebraic relation holds true:

$$\delta x^2_{sut} = x^2_{st} - x^2_{su} - x^2_{ut} = x^1_{su} \otimes x^1_{ut} \qquad s \le u \le t, \tag{2.3}$$

where we have invoked (2.1) for the definition of δx^2. For a γ-rough path $S_2(x)$, we define a γ-Hölder semi-norm as follows:

$$\|S_2(x)\|_{\gamma} := \|x^1\|_{\gamma} + \|x^2\|_{2\gamma}^{\frac{1}{2}}. \tag{2.4}$$

An important subclass of rough paths are the so-called *geometric γ-Hölder rough paths*. A geometric γ-Hölder rough path is a γ-rough path (x, x^2) such that there exists a sequence of smooth $\mathbb{R}^d$-valued paths $(x^n, x^{2,n})$ verifying:

$$\|x - x^n\|_{\gamma} + \|x^2 - x^{2,n}\|_{2\gamma} \to 0, \quad \text{as } n \to \infty. \tag{2.5}$$

We will mainly consider geometric rough paths in the remainder of the article.

Let x be a rough path as given in Definition 2.1. We shall interpret equation (1.1) in a way introduced by Davie in [7], which is conveniently compatible with numerical approximations.

Definition 2.2. Let (x, x^2) be a γ-rough path with $\gamma > 1/3$. We say that y is a solution of (1.1) on $[0,T]$ if $y_0 = a$ and there exists a constant $K > 0$ and $\mu > 1$ such that

$$\left|\delta y_{st} - \int_s^t V_0(y_u)\,du - V(y_s)x^1_{st} - \sum_{i,j=1}^{d} \partial V_i V_j(y_s) x^{2,ij}_{st}\right| \le K|t-s|^{\mu} \tag{2.6}$$

for all $(s,t) \in \mathcal{S}_2([0,T])$, where we recall that δy is defined by (2.1) and the notation $\partial V_i V_j$ is introduced in (1.3).

According to [7] there exists a unique RDE solution to equation (1.1), understood as in Definition 2.2.

In the following we recall a sewing map lemma with respect to discrete control functions. It is an elaboration of [18, Lemma 2.5] and proves to be useful in the analysis of the numerical scheme. Let $\pi : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ be a generic partition of the interval $[0,T]$ for $n \in \mathbb{N}$. We denote by $[\![s,t]\!]$ the discrete interval $\{t_k : s \le t_k \le t\}$ for $0 \le s < t \le T$. In this paper, a two variable function $\omega : \mathcal{S}_2([\![0,T]\!]) \to [0,\infty)$ is called a control on $[\![0,T]\!]$ if it satisfies the super-additivity condition. That is, $\omega(s,u) + \omega(u,t) \le \omega(s,t)$ for $s,u,t \in [\![0,T]\!]$ such that $s \le u \le t$.

Lemma 2.3. *Suppose that ω is a control on $[\![0,T]\!]$. Consider a Banach space $\mathcal{B}$ and an increment $R : \mathcal{S}_2([\![0,T]\!]) \to \mathcal{B}$. Suppose that $|R_{t_k t_{k+1}}| \le \omega(t_k,t_{k+1})^{\mu}$ for all $t_k \in [\![0,T]\!]$ and that $|\delta R_{sut}| \le \omega(s,t)^{\mu}$ with an exponent $\mu > 1$, where recall that $\delta R_{sut} = R_{st} - R_{su} - R_{ut}$. Then the following relation holds:*

$$|R_{st}| \le K_{\mu}\omega(s,t)^{\mu}, \quad \textit{where} \quad K_{\mu} = 2^{\mu}\sum_{l=1}^{\infty} l^{-\mu}. \tag{2.7}$$

We now specialize our setting to a path $x = (x^1, \dots, x^d)$ defined as a standard m-dimensional fBm on $[0, T]$ with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$. This fBm is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and we assume that the σ-algebra $\mathcal{F}$ is generated by x. In this situation, recall that the covariance function of each coordinate of x is defined on $S_2([0,T])$ by:

$$R(s,t) = \frac{1}{2}\left[|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right]. \tag{2.8}$$

It is established in [10, Chapter 15] that the geometric rough path $S_2(x)$ of x via the piecewise linear approximation is well defined for $\frac{1}{3} < \gamma < H$ in the sense of Definition 2.1.

2.2. Malliavin calculus for $\mathbf{x}$

In this subsection we recall some concepts of Malliavin calculus which will be used later in the paper. Recall that R is the covariance function of the fBm x defined in (2.8). Denote by $\mathcal{E}$ the set of step functions on the interval $[0,T]$. We define the Hilbert space $\mathcal{H}$ as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle \mathbf{1}_{[u,v]}, \mathbf{1}_{[s,t]} \rangle_{\mathcal{H}} = R([u,v],[s,t]) \equiv R(v,t) - R(v,s) - R(u,t) + R(u,s). \tag{2.9}$$

The space $\mathcal{H}$ is very useful in order to define Wiener integrals with respect to x. However, in the current paper we also need to introduce the Cameron–Martin space $\bar{\mathcal{H}}$ related to our driving process. The latter space is the one allowing to identify pathwise derivatives with respect to x and Malliavin derivatives. In order to construct $\bar{\mathcal{H}}$, let first $\mathcal{R}$ be the linear operator on $\mathcal{E}$ such that

$$\mathcal{R}(\mathbf{1}_{[0,t]}) = R(t,\cdot), \tag{2.10}$$

and we also set $\bar{\mathcal{E}} = \mathcal{R}(\mathcal{E})$. Then we can define the Cameron–Martin space $\bar{\mathcal{H}}$ as the closure of $\bar{\mathcal{E}}$ with respect to the inner product

$$\langle \mathcal{R}(\mathbf{1}_{[0,t]}), \mathcal{R}(\mathbf{1}_{[0,s]}) \rangle_{\bar{\mathcal{H}}} = \langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[0,t]} \rangle_{\mathcal{H}}.$$

It is clear that $\mathcal{R}$ is an isometry between the two Hilbert spaces $\mathcal{H}$ and $\bar{\mathcal{H}}$. Note that according to (2.10) we have

$$\mathcal{R}(h)(t) = \langle h, \mathbf{1}_{[0,t]} \rangle_{\mathcal{H}} \tag{2.11}$$

for $h \in \mathcal{E}$. By the isometry property of $\mathcal{R}$ we see that (2.11) holds for all $h \in \mathcal{H}$. We refer to [1,20] for more details about the spaces $\mathcal{H}, \bar{\mathcal{H}}$.

For the sake of conciseness, we refer to [19] for a proper definition of Malliavin derivatives in the Hilbert space $\mathcal{H}$ and related Sobolev spaces in Gaussian analysis. Let us just mention that for a functional F of x we will denote its Malliavin derivative by DF, the Sobolev spaces by $\mathbb{D}^{k,p}$ and the corresponding norms by $\|F\|_{k,p}$.

As mentioned above, in this paper we will mainly focus on a more pathwise Malliavin derivative taking values in $\bar{\mathcal{H}}$. Namely we define the Malliavin derivative in the Cameron–Martin $\bar{\mathcal{H}}$ space via the isometry $\mathcal{R}$. Precisely, we define $\bar{D}$ such that $\bar{D}F = \mathcal{R}(DF)$. In other words, for $h \in \mathcal{H}$ and a functional F of x we have

$$\bar{D}_{\mathcal{R}(h)}F := \langle \bar{D}F, \mathcal{R}(h) \rangle_{\bar{\mathcal{H}}} = \langle DF, h \rangle_{\mathcal{H}} =: D_h F.$$

This Malliavin derivative can be expressed easily for cylindrical functionals of x. Namely suppose that $F = f(x_{t_1}, \dots, x_{t_\ell})$ for $f \in C_p^1(\mathbb{R}^\ell)$. According to the definition of $\bar{D}$, for $h \in \mathcal{H}$ we have

$$\begin{aligned}\langle \bar{D}F, \mathcal{R}(h) \rangle_{\bar{\mathcal{H}}} &= \langle DF, h \rangle_{\mathcal{H}} = \sum_{i=1}^{\ell} \partial_i f(x_{t_1}, \dots, x_{t_\ell}) \langle \mathbf{1}_{[0,t_i]}, h \rangle_{\mathcal{H}} \\ &= \sum_{i=1}^{\ell} \partial_i f(x_{t_1}, \dots, x_{t_\ell}) \mathcal{R}(h)(t_i). \end{aligned} \tag{2.12}$$

Notice that the computation in (2.12) shows that $\langle \bar{D}F, \mathcal{R}(h) \rangle_{\bar{\mathcal{H}}}$ can be interpreted as an extension in the Fréchet derivative of F of the $\mathcal{R}(h)$ direction. Indeed, for the quantity in the right-hand side of (2.12) we have

$$\frac{d}{d\varepsilon} f(x_{t_1} + \varepsilon \mathcal{R}(h)(t_1), \dots, x_{t_\ell} + \varepsilon \mathcal{R}(h)(t_\ell)) \Big|_{\varepsilon=0} = \sum_{i=1}^{\ell} \partial_i f(x_{t_1}, \dots, x_{t_\ell}) \mathcal{R}(h)(t_i). \tag{2.13}$$

This pathwise interpretation of Malliavin derivatives is also the one adopted in [10].

In this paper, we denote by $\bar{D}^k F$ the kth iteration of the Malliavin derivative $\bar{D}$ applied on F. Also notice that we are considering a d-dimensional fBm $x = (x^1, \dots, x^d)$. Therefore, we shall consider partial Malliavin derivatives with respect to each coordinate x^i in the sequel. Those partial derivatives will be denoted by $\bar{D}^{(i)}$. Then for $\bar{h} = (\bar{h}^1, \dots, \bar{h}^d) \in \bar{\mathcal{H}}^d$ we write $\bar{D}_{\bar{h}} F = \sum_{i=1}^d \langle \bar{D}^{(i)} F, \bar{h}^i \rangle_{\bar{\mathcal{H}}}$. For $L \geq 2$ we denote by $\bar{D}_{\bar{h}}^L$ the iterated versions of $\bar{D}_{\bar{h}}$. Namely we set

$$\bar{D}_{\bar{h}}^L F = \bar{D}_{\bar{h}} \circ \cdots \circ \bar{D}_{\bar{h}} F. \tag{2.14}$$

The Sobolev spaces related to the Malliavin derivatives in the Cameron–Martin space are denoted by $\bar{\mathbb{D}}^{k,p}$ and the corresponding norms are written $\|\cdot\|_{\bar{\mathbb{D}}^{k,p}}$.

Let us now review some results on the Malliavin differentiability of Eq. (1.1). In the following we assume that the vector fields $V_0, \dots, V_d$ are at least in $C_b^3(\mathbb{R}^m)$ (bounded together with their derivatives up to order 3), although later on we will have to introduce further smoothness conditions in order to estimate higher order Malliavin derivatives. We shall express the first order Malliavin derivative of y_t in terms of the Jacobian Φ of the equation, which is defined by the relation $\Phi_t^{ij} = \partial_{a_j} y_t^{(i)}$. Setting ∂V_j for the Jacobian of V_j seen as a function from $\mathbb{R}^n$ to $\mathbb{R}^n$, let us recall that Φ is the unique solution to the linear equation

$$\Phi_t = \mathrm{Id}_n + \int_0^t \partial V_0(y_s)\Phi_s\, ds + \sum_{j=1}^d \int_0^t \partial V_j(y_s)\Phi_s\, dx_s^j, \tag{2.15}$$

The following result (for which we refer to [3]) holds true:

Proposition 2.4. *Let y be the solution to Eq. (1.1). Then for every $i = 1, \dots, m$, $t > 0$, and $a \in \mathbb{R}^m$, we have $y_t^{(i)} \in \mathbb{D}(\mathcal{H})$ and*

$$D_s y_t = \left(\Phi_{s,t} V_j(y_s), j = 1, \dots, d\right), \quad 0 \le s \le t,$$

where $\Phi_t = \partial_a y_t$ solves Eq. (2.15) and $\Phi_{s,t} = \Phi_t \Phi_s^{-1}$.

Let us now quote the result [5], which gives a useful estimate for moments of the Jacobian of rough differential equations driven by Gaussian processes. Note that this result is expressed in terms of p-variations, for which we refer to [10].

Proposition 2.5. *Consider a fractional Brownian motion x with Hurst parameter $H \in (1/4, 1/2]$ and $p > 1/H$. Then for any $\eta \ge 1$, there exists a finite constant c_η such that the Jacobian Φ defined by (2.15) satisfies:*

$$\mathbb{E}\left[\|\Phi\|^\eta_{p\text{-}var;[0,1]}\right] = c_\eta. \tag{2.16}$$

3. Malliavin derivatives of the Euler scheme

The estimates for the derivatives of the Euler scheme approximation y^n require a substantial amount of algebraic and analytic efforts. In this section we focus on the algebraic aspect of the problem. Precisely, we apply a tree argument to derive a representation for Malliavin derivatives of y^n. This will be useful for our main bound of the derivatives of y^n in the next section (see Theorem 4.10).

3.1. A directed rooted tree

The higher order Malliavin derivatives of the Euler scheme y^n are better understood thanks to a tree type encoding. We introduce the necessary notation in this section. Let us start with the definition of rooted trees which will be used in the sequel.

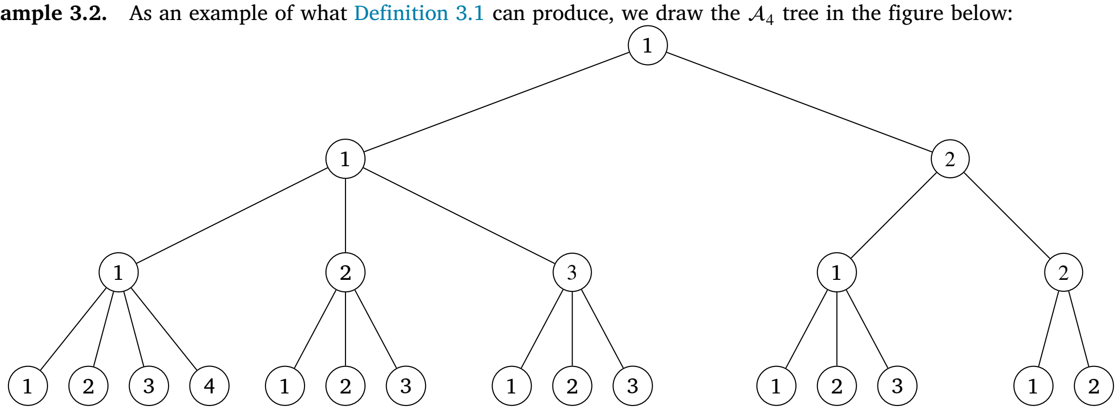
Definition 3.1. In the remainder of the paper we consider rooted trees $\mathcal{A}_N$ of height N defined recursively as follows:

(i) $\mathcal{A}_1$ contains one branch with length 1 and with root labeled 1. Namely, $\mathcal{A}_1 = \{(1)\}$.

(ii) For each $N \in \mathbb{N}$ we define $\mathcal{A}_{N+1}$ such that its first N generations coincides with $\mathcal{A}_N$. Its $(N+1)$th generation is defined as follows: Take a branch i in $\mathcal{A}_N$. We call ℓ_1^i the number of 1's in i and we also set $\alpha_i = \ell_1^i + 1$. Then $\mathcal{A}_{N+1}$ is constructed by adding the branches $(i, 1), \dots, (i, \alpha_i)$ to $\mathcal{A}_N$. Specifically, one can also define $\mathcal{A}_{N+1}$ recursively as

$$\left\{(i, r) : i \in \mathcal{A}_N, r = 1, \dots, \alpha_i\right\}.$$

Example 3.2. As an example of what Definition 3.1 can produce, we draw the $\mathcal{A}_4$ tree in the figure below:



In the following we introduce some additional notation about the trees $\mathcal{A}_N$ which will be useful for our future computations.

Notation 3.3. *With a slight abuse of notation, we will write $\mathcal{A}_N$ for both the tree $\mathcal{A}_N$ and the collection of its branches. For each branch i in $\mathcal{A}_N$ we denote $|i|$ the number of vertices in i and denote i_τ the τth label in the branch i. We set*

$$\ell_1^i = \#\{i_\tau = 1 : \tau = 1,\dots,|i|\} \quad \text{and} \quad \ell_r^i = \#\{i_\tau = r : \tau = 1,\dots,|i|\} + 1 \quad \text{for } r = 2,\dots,|i|.$$

Also recall that we denote $\alpha_i = \ell_1^i + 1$.

Remark 3.4. In the sequel we shall use the relation

$$\sum_{r=2}^{\alpha_i} \ell_r^i = N, \tag{3.1}$$

valid for every tree $\mathcal{A}_N$ from Definition 3.1. Let us give a brief proof of this fact. According to the definition of $\mathcal{A}_N$, for each branch i the vertices of i are labeled by the numbers $1,2,\dots,\alpha_i$. No vertex of i is labeled a number $\alpha_i + 1$ or larger. Therefore, for all $r > \alpha_i$ we have

$$\#\{i_\tau = r : \tau = 1,\dots,|i|\} = 0, \qquad \text{and} \qquad \ell_r^i = 1.$$

Moreover, by construction every branch i in $\mathcal{A}_N$ has length $|i| = N$. Thus

$$\sum_{r=1}^{\alpha_i} \#\{i_\tau = r : \tau = 1,\dots,|i|\} = |i| = N.$$

Because $\alpha_i = \ell_1^i + 1 = \#\{i_\tau = 1 : \tau = 1,\dots,|i|\} + 1$, the above becomes

$$\sum_{r=2}^{\alpha_i} \big(\#\{i_\tau = r : \tau = 1,\dots,|i|\}\big) + \big(\alpha_i - 1\big) = \sum_{r=2}^{\alpha_i} \big(\#\{i_\tau = r : \tau = 1,\dots,|i|\} + 1\big) = N.$$

Otherwise stated, according to Notation 3.3 we obtain relation (3.1).

Example 3.5. Let us follow up on Example 3.2, and see how Notation 3.3 works on $\mathcal{A}_4$. Namely $(1,2,1,3)$ and $(1,2,1,1)$ are both branches in $\mathcal{A}_4$. For those branches, the reader can easily check that we have

$$\ell_1^{(1,2,1,3)} = 2, \quad \ell_2^{(1,2,1,3)} = 2, \quad \ell_3^{(1,2,1,3)} = 2, \quad \alpha_{(1,2,1,3)} = 3$$
$$\ell_1^{(1,2,1,1)} = 3, \quad \ell_2^{(1,2,1,1)} = 2, \quad \ell_3^{(1,2,1,1)} = 1, \quad \ell_4^{(1,2,1,1)} = 1, \quad \alpha_{(1,2,1,1)} = 4.$$

It is easily checked that the identity (3.1) holds for these two branches. Namely we have $\ell_2^{(1,2,1,3)} + \ell_3^{(1,2,1,3)} = 4$ and $\ell_2^{(1,2,1,1)} + \ell_3^{(1,2,1,1)} + \ell_4^{(1,2,1,1)} = 4$.

In order to state our differentiation rule for Malliavin derivatives, let us also label some notation about partial differentiation in $\mathbb{R}^m$.

Notation 3.6. *For a constant*

$$A = (A^{p_1,\dots,p_k};\ p_1,\dots,p_k = 1\dots,m) \in (\mathbb{R}^m)^{\otimes k} = \mathbb{R}^{m^k},$$

and for $a^{(1)},\ \dots,\ a^{(k)} \in \mathbb{R}^m$ we set

$$\langle A, a^{(1)} \otimes \cdots \otimes a^{(k)} \rangle := \sum_{p_1,\dots,p_k=1}^{m} a_{p_k}^{(k)} \cdots a_{p_1}^{(1)} A^{p_1,\dots,p_k}.$$

Notation 3.7. *Let $f : y \to f(y)$ be a continuous function from $\mathbb{R}^m$ to $\mathbb{R}$. We denote by ∂ the differential operator from $C^1(\mathbb{R}^m)$ to $\mathcal{L}(\mathbb{R}^m,\mathbb{R}) = \mathcal{L}(\mathbb{R}^m)$. That is for $a = (a_1,\dots,a_m) \in \mathbb{R}^m$, we define $\langle \partial f, a\rangle = \sum_{i=1}^m a_i \frac{\partial f}{\partial y^{(i)}}$. Note that the space $\mathcal{L}(\mathbb{R}^m)$ can also be identified with $\mathbb{R}^m$. Namely, we can write*

$$\partial f := (\partial_1 f,\dots,\partial_m f) := \left(\frac{\partial f}{\partial y^{(1)}},\dots,\frac{\partial f}{\partial y^{(m)}} \right).$$

One can generalize this notation to higher order derivatives. Specifically, we denote by ∂^k the differential operator from $C^k(\mathbb{R}^m)$ to $\mathcal{L}((\mathbb{R}^m)^{\otimes k},\mathbb{R})$. Otherwise stated for $a^{(1)},\ \dots,\ a^{(k)} \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$ we define a vector

$$\partial^k f(y) = \left\{ \frac{\partial^k f(y)}{\partial y^{(p_k)} \dots \partial y^{(p_1)}};\ p_1,\dots,p_k = 1\dots,m \right\},$$

so that one can write

$$\langle \partial^k f(y), a^{(1)} \otimes \cdots \otimes a^{(k)} \rangle = \sum_{p_1,\dots,p_k=1}^{m} a_{p_k}^{(k)} \cdots a_{p_1}^{(1)} \frac{\partial^k f(y)}{\partial y^{(p_k)} \dots \partial y^{(p_1)}}. \tag{3.2}$$

Notice that for $k = 2$, $\partial^2 f$ can also be identified with the matrix $(\frac{\partial^2 f}{\partial y^{(i)} \partial y^{(j)}})_{i,j=1,\dots,m}$.

3.2. A differentiation rule

With the preparation in the previous subsection, we are now ready to state a lemma allowing to compute iterated Malliavin derivatives for a functional of the form $f(F)$, with a smooth enough function f and random variable F. Recall that the Malliavin derivative operator $\bar{D}$ in the Cameron–Martin space $\bar{\mathcal{H}}$ is introduced in Section 2.2.

Lemma 3.8. *Let f, g be continuous functions in $C^N(\mathbb{R}^m)$ and let $F \in \bar{\mathbb{D}}^{N,2}(\mathbb{R}^m)$, where the space $\bar{\mathbb{D}}^{N,2}$ is introduced in Section 2.2. Then we have the following identity:*

$$\bar{D}^N_{\bar{h}} f(F) = \sum_{i\in\mathcal{A}_N} \langle \partial^{\ell^i_1} f(F), \bar{D}^{\ell^i_2}_{\bar{h}} F \otimes \cdots \otimes \bar{D}^{\ell^i_{\alpha_i}}_{\bar{h}} F\rangle \tag{3.3}$$

for $\bar{h} \in \bar{\mathcal{H}}^m$, where the sum in the right side of (3.3) runs over the branches of $\mathcal{A}_N$ as specified in Notations 3.3–3.7. As far as the product $f(F)g(F)$ is concerned, we get the following differentiation rule:

$$\bar{D}^N_{\bar{h}}(f\cdot g)(F) = M_1(N) + M_2(N), \tag{3.4}$$

where

$$M_1(N) = \sum_{i\in\mathcal{A}_N} \left\langle (\partial^{\ell^i_1} f\cdot g)(F), \bar{D}^{\ell^i_2}_{\bar{h}} F \otimes \cdots \otimes \bar{D}^{\ell^i_{\alpha_i}}_{\bar{h}} F\right\rangle$$

$$M_2(N) = \sum_{i\in\mathcal{A}_N} \sum_{r=2}^{\alpha_i} \left\langle \partial^{\ell^i_1-1} f(F), \bar{D}^{\ell^i_2}_{\bar{h}} F \otimes \cdots \otimes \bar{D}^{\ell^i_r}_{\bar{h}} g(F) \otimes \cdots \otimes \bar{D}^{\ell^i_{\alpha_i}}_{\bar{h}} F\right\rangle.$$

Remark 3.9. Let us check the dimension compatibilities in the right side of (3.3). Since F is $\mathbb{R}^m$-valued, according to Notation 3.7 we have $\partial^{\ell^i_1} f(F) \in \mathcal{L}((\mathbb{R}^m)^{\otimes \ell^i_1};\mathbb{R})$. Next each term $\bar{D}^{\ell^i_r}_{\bar{h}} F$ sits in $\mathbb{R}^m$. Therefore, $\bar{D}^{\ell^i_2}_{\bar{h}} F \otimes \cdots \otimes \bar{D}^{\ell^i_{\alpha_i}}_{\bar{h}} F \in \mathcal{L}((\mathbb{R}^m)^{\otimes(\alpha_i-1)})$. The compatibility of dimensions in (3.3) thus stems from the relation $\alpha_i = \ell^i_1 + 1$ (see Notation 3.3). Similar considerations are also valid for Eq. (3.4), taking into account the fact that $\bar{D}^{\ell^i_r}_{\bar{h}} F \in \mathbb{R}^m$ and $\bar{D}^{\ell^i_r}_{\bar{h}} g(F) \in \mathbb{R}$.

Remark 3.10. In order to get a formula for $\bar{D}^N_{\bar{h}} f(F)$, we could have invoked some multivariate elaboration of Faà Di Bruno's formula; see [6,16]. However, our tree type formulation is required in order to handle the computations in Lemma 3.18 below.

Remark 3.11. Lemma 3.8 can easily be generalized in three directions:

(i) We have stated (3.3) using the directional derivative $\bar{D}_{\bar{h}}$. The same formula holds true for the function-valued derivative $D_r f(F)$.

(ii) Instead of the Malliavin derivative, we could have obtained (3.3) as a chain rule for any operator A satisfying a Leibniz type rule of the form $A(f(F)) = f'(F)AF$.

(iii) Instead of considering iterations of the same operation A, that is a formula for $A^N(f(F))$, one can obtain a formula like (3.3) for quantities of the form $A_1\circ\cdots\circ A_N(f(F))$, where each of the operators A_j satisfies $A_j(f(F)) = f'(F)A_jF$.

Proof of Lemma 3.8. We first show by induction that (3.3) is true. To this aim, note that $\bar{D}_{\bar{h}} f(F) = \langle \partial f(F), \bar{D}_{\bar{h}} F\rangle$. On the other hand, by Definition 3.1 we have $\mathcal{A}_1 = \{(1)\}$ and according to Notation 3.3 we have $\ell_1 = \ell_2 = 1$ and $\alpha_{(1)} = 2$. This concludes (3.3) for $N = 1$.

Now suppose that (3.3) is true for $N = L$. This means that $\bar{D}^L_{\bar{h}} f(F)$ is equal to the summation over $i \in \mathcal{A}_N$ of quantities of the form $\langle \partial^{\ell_1} f(F), \bar{D}^{\ell_2}_{\bar{h}} F \otimes \cdots \otimes \bar{D}^{\ell_{\alpha_i}}_{\bar{h}} F\rangle$ which are one-to-one corresponding to the branches in $\mathcal{A}_L$. Consider a generic term in this summation and differentiate it in a direction $h \in \bar{\mathcal{H}}$. Dropping the superscript i in ℓ^i_r for notational sake, we get

$$\begin{aligned}
&\bar{D}_{\bar{h}}\left(\langle \partial^{\ell_1} f(F), \bar{D}^{\ell_2}_{\bar{h}} F \otimes \cdots \otimes \bar{D}^{\ell_{\alpha_i}}_{\bar{h}} F\rangle\right) \\
&= \langle \partial^{\ell_1+1} f(F), \bar{D}^{\ell_2}_{\bar{h}} F \otimes \cdots \otimes \bar{D}^{\ell_{\alpha_i}}_{\bar{h}} F \otimes \bar{D}_{\bar{h}} F\rangle \\
&\quad + \langle \partial^{\ell_1} f(F), \bar{D}^{\ell_2+1}_{\bar{h}} F \otimes \cdots \otimes \bar{D}^{\ell_{\alpha_i}}_{\bar{h}} F\rangle + \cdots + \langle \partial^{\ell_1} f(F), \bar{D}^{\ell_2}_{\bar{h}} F \otimes \cdots \otimes \bar{D}^{\ell_{\alpha_i}+1}_{\bar{h}} F\rangle.
\end{aligned} \tag{3.5}$$

One can relate relation (3.5) to our tree Definition 3.1 in the following way. Namely in the recursive step (ii) in Definition 3.1, from $\mathcal{A}_L$ we have created a new tree by adding labeled offsprings to the branch i. Specifically we add the branches $(i,1)$, $(i,2)$, …, (i,α_i), and we set $\alpha_{(i,1)} = \alpha_i + 1$ and $\alpha_{(i,r)} = \alpha_i$ for $r = 2,\ldots,\alpha_i$. This shows that after differentiation, the term corresponding to the branch i is replaced by α_i terms corresponding to the branches: $(i,1)$, $(i,2)$,…, (i,α_i). These are exactly the branches in $\mathcal{A}_{L+1}$ which overlap with i in the first $|i|$ vertices. Here the sum (3.5) can also be written as

$$\sum_{j\in K_i} \langle \partial^{\ell^j_1} f(F), \bar{D}^{\ell^j_2}_{\bar{h}} F \otimes \cdots \otimes \bar{D}^{\ell^j_{\alpha_j}}_{\bar{h}} F\rangle,$$

where the branches j sit in a set K_i defined by

$$K_i = \{(i,r) : 1 \le r \le \alpha_i\}.$$

Summing all those contributions we get

$$\bar{D}_{\bar{h}}^{L+1} f(F) = \sum_{i\in\mathcal{A}_L} \sum_{j\in K_i} \langle \partial^{\ell_1^j} f(F), \bar{D}_{\bar{h}}^{\ell_2^j} F \otimes \cdots \otimes \bar{D}_{\bar{h}}^{\ell_{\alpha_j}^j} F\rangle, \tag{3.6}$$

from which it is easily seen that (3.3) holds up to order $L+1$. This finishes our induction procedure for (3.3).

Let us now consider (3.4). First, it is easy to verify that (3.4) holds for $N=1$. Next suppose that (3.4) is true for $N=L$. Let us now differentiate the terms in $M_1(L)$ on the right-hand side of (3.4). Still writing ℓ_r instead of ℓ_r^i for notational sake, we get:

$$\bar{D}_{\bar{h}} \left(\langle (\partial^{\ell_1} f \cdot g)(F), \bar{D}_{\bar{h}}^{\ell_2} F \otimes \cdots \otimes \bar{D}_{\bar{h}}^{\ell_{\alpha_i}} F\rangle \right) = H_1^i + H_2^i, \tag{3.7}$$

where the term H_1^i takes care of the differentiation of $g(F)$ in the left side of (3.7), while H_2^i corresponds to the differentiation of $\partial^{\ell_1} f(F)$ and $\bar{D}_{\bar{h}}^{\ell_r} F$. Specifically, we get

$$H_1^i = \langle \partial^{\ell_1} f(F), \bar{D}_{\bar{h}}^{\ell_2} F \otimes \cdots \otimes \bar{D}_{\bar{h}}^{\ell_{\alpha_i}} F\rangle \cdot \bar{D}_{\bar{h}} g(F),$$

while H_2^i is obtained similarly to (3.5) as

$$\begin{aligned} H_2^i &= \langle (\partial^{\ell_1+1} f \cdot g)(F), \bar{D}_{\bar{h}}^{\ell_2} F \otimes \cdots \otimes \bar{D}_{\bar{h}}^{\ell_{\alpha_i}} F \otimes \cdots \otimes \bar{D}_{\bar{h}} F\rangle \\ &\quad + \langle (\partial^{\ell_1} f \cdot g)(F), \bar{D}_{\bar{h}}^{\ell_2+1} F \otimes \cdots \otimes \bar{D}_{\bar{h}}^{\ell_{\alpha_i}} F\rangle + \cdots + \langle (\partial^{\ell_1} f \cdot g)(F), \bar{D}_{\bar{h}}^{\ell_2} F \otimes \cdots \otimes \bar{D}_{\bar{h}}^{\ell_{\alpha_i}+1} F\rangle. \end{aligned}$$

Now we follow the same argument as the one leading to (3.6) to get

$$\sum_{i\in\mathcal{A}_L} H_2^i = \sum_{i\in\mathcal{A}_{L+1}} \langle (\partial^{\ell_1} f \cdot g)(F), \bar{D}_{\bar{h}}^{\ell_2} F \otimes \cdots \otimes \bar{D}_{\bar{h}}^{\ell_{\alpha_i}} F\rangle = M_1(L+1). \tag{3.8}$$

In order to complete the induction proof it remains to show that

$$\sum_{i\in\mathcal{A}_L} H_1^i + \bar{D}_{\bar{h}} M_2(L) = M_2(L+1). \tag{3.9}$$

For this purpose we consider the map from $i \in \mathcal{A}_L$ to $(i,1)$. It is clear that this is a one-to-one mapping. We conclude from this one-to-one correspondence that

$$\sum_{i\in\mathcal{A}_L} H_1^i = \sum_{j=(i,1):\, i\in\mathcal{A}_L} \langle \partial^{\ell_1^j - 1} f(F), \bar{D}_{\bar{h}}^{\ell_2^j} F \otimes \cdots \otimes \bar{D}_{\bar{h}}^{\ell_{\alpha_j-1}^j} F \otimes \bar{D}_{\bar{h}}^{\ell_{\alpha_j}^j} g(F)\rangle, \tag{3.10}$$

where we recall that according to Notation 3.3 we have $\ell_{\alpha_j}^j = 1$ for $j=(i,1)$.

We turn to the second summation $M_2(L)$ in (3.4). Note that each term in (3.4) corresponds to a couple $(i;r)$ where i is a branch in $\mathcal{A}_L$ and $r \in \{2,\dots,\alpha_i\}$ denotes the position for which a term of the form $\bar{D}_{\bar{h}}^{\ell_r^i} g(F)$ shows up. By differentiating $M_2(L)$ we see that the term corresponding to the couple $(i;r)$ is replaced by the terms corresponding to $((i,1);r)$, $((i,2);r)$, …, $((i,\alpha_i);r)$. Precisely, we have

$$\bar{D}_{\bar{h}} M_2(L) = \sum_{(j;r)\in\mathcal{B}_{2,L}} \langle \partial^{\ell_1 - 1} f(F), \bar{D}_{\bar{h}}^{\ell_2} F \otimes \cdots \otimes \bar{D}_{\bar{h}}^{\ell_r} g(F) \otimes \cdots \otimes \bar{D}_{\bar{h}}^{\ell_{\alpha_j}} F\rangle, \tag{3.11}$$

where

$$\mathcal{B}_{2,L} = \bigcup_{i\in\mathcal{A}_L} \bigcup_{r=2}^{\alpha_i} \{((i,1);r), ((i,2);r), \dots, ((i,\alpha_i);r)\}.$$

Observe that in a similar way we can also write (3.10) as

$$\sum_{i\in\mathcal{A}_L} H_1^i = \sum_{(j;r)\in\mathcal{B}_{1,L}} \langle \partial^{\ell_1^j - 1} f(F), \bar{D}_{\bar{h}}^{\ell_2^j} F \otimes \cdots \otimes \bar{D}_{\bar{h}}^{\ell_{\alpha_j-1}^j} F \otimes \bar{D}_{\bar{h}}^{\ell_{\alpha_j}^j} g(F)\rangle, \tag{3.12}$$

where

$$\mathcal{B}_{1,L} = \bigcup_{i\in\mathcal{A}_L} \{((i,1);\alpha_i+1)\} = \bigcup_{i\in\mathcal{A}_L} \{((i,1);\alpha_{(i,1)})\}.$$

On the other hand, note that $\alpha_{(i,2)} = \cdots = \alpha_{(i,\alpha_i)} = \alpha_i$, and $\alpha_{(i,1)} = \alpha_i + 1$. Therefore, we can express the tree $\mathcal{A}_{L+1}$ as follows:

$$\bigcup_{i\in\mathcal{A}_{L+1}} \bigcup_{r=2}^{\alpha_i} \{(i;r)\} = \mathcal{B}_{2,L} \cup \mathcal{B}_{1,L}. \tag{3.13}$$

Observe that $\mathcal{B}_{1,L}$ and $\mathcal{B}_{2,L}$ corresponds to the terms in H_2^i and $\bar{D}_{\bar{h}} M_2(L)$ thanks to (3.12) and (3.11), while the set $\bigcup_{i\in\mathcal{A}_{L+1}} \bigcup_{r=2}^{\alpha_i} \{(i;r)\}$ corresponds to the terms in $M_2(L+1)$. We conclude from identity (3.13) that relation (3.9) holds. The proof is now complete. □

3.3. An expression for the Malliavin derivatives of the Euler scheme

In this subsection we come back to the solution y of our rough differential equation (1.1). However, for notational sake, we shall omit from now the drift term V_0 in (1.1). Therefore we are reduced to an equation of the form

$$y_t = a + \sum_{i=1}^{d} \int_0^t V_i(y_s) dx_s^i . \tag{3.14}$$

The corresponding Euler scheme y^n (given by (1.2)) can now be expressed as:

$$\delta y^n_{t_k t_{k+1}} = V(y^n_{t_k}) x^1_{t_k t_{k+1}} + \frac{1}{2} \sum_{j=1}^{d} \partial V_j V_j(y^n_{t_k}) \Delta^{2H}, \tag{3.15}$$

where we recall that $\Delta = T/n$ and $t_k = k\Delta$, where the notation $x^1 = \delta x$ is introduced in Definition 2.1, and where we have used the notation in (1.3) for the quantities $\partial V_j V_j(y)$. Notice that for $t \in [\![0, T]\!]$, the approximation y^n_t can also be written as

$$y^n_t = y_0 + \sum_{0 \le t_k < t} V(y^n_{t_k}) x^1_{t_k t_{k+1}} + \frac{1}{2} \sum_{0 \le t_k < t} \sum_{j=1}^{d} \partial V_j V_j(y^n_{t_k}) \Delta^{2H}. \tag{3.16}$$

Remark 3.12. In the general case taking V_0 into account, Eq. (1.1) can be written as

$$d y_t = \tilde{V}(y_t) d\tilde{x}, \qquad y_0 = a,$$

where $\tilde{V}(y_t) = (V_0(y_t), V(y_t))$ and $\tilde{x}_t = (t, x_t)$. Similarly, the Euler scheme (3.15) becomes:

$$\delta y^n_{t_k t_{k+1}} = \tilde{V}(y^n_{t_k}) \delta \tilde{x}_{t_k t_{k+1}} + \frac{1}{2} \sum_{j=1}^{d} \partial V_j V_j(y^n_{t_k}) \Delta^{2H}.$$

Therefore, our discussions in this paper stays unchanged except that the products $V(y_t) dx_t$ and $V(y^n_{t_k}) \delta x_{t_k t_{k+1}}$ are replaced by $\tilde{V}(y_t) d\tilde{x}_t$ and $\tilde{V}(y^n_{t_k}) \delta \tilde{x}_{t_k t_{k+1}}$.

The aim of this section is to find a proper expression for the Malliavin derivatives of y^n.

Remark 3.13. In order to show upper bounds on Malliavin derivatives $\| \bar{D}^L y^n_t \|_{\bar{\mathcal{H}}^{\otimes L}}$ of the Euler scheme we borrow Inamaha's approach in [15]. However, a very special attention to combinatoric issues will have to be paid, due to the fact that we are considering a discrete equation. We have prepared the ground for this in Sections 3.1 and 3.2. Furthermore, note that the uniform continuity in n of Lyons-Itô's map fails in our discrete context. Hence the upper-bound estimates for Eq. (3.15) have to be treated differently for small and large step sizes of the Euler scheme; see Section 4.2.

Remark 3.14. Note that since the Euler scheme y^n is the result of a finite iteration, the existence of Malliavin derivatives is easily obtained via an induction argument. Precisely, assuming that $y^n_{t_k} \in \mathbb{D}^{L,2}$, then by relation (3.15) we have:

$$D^L y^n_{t_{k+1}} = D^L y^n_{t_k} + D^L [V(y^n_{t_k}) x^1_{t_k t_{k+1}}] + \frac{1}{2} \sum_{j=1}^{d} D^L [\partial V_j V_j(y^n_{t_k})] \Delta^{2H}. \tag{3.17}$$

Given that V and its derivatives up to order L are continuous and bounded, the right-hand side of (3.17) also belongs to $\mathbb{D}^{L,2}$. It follows that we have $y^n_{t_{k+1}} \in \mathbb{D}^{L,2}$.

One of the basic ideas in [15] is to use an independent copy b of the fBm x in order to obtain norms in the Cameron–Martin space $\bar{\mathcal{H}}$. With this consideration in mind, we now define a family of processes which will be at the heart of our computations of Malliavin derivatives. We start by introducing a family of operators which will be useful for our future definitions.

Notation 3.15. *Let y^n be the numerical scheme defined in (3.15). Let $f : \mathbb{R}^m \to \mathbb{R}$ be a smooth function. For each $l = 1, \dots, L$, we let ξ^l_t, $t \in [\![0, T]\!]$ be a process with values in $\mathbb{R}^m$. We denote the process $\xi_t = (\xi^1_t, \dots, \xi^L_t) \in \mathbb{R}^{Lm}$, $t \in [\![0, T]\!]$.*

Recall that the trees $\mathcal{A}_L$ are introduced in Definition 3.1. For each $i \in \mathcal{A}_L$ we denote by $c_{L,i}$ and $\tilde{c}_{L,i}$ some constant depending on L and i. Also recall our Notations 3.6–3.7. For $\ell_1 = \ell^i_1$ corresponding to a branch $i \in \mathcal{A}_L$, we have to consider $\partial^{\ell_1} f(y)$ as an element of $\mathcal{L}((\mathbb{R}^m)^{\otimes \ell_1}; \mathbb{R})$. In addition, still for a branch $i \in \mathcal{A}_L$, owing to the relation $\ell_1 = \alpha_i - 1$ we have $\xi^{\ell_2}_s \otimes \cdots \otimes \xi^{\ell_{\alpha_i}}_s \in (\mathbb{R}^m)^{\otimes(\alpha_i - 1)} = (\mathbb{R}^m)^{\otimes \ell_1}$. With these elementary algebra considerations in mind, we define the following notation

$$\begin{aligned} \mathcal{L}^L_{\xi,c} f(y^n_{t_k}) &:= \sum_{i \in \mathcal{A}_L} c_{L,i} \langle \partial^{\ell_1} f(y^n_{t_k}), \xi^{\ell_2}_{t_k} \otimes \cdots \otimes \xi^{\ell_{\alpha_i}}_{t_k} \rangle \\ &= \sum_{i \in \mathcal{A}_L} c_{L,i} \sum_{p_1, \dots, p_{\ell_1} = 1}^{m} \xi^{\ell_2, p_1}_{t_k} \cdots \xi^{\ell_{\alpha_i}, p_{\ell_1}}_{t_k} \frac{\partial^{\ell_1} f(y^n_{t_k})}{\partial y^{(p_{\ell_1})} \cdots \partial y^{(p_1)}}. \end{aligned} \tag{3.18}$$

Moreover, for smooth functions $f : \mathbb{R}^m \to \mathbb{R}$ *and* $g : \mathbb{R}^m \to \mathbb{R}^m$ *we set*

$$\bar{\mathcal{L}}^L_{\xi,c}(\partial f \cdot g)(y^n_{t_k}) := \sum_{i\in\mathcal{A}_L} c_{L,i}\langle(\partial^{\ell_1+1} f \cdot g)(y^n_{t_k}), \xi^{\ell_2}_{t_k} \otimes \cdots \otimes \xi^{\ell_{\alpha_i}}_{t_k}\rangle + \sum_{i\in\mathcal{A}_L}\sum_{r=2}^{\alpha_i} c_{L,i}\langle\partial^{\ell_1} f(y^n_{t_k}), \xi^{\ell_2}_{t_k} \otimes \cdots \otimes \mathcal{L}^{\ell_r}_{\xi,c} g(y^n_{t_k}) \otimes \cdots \otimes \xi^{\ell_{\alpha_i}}_{t_k}\rangle, \tag{3.19}$$

$$\tilde{\mathcal{L}}^L_{\xi,c,\tilde{c}}(\partial f \cdot g)(y^n_{t_k}) = \sum_{i\in\mathcal{A}_L}\sum_{r=2}^{\alpha_i} c_{L,i}\langle\partial^{\ell_1} f(y^n_{t_k}), \xi^{\ell_2}_{t_k} \otimes \cdots \otimes \mathcal{L}^{\ell_r-1}_{\xi,\tilde{c}} g(y^n_{t_k}) \otimes \cdots \otimes \xi^{\ell_{\alpha_i}}_{t_k}\rangle, \tag{3.20}$$

where

$$\partial^{\ell_1+1} f \cdot g(y) = \left(\sum_{p_{\ell_1+1}=1}^{m} \frac{\partial^{\ell_1+1} f(y)}{\partial y^{(p_{\ell_1+1})}\cdots\partial y^{(p_1)}} g^{p_{\ell_1+1}}(y), \quad p_1,\ldots,p_{\ell_1} = 1,\ldots,m\right)$$

and

$$\langle\partial^{\ell_1+1} f \cdot g(y^n_{t_k}), \xi^{\ell_2}_{t_k} \otimes \cdots \otimes \xi^{\ell_{\alpha_i}}_{t_k}\rangle = \sum_{p_1,\ldots,p_{\ell_1+1}=1}^{m} \xi^{\ell_2,p_1}_{t_k}\cdots\xi^{\ell_{\alpha_i},p_{\ell_1}}_{t_k} \frac{\partial^{\ell_1+1} f(y^n_{t_k})}{\partial y^{(p_{\ell_1+1})}\cdots\partial y^{(p_1)}} g^{p_{\ell_1+1}}(y^n_{t_k}).$$

We also set $\mathcal{L}^0_{\xi,c} = \bar{\mathcal{L}}^0_{\xi,c} = Id$ *and* $\tilde{\mathcal{L}}^0_{\xi,c,\tilde{c}} = \tilde{\mathcal{L}}^{-1}_{\xi,c,\tilde{c}} = \mathcal{L}^{-1}_{\xi,c} = \bar{\mathcal{L}}^{-1}_{\xi,c} = 0$.

In order to be able to differentiate our processes of interest in the Malliavin calculus sense, let us label the following regularity assumption on the vector fields V_i.

Hypothesis 3.16. The vector fields $V_1,\ldots,V_d$ are $C_b^{(L+2)\vee 3}(\mathbb{R}^m)$ (bounded together with all their derivatives up to order $(L+2)\vee 3$) for $L \geq 0$.

We can now define a family of paths which will encode the expressions for the Malliavin derivatives of the Euler scheme.

Definition 3.17. For $n \geq 1$ we consider the Euler scheme y^n given by (3.15), and recall that b designates a fBm independent of x. Let $\mathcal{L}^L$, $\bar{\mathcal{L}}^L$, $\tilde{\mathcal{L}}^L$ be the operators introduced in Notation 3.15. Then for $L \geq 0$ we define a discrete process Ξ^L defined for $t = t_k$ and taking values in $\mathbb{R}^m$, given similarly to (3.16) by the iterative equation

$$\delta\Xi^L_{t_kt_{k+1}} = \mathcal{L}^L_{\Xi,c}V(y^n_{t_k})\delta x_{t_kt_{k+1}} + \mathcal{L}^{L-1}_{\Xi,\tilde{c}}V(y^n_{t_k})\delta b_{t_kt_{k+1}} + \frac{1}{2}\sum_{j=1}^{d}\bar{\mathcal{L}}^L_{\Xi,c}\left(\partial V_j \cdot V_j\right)(y^n_{t_k})\Delta^{2H} + \frac{1}{2}\sum_{j=1}^{d}\tilde{\mathcal{L}}^{L-1}_{\Xi,c,\tilde{c}}\left(\partial V_j \cdot V_j\right)(y^n_{t_k})\Delta^{2H}, \tag{3.21}$$

or in the integral form

$$\Xi^L_t = \Xi^L_{t_0} + \sum_{t_0\leq t_k<t}\mathcal{L}^L_{\Xi,c}V(y^n_{t_k})\delta x_{t_kt_{k+1}} + \sum_{t_0\leq t_k<t}\mathcal{L}^{L-1}_{\Xi,\tilde{c}}V(y^n_{t_k})\delta b_{t_kt_{k+1}} + \frac{1}{2}\sum_{t_0\leq t_k<t}\sum_{j=1}^{d}\bar{\mathcal{L}}^L_{\Xi,c}\left(\partial V_j \cdot V_j\right)(y^n_{t_k})\Delta^{2H} + \frac{1}{2}\sum_{t_0\leq t_k<t}\sum_{j=1}^{d}\tilde{\mathcal{L}}^{L-1}_{\Xi,c,\tilde{c}}\left(\partial V_j \cdot V_j\right)(y^n_{t_k})\Delta^{2H},$$

where $t_0 \in [\![0,T]\!]$ is the initial time of the iteration equation and $c = (c_{L,i}, i \in \mathcal{A}_L)$ and $c = (\tilde{c}_{L,i}, i \in \mathcal{A}_L)$ are some constants.

Note that we apply $\mathcal{L}^L_{\Xi,c}$ to every component of V (i.e. $f = V^i_j$ for each i and j) in order to get a $\mathbb{R}^m$-valued element $\mathcal{L}^L_{\Xi,c}V(y^n_{t_k})\delta x_{t_kt_{k+1}}$ in the right-hand side of (3.21). Precisely, we have $\mathcal{L}^L_{\Xi,c}V(y) = (\mathcal{L}^L_{\Xi,c}V^i_j, i = 1\ldots,m, j = 1,\ldots,d)$. $\bar{\mathcal{L}}^L_{\Xi,c}(\partial V_j \cdot V_j)(y)$ and $\tilde{\mathcal{L}}^L_{\Xi,c,\tilde{c}}(\partial V_j \cdot V_j)(y)$ should be interpreted in the same way.

We now state our general expression for the Malliavin derivatives of the Euler scheme. In the following, for conciseness we drop the subscript (Ξ, c) of $\mathcal{L}^L_{\Xi,c}$ and simply write $\mathcal{L}^L$. This simplification is also applied to $\bar{\mathcal{L}}^L_{\Xi,c}$ and $\tilde{\mathcal{L}}^L_{\Xi,c,\tilde{c}}$.

Lemma 3.18. *For* $n \geq 1$ *let* y^n *be the Euler scheme defined by* (3.16). *Assume that Hypothesis 3.16 holds for* $L \geq 1$. *Recall that a notation* $\bar{D}_{\bar{h}}$ *has been introduced in Section 2.2 for the Malliavin derivative with respect to the fBm* x. *We also write* $\hat{D}_{\bar{h}}$ *for the directional derivative with respect to the independent fBm* b. *Let* Ξ^L *be the process introduced in Definition 3.17 with* $c_{L,i} = \tilde{c}_{L,i} = \frac{\ell_2!\cdots\ell_{\alpha_i}!}{L!}$ *for all* $i \in \mathcal{A}_L$. *Then for all* $t \in [\![0,T]\!]$ *the iterated derivative* (2.14) *of* y^n_t *can be expressed as*

$$\bar{D}^L_{\bar{h}}y^n_t = \hat{D}^L_{\bar{h}}\Xi^L_t. \tag{3.22}$$

Proof. According to Definition 3.17 and recalling our convention $\mathcal{L}^{-1} = \tilde{\mathcal{L}}^{-1} = 0$, it is straightforward to see that $\Xi^0 = y^n$.

For $L \geq 1$, consider the process Ξ^L defined by (3.21). Our next endeavor is to find a difference equation satisfied by $\hat{D}^L\Xi^L$. To this aim we differentiate the terms in the right-hand side of (3.21). Note that y^n does not depend on b and therefore $\hat{D}\left[\partial^{\ell_1}V(y^n_{t_k})\right] = 0$. We now prove that Ξ^ℓ belongs to the ℓth Wiener chaos of b, which will be denoted by $\mathcal{K}^b_\ell$ (similarly to that of x in Section 2.2).

This can be done recursively on L using relation (3.21). Namely we assume that $\Xi^p_{t_k} \in \mathcal{K}^b_p$ for all $p \leq L-1$. It can be checked that the terms on the right-hand side of (3.21) belongs to $\mathcal{K}^b_L$. For sake of conciseness, let us focus on the following term of (3.21) (the other terms being left to the patient reader):

$$\mathcal{L}^{L-1}_{\Xi,\tilde{c}} V(y^n_{t_k})\delta b_{t_k t_{k+1}} = \sum_{i\in\mathcal{A}_{L-1}} \tilde{c}_{L-1,i} \left\langle \partial^{\ell_1} V(y^n_{t_k}),\ \Xi^{\ell_2}_{t_k} \otimes \cdots \otimes \Xi^{\ell_{\alpha_i}}_{t_k} \right\rangle \delta b_{t_k t_{k+1}}. \tag{3.23}$$

By the induction assumption the generic term on the right-hand side of (3.23) is in the chaos

$$\mathcal{K}^b_{\sum_{p=2}^{\alpha_i}\ell_p+1} = \mathcal{K}^b_{(L-1)+1} = \mathcal{K}^b_L, \tag{3.24}$$

where we have invoked Remark 3.4 for the second equation. Relation (3.24) thus proves that $\Xi^{\ell}_{t_k} \in \mathcal{K}^b_\ell$ by induction. In particular, $\hat{D}^{\ell'}_{\bar{h}}\Xi^{\ell} = 0$ if $\ell' > \ell$.

In order to differentiate the right-hand side of (3.21), we need to differentiate generic terms of the form $\Xi^{\ell_2}_{t_k} \otimes \cdots \otimes \Xi^{\ell_{\alpha_i}}_{t_k}$ for $i \in \mathcal{A}_L$. It is easily seen that

$$\hat{D}^L_{\bar{h}}\left(\Xi^{\ell_2}_{t_k} \otimes \cdots \otimes \Xi^{\ell_{\alpha_i}}_{t_k}\right) = \sum_{(p_2,\ldots,p_{\alpha_i}):p_2+\cdots+p_{\alpha_i}=L} \frac{L!}{p_2!\cdots p_{\alpha_i}!}\hat{D}^{p_2}_{\bar{h}}\Xi^{\ell_2}_{t_k} \otimes \cdots \otimes \hat{D}^{p_{\alpha_i}}_{\bar{h}}\Xi^{\ell_{\alpha_i}}_{t_k}.$$

However, if $(p_2,\ldots,p_{\alpha_i}) \neq (\ell^i_2,\ldots,\ell^i_{\alpha_i})$, at least one of the ℓ^i_j will be larger than p_j, yielding a null contribution. Therefore the only surviving term in the sum above is

$$\hat{D}^L_{\bar{h}}[\Xi^{\ell_2}_{t_k} \otimes \cdots \otimes \Xi^{\ell_{\alpha_i}}_{t_k}] = \frac{L!}{\ell_2!\cdots\ell_{\alpha_i}!}\cdot \hat{D}^{\ell_2}_{\bar{h}}\Xi^{\ell_2}_{t_k} \otimes \cdots \otimes \hat{D}^{\ell_{\alpha_i}}_{\bar{h}}\Xi^{\ell_{\alpha_i}}_{t_k}. \tag{3.25}$$

The reader is referred to [15] for more details about the above computation. Note that the number of ways to assign the L operators $\hat{D}_{\bar{h}},\ldots,\hat{D}_{\bar{h}}$ into groups of sizes $\ell_2,\ldots,\ell_{\alpha_i}$ is $\frac{L!}{\ell_2!\cdots\ell_{\alpha_i}!}$, which explains the multiplicative constant in the equation.

With (3.25) in hand, we are now ready to differentiate the right-hand side of (3.21). For the first term, using definition (3.18) of $\mathcal{L}^L$ we get

$$\hat{D}^L_{\bar{h}}\mathcal{L}^L V(y^n_{t_k}) = \sum_{i\in\mathcal{A}_L}\langle\partial^{\ell_1}V(y^n_{t_k}), \hat{D}^{\ell_2}_{\bar{h}}\Xi^{\ell_2}_{t_k} \otimes \cdots \otimes \hat{D}^{\ell_{\alpha_i}}_{\bar{h}}\Xi^{\ell_{\alpha_i}}_{t_k}\rangle.$$

Along the same lines and resorting to (3.19) for the definition of $\bar{\mathcal{L}}^L$, it is easily checked that

$$\begin{aligned}\hat{D}^L_{\bar{h}}\bar{\mathcal{L}}^L(\partial V\cdot V)(y^n_{t_k}) = &\sum_{i\in\mathcal{A}_L}\langle(\partial^{\ell_1+1}V\cdot V)(y^n_{t_k}), \hat{D}^{\ell_2}_{\bar{h}}\Xi^{\ell_2}_{t_k} \otimes \cdots \otimes \hat{D}^{\ell_{\alpha_i}}_{\bar{h}}\Xi^{\ell_{\alpha_i}}_{t_k}\rangle \\ &+ \sum_{i\in\mathcal{A}_L}\sum_{r=2}^{\alpha_i}\langle\partial^{\ell_1}V(y^n_{t_k}), \hat{D}^{\ell_2}_{\bar{h}}\Xi^{\ell_2}_{t_k} \otimes \cdots \otimes \hat{D}^{\ell_r}_{\bar{h}}\mathcal{L}^{\ell_r}V(y^n_{t_k}) \otimes \cdots \otimes \hat{D}^{\ell_{\alpha_i}}_{\bar{h}}\Xi^{\ell_{\alpha_i}}_{t_k}\rangle.\end{aligned}$$

Similarly, for the second term on the right-hand side of (3.21), we end up with

$$\hat{D}^L_{\bar{h}}\mathcal{L}^{L-1}V(y^n_{t_k})\delta b_{t_k t_{k+1}} = L\sum_{i\in\mathcal{A}_{L-1}}\langle\partial^{\ell_1}V(y^n_{t_k}), \hat{D}^{\ell_2}_{\bar{h}}\Xi^{\ell_2}_{t_k} \otimes \cdots \otimes \hat{D}^{\ell_{\alpha_i}}_{\bar{h}}\Xi^{\ell_{\alpha_i}}_{t_k} \otimes \delta\bar{h}_{t_k t_{k+1}}\rangle.$$

Note also that the fourth term on the right-hand side of (3.21) is in the $(L-2)$th chaos of b and thus has zero $\hat{D}^L$ derivative. Differentiating both sides of (3.21) and taking into account the above computations, we have thus obtained

$$\begin{aligned}\hat{D}^L_{\bar{h}}\Xi^L_t = &\sum_{0\leq t_k<t}\sum_{i\in\mathcal{A}_L}\langle\partial^{\ell_1}V(y^n_{t_k}), \hat{D}^{\ell_2}_{\bar{h}}\Xi^{\ell_2}_{t_k} \otimes \cdots \otimes \hat{D}^{\ell_{\alpha_i}}_{\bar{h}}\Xi^{\ell_{\alpha_i}}_{t_k}\rangle\delta x_{t_k t_{k+1}} \\ &+ L\sum_{0\leq t_k<t}\sum_{i\in\mathcal{A}_{L-1}}\langle\partial^{\ell_1}V(y^n_{t_k}), \hat{D}^{\ell_2}_{\bar{h}}\Xi^{\ell_2}_{t_k} \otimes \cdots \otimes \hat{D}^{\ell_{\alpha_i}}_{\bar{h}}\Xi^{\ell_{\alpha_i}}_{t_k} \otimes \delta\bar{h}_{t_k t_{k+1}}\rangle \\ &+ \frac{1}{2}\sum_{0\leq t_k<t}\sum_{j=1}^{d}\sum_{i\in\mathcal{A}_L}\Big(\langle(\partial^{\ell_1+1}V_j\cdot V_j)(y^n_{t_k}), \hat{D}^{\ell_2}_{\bar{h}}\Xi^{\ell_2}_{t_k} \otimes \cdots \otimes \hat{D}^{\ell_{\alpha_i}}_{\bar{h}}\Xi^{\ell_{\alpha_i}}_{t_k}\rangle \\ &\qquad + \sum_{r=2}^{\alpha_i}\langle\partial^{\ell_1}V_j(y^n_{t_k}), \hat{D}^{\ell_2}_{\bar{h}}\Xi^{\ell_2}_{t_k} \otimes \cdots \otimes \hat{D}^{\ell_r}_{\bar{h}}\mathcal{L}^{\ell_r}V_j(y^n_{t_k}) \otimes \cdots \otimes \hat{D}^{\ell_{\alpha_i}}_{\bar{h}}\Xi^{\ell_{\alpha_i}}_{t_k}\rangle\Big)\Delta^{2H}.\end{aligned} \tag{3.26}$$

Let us now differentiate y^n_t according to its definition (3.16). To this aim resorting to the fact that $\delta x_{t_k t_{k+1}}$ is in the first chaos of x, we get

$$\begin{aligned}\bar{D}^L_{\bar{h}}y^n_t = &\sum_{0\leq t_k<t}\bar{D}^L_{\bar{h}}V(y^n_{t_k})\delta x_{t_k t_{k+1}} + L\sum_{0\leq t_k<t}\bar{D}^{L-1}_{\bar{h}}V(y^n_{t_k})\delta\bar{h}_{t_k t_{k+1}} \\ &+\frac{1}{2}\sum_{0\leq t_k<t}\sum_{j=1}^{d}\bar{D}^L_{\bar{h}}\left[\left(\partial V_j\cdot V_j\right)\left(y^n_{t_k}\right)\right]\Delta^{2H}.\end{aligned} \tag{3.27}$$

Next we differentiate the terms $V(y^n_{t_k})$ and $(\partial V_j\cdot V_j)(y^n_{t_k})$ thanks to Lemma 3.8. It is readily checked that we get exactly the same expression as (3.26). This shows our claim (3.22) and finishes our proof. □

4. Upper-bound estimates of Malliavin derivatives of the Euler scheme

In Section 2.2 we have recalled some known results on the Malliavin derivative of the solution y to (1.1). Note that upper bounds for higher order derivatives of y are obtained in [4,15]. With the preparation in Section 3 we are ready to extend those estimates to the Euler scheme approximations y^n.

4.1. Some auxiliary results

This subsection is dedicated to some necessary auxiliary results. Throughout the subsection we fix an integer $N > 0$. Recall that Ξ^L is defined by the iterative equation in Definition 3.17. We start by introducing a process related to Ξ^L, $L = 0, 1, \dots, N$. In the following we assume that Hypothesis 3.16 holds with L replaced by N. Namely, we assume that $V \in C^{(N+2)\vee 3}(\mathbb{R}^m)$.

Definition 4.1. For each $s \in [0, T]$ we define P_s^L to be the maximum among the quantities of the form $|\Xi_s^{i_1}| \times \cdots \times |\Xi_s^{i_{N_0}}|$ with $N_0 > 0$ such that $i_1, \dots, i_{N_0} \in \{1, \dots, L\}$ and $i_1 + \cdots + i_{N_0} \le L$. We also define $P^0 \equiv 1$ and $P^{-1} \equiv 0$.

The following result follows immediately from the definition of P^L:

Lemma 4.2. *The following three inequalities hold for $s \in [\![0, T]\!]$:*

$$P^L \ge P^{L'}, \qquad P_s^L \times P_s^{L'} \le P_s^{L+L'} \quad \text{for } L \ge L' \ge 0, \quad \text{and} \quad |\Xi_s^L| \le P_s^L \quad \text{for } L \ge 1.$$

Let us now fix some constants which we will make extensive use of: For $f \in C_b^{N+2}$, $N \in \mathbb{N}$ we set

$$C_f^0 = \sup_{\tau \le N+2} \|\partial_\tau f\|_\infty, \tag{4.1}$$

where $\|\cdot\|_\infty$ denotes the sup norm for continuous functions. Also recall that for $i \in \mathcal{A}_L$ the constants α_i and $c_{L,i}$ are defined respectively in Notation 3.3 and Lemma 3.18, and C_V^0 is defined in (4.1). Then for $L = 1, 2, \dots, N$ we define

$$C_{L,V}^1 = \sum_{i \in \mathcal{A}_L} c_{L,i} C_V^0, \quad C_{0,V}^1 = C_V^0, \quad C_{-1,V}^1 = 0, \tag{4.2}$$

$$C_{L,V}^2 = \sum_{i \in \mathcal{A}_L} c_{L,i} C_V^0 \left(C_V^0 + \sum_{r=2}^{\alpha_i} C_{\ell_r,V}^1 \right), \quad C_{0,V}^2 = 2(C_V^0)^2, \quad C_{-1,V}^2 = 0, \tag{4.3}$$

$$C_{L,V}^3 = \sum_{i \in \mathcal{A}_L} \sum_{r=2}^{\alpha_i} c_{L,i} C_V^0 C_{\ell_r - 1,V}^1, \quad C_{0,V}^3 = 0, \quad C_{-1,V}^3 = 0. \tag{4.4}$$

We will also resort to the following constants:

$$K_1^L = C_{L,V}^1 + C_{L-1,V}^1 + 1, \qquad K_2^L = C_{L,V}^2 + C_{L-1,V}^2 + C_{L,V}^3 + C_{L-1,V}^3 + 1. \tag{4.5}$$

Next we introduce a family of sets in the following way, for $L \ge 1$:

$$S_L = \{(L', \ell_1, \dots, \ell_{L'}) : L' \in \mathbb{N}_+, \ell_1, \dots, \ell_{L'} \in \mathbb{N}_+, \ \ell_1 + \cdots + \ell_{L'} = L\}.$$

Related to this definition, we define another family of constants:

$$C_L^4 = \max_{(L', \ell_1, \dots, \ell_{L'}) \in S_L} 2^{L+1} \times K_1^{\ell_1} \times \cdots \times K_1^{\ell_{L'}}, \tag{4.6}$$

and

$$C_{f,L}^5 = 2 \sum_{i \in \mathcal{A}_L} c_{L,i} C_f^0 \left(K_1^1 + \sum_{r=2}^{\alpha_i} K_1^{\ell_r} \right) (1 + C_L^4), \qquad C_{f,0}^5 = C_f^0 K_1^0. \tag{4.7}$$

Remark 4.3. The constants introduced above will appear in our proof for the upper bound of Ξ^L. We will see that because our proof is an induction argument, it is important to keep track of these constants.

With this additional notation, in the following we derive an upper-bound estimate for the product of Ξ^L.

Lemma 4.4. *Let ω be a control function on $[\![0, T]\!]$. Recalling our definition (2.1) for the operator δ, let $(s, u) \in \mathcal{S}_2([\![0, T]\!])$ be such that*

$$\omega(s, u)^{1/p} \le 1/2, \qquad \text{and} \qquad |\delta \Xi_{su}^L| \le K_1^L P_s^L \omega(s, u)^{1/p}, \quad L = 1, \dots, N. \tag{4.8}$$

Let $L_0 \le N$ and $(N', \ell_1, \dots, \ell_{N'}) \in S_{L_0}$. Then we have

$$|\delta \left(\Xi^{\ell_1} \otimes \cdots \otimes \Xi^{\ell_{N'}} \right)_{su}| \le C_{L_0}^4 \cdot P_s^{L_0} \cdot \omega(s, u)^{1/p}. \tag{4.9}$$

Proof. We first note that a straightforward computation shows that $\delta\left(\Xi^{\ell_1}\otimes\cdots\otimes\Xi^{\ell_{N'}}\right)_{su}$ is equal to the summation of products of the quantities of the forms $\Xi_s^{\ell_r}$ and $\delta\Xi_{su}^{\ell_r}$ with $r=1,\dots,N'$. Apply Lemma 4.2 to Ξ^{ℓ_r} and condition (4.8) to $|\delta\Xi_{st}^{\ell_r}|$. Also take into account that $\omega(s,u)^{1/p}\le 1/2$, according to (4.8). We obtain that each product in the summation is bounded by

$$\left(\max_{(N',\ell_1,\dots,\ell_{N'})\in S_{L_0}} K_1^{\ell_1}\times\cdots\times K_1^{\ell_{N'}}\right)\cdot P_s^{L_0}\cdot 2\omega(s,u)^{1/p}.$$

Note that there are at most $2^{L_0}-1$ terms in the summation. Hence owing to the definition of $C^4_{L_0}$ in (4.6) we obtain the desired estimate (4.9). □

Following is an estimate for the Euler scheme y^n:

Lemma 4.5. *Let ω be a control function on $[\![0,T]\!]$ and consider the Euler scheme in (3.15). Let $(s,u)\in\mathcal{S}_2([\![0,T]\!])$ be such that*

$$|\delta y^n_{su}|\le K_1^0\omega(s,u)^{1/p}. \tag{4.10}$$

Then for $0\le L\le N$ and recalling that C_f^0 is defined by (4.1), we have:

$$|\delta\left(\partial^L f(y^n_\cdot)\right)_{su}|\le C_f^0K_1^0\omega(s,u)^{1/p}.$$

Proof. The lemma follows immediately from the mean value theorem and the condition (4.10). □

Recall that we have defined the solution to (3.14) as the controlled process in (2.6). The following lemma improves our Lemma 4.5 when y is a discrete controlled process.

Lemma 4.6. *Let ω be a control function on $[\![0,T]\!]$. Let $(s,u)\in\mathcal{S}_2([\![0,T]\!])$ be such that*

$$|\delta y^n_{su}|\le K_1^0\omega(s,u)^{1/p}\qquad\textit{and}\qquad|\delta y^n_{su}-V(y^n_s)\delta x_{su}|\le K_2^0\omega(s,u)^{2/p}. \tag{4.11}$$

Then the following relation holds true for $0\le L\le N$:

$$|\delta\left(\partial^L V(y^n_\cdot)\right)_{su}-(\partial^{L+1}VV)(y^n_s)\delta x_{su}|\le C_V^0\left(K_1^0K_1^0+K_2^0\right)\omega(s,u)^{2/p}. \tag{4.12}$$

Proof. The lemma follows from the application of an obvious second order Taylor expansion, as well as the conditions in (4.11). Precisely, let f be a continuous function from $\mathbb{R}^m\mapsto\mathbb{R}$ whose first- and second-order derivatives exist and are continuous. Then the elementary mean value theorem shows that we have the relation:

$$|f(b)-f(a)-\partial f(a)(b-a)|\le\|\partial^2 f\|_\infty\cdot|b-a|^2. \tag{4.13}$$

Taking $f=\partial^L V$, $a=y^n_s$ and $b=y^n_u$ in (4.13) we obtain the relation:

$$\begin{aligned}|\delta\left(\partial^L V(y^n_\cdot)\right)_{su}-\partial^{L+1}V(y^n_s)\delta y^n_{su}|&\le\|\partial^{L+2}V\|_\infty\cdot|\delta y^n_{su}|^2\\&\le C_V^0\cdot|K_1^0|^2\omega(s,u)^{2/p},\end{aligned}$$

where in the second inequality we have used the first condition in (4.11) and the fact that $\|\partial^{L+2}V\|_\infty\le C_V^0$. In order to prove (4.12) it thus remains to show that

$$|\partial^{L+1}V(y^n_s)\delta y^n_{su}-(\partial^{L+1}VV)(y^n_s)\delta x_{su}|\le C_V^0\cdot K_2^0\omega(s,u)^{2/p}. \tag{4.14}$$

It is easy to see that (4.14) follows by applying the second condition in (4.11). We thus conclude that (4.12) holds. □

Recall that $\mathcal{L}^L_{\Xi,c}$, $\bar{\mathcal{L}}^L_{\Xi,c}$, $\tilde{\mathcal{L}}^L_{\Xi,c,\tilde c}$ are defined in Notation 3.15. For the sake of simplicity we will drop the subscript and write $\mathcal{L}^L$, $\bar{\mathcal{L}}^L$, $\tilde{\mathcal{L}}^L$ in the following series of lemmas.

Lemma 4.7. *Recall that y^n is the numerical scheme given by (3.15), and that Hypothesis 3.16 holds true. For $L\ge 0$ let $C^1_{L,V}$, $C^2_{L,V}$ and $C^3_{L,V}$ be the constants defined by (4.2)–(4.4) and recall that P_s^L is introduced in Definition 4.1. Then the following holds true for all $s\in[\![0,T]\!]$ and $L\ge 0$:*

$$|\mathcal{L}^L V(y^n_s)|\le C^1_{L,V}P_s^L, \tag{4.15}$$

$$|\bar{\mathcal{L}}^L(\partial VV(y^n_s))|\le C^2_{L,V}P_s^L, \tag{4.16}$$

$$|\tilde{\mathcal{L}}^L(\partial VV)(y^n_s)|\le C^3_{L,V}P_s^L. \tag{4.17}$$

Proof. An application of Lemma 4.2 to (3.18) yields

$$|\mathcal{L}^L V(y^n_s)|\le\sum_{i\in\mathcal{A}_L}c_{L,i}C_V^0P_s^L.$$

Relation (4.15) then follows immediately from the definition of $C^1_{L,V}$ in (4.2). Now apply Lemma 4.2 to (3.19) as before, and then apply (4.15) in order to handle the terms $\mathcal{L}^{\ell_r} g(y^n_{t_k})$ in the right-hand side of (3.19). We obtain:

$$|\bar{\mathcal{L}}^L(\partial V V(y^n_s))| \le \sum_{i\in\mathcal{A}_L} c_{L,i} C^0_V \left(C^0_V P^L_s + \sum_{r=2}^{\alpha_i} C^1_{\ell_r,V} P^L_s \right).$$

Hence resorting to the definition of $C^2_{L,V}$ in (4.3) we obtain relation (4.16). The last relation (4.17) can be shown in a similar way. We have

$$|\tilde{\mathcal{L}}^L(\partial V V)(y^n_s)| \le \sum_{i\in\mathcal{A}_L} \sum_{r=2}^{\alpha_i} c_{L,i} C^0_V C^1_{\ell_r-1,V} P^L_s.$$

Relation (4.17) then follows from the definition of $C^3_{L,V}$ in (4.4). □

In the following we consider the increments of processes in Lemma 4.7.

Lemma 4.8. *Let ω be a control function on $[\![0,T]\!]$. Recall that we write $\mathcal{L}^L$, $\bar{\mathcal{L}}^L$, $\tilde{\mathcal{L}}^L$ for $\mathcal{L}^L_{\Xi,c}$, $\bar{\mathcal{L}}^L_{\Xi,c}$, $\tilde{\mathcal{L}}^L_{\Xi,c,\tilde{c}}$. Assume that (4.8) and (4.10) hold (notice that (4.8) for $L=0$ in fact implies (4.10)) for some $(s,u)\in\mathcal{S}_2([\![0,T]\!])$. Then we have the following relations $L\ge 0$:*

$$|\delta\left(\mathcal{L}^L f(y^n_\cdot)\right)_{su}| \le C^5_{f,L} P^L_s \omega(s,u)^{1/p}, \tag{4.18}$$

$$\left|\delta\left(\bar{\mathcal{L}}^L(\partial V V)(y^n_\cdot)\right)_{su}\right| \le C^6_{V,L} P^L_s \omega(s,u)^{1/p}, \tag{4.19}$$

$$\left|\delta\left(\tilde{\mathcal{L}}^L(\partial V V)(y^n_\cdot)\right)_{su}\right| \le C^7_{V,L} P^L_s \omega(s,u)^{1/p}. \tag{4.20}$$

Proof. Recall that $\mathcal{L}^L$ is defined in (3.18). For two functions f, g: $[0,T]\to\mathbb{R}$ and for the operator δ defined by (2.1), it is easily seen that

$$\delta(fg)_{su} = \delta f_{su}\, g_u + f_s\, \delta g_{su}. \tag{4.21}$$

Invoking repeatedly this relation and consistently replacing the terms g_u above by g_s we end up with the relation

$$\delta\left(\mathcal{L}^L f(y^n_\cdot)\right)_{su} = J^1_{su} + J^2_{su}, \tag{4.22}$$

where the terms J^1_{su} and J^2_{su} are defined by

$$J^1_{su} = \sum_{i\in\mathcal{A}_L} c_{L,i}\Big(\langle \delta\left(\partial^{\ell_1} f(y^n_\cdot)\right)_{su}, \Xi^{\ell_2}_s \otimes\cdots\otimes \Xi^{\ell_{\alpha_i}}_s\rangle + \sum_{r=2}^{\alpha_i} \langle \partial^{\ell_1} f(y^n_s), \Xi^{\ell_2}_s\otimes\cdots\otimes\delta\Xi^{\ell_r}_{su}\otimes\cdots\otimes\Xi^{\ell_{\alpha_i}}_s\rangle\Big), \tag{4.23}$$

$$J^2_{su} = \sum_{i\in\mathcal{A}_L} c_{L,i}\Big\{ \Big\langle \delta\left(\partial^{\ell_1} f(y^n_\cdot)\right)_{su}, \delta\left(\Xi^{\ell_2}_\cdot\otimes\cdots\otimes\Xi^{\ell_{\alpha_i}}_\cdot\right)_{su}\Big\rangle + \sum_{r=2}^{\alpha_i} \Big\langle \partial^{\ell_1} f(y^n_s), \Xi^{\ell_2}_s\otimes\cdots\otimes\delta\Xi^{\ell_r}_{su}\otimes\delta\left(\Xi^{\ell_{r+1}}_\cdot\otimes\cdots\otimes\Xi^{\ell_{\alpha_i}}_\cdot\right)_{su}\Big\rangle\Big\}. \tag{4.24}$$

In order to bound J^1_{su} above we apply Lemma 4.2 to the quantities Ξ^r, and Lemma 4.5 to $\delta\left(\partial^{\ell_1} f(y^n_\cdot)\right)_{su}$ in (4.23). We get

$$|J^1_{su}| \le \sum_{i\in\mathcal{A}_L} c_{L,i}\left(C^0_f K^0_1 P^L_s + \sum_{r=2}^{\alpha_i} C^0_f K^{\ell_r}_1 P^L_s\right)\omega(s,u)^{1/p}. \tag{4.25}$$

We can bound J^2_{su}, in a similar way. As before we apply Lemmas 4.2 and 4.5 respectively to Ξ^r_s and $\delta\left(\partial^{\ell_1} f(y^n_\cdot)\right)_{su}$, and then apply Lemma 4.4 to the quantity $\delta(\Xi^{\ell_{r+1}}_\cdot\otimes\cdots\otimes\Xi^{\ell_{\alpha_i}}_\cdot)_{su}$. Taking into acount the assumption $\omega(s,u)^{1/p}<1/2$ in (4.8), we obtain

$$|J^2_{su}| \le 2\sum_{i\in\mathcal{A}_L} c_{L,i} C^0_f \left(K^0_1 C^4_L P^L_s + \sum_{r=2}^{\alpha_i} K^{\ell_r}_1 C^4_L P^L_s\right)\cdot\omega(s,u)^{1/p}. \tag{4.26}$$

Combining the estimates (4.25) and (4.26) in (4.22) and recalling the definition of $C^5_{f,L}$ in (4.7), relation (4.18) is now easily obtained. □

We end this subsection with a result on the remainder of $\mathcal{L}^L V(y^n_s)$ considered as a controlled process:

Lemma 4.9. *Let ω be a control function on $[\![0,T]\!]$. Suppose that*

$$|\delta\Xi^L_{su} - \mathcal{L}^L V(y^n_s)\delta x_{su} - \mathcal{L}^{L-1}V(y^n_s)\delta b_{su}| \le K^L_2 P^L_s \omega(s,u)^{2/p}, \qquad L=0,1,\ldots,N \tag{4.27}$$

for some $(s,u)\in\mathcal{S}_2([\![0,T]\!])$, where P_s^L is the quantity given in Definition 4.1. Suppose that (4.8) and (4.11) holds for the same (s,u). Then we have the following relation for $L\geq -1$:

$$\left|\delta\left(\mathcal{L}^L V(y_\cdot^n)\right)_{su}-\bar{\mathcal{L}}^L\,(\partial VV)\,(y_s^n)\delta x_{su}-\tilde{\mathcal{L}}^L\,(\partial VV)\,(y_s^n)\delta b_{su}\right|\leq C_{V,L}^8 P_s^L\omega(s,u)^{2/p},$$

where we define the constants $\{C_{V,L}^8, L\geq -1\}$ by

$$C_{V,L}^8=2\sum_{i\in\mathcal{A}_L}c_{L,i}C_V^0\left(K_1^0K_1^0+K_2^0+K_0^1C_L^4+\sum_{r=2}^{\alpha_i}(K_2^{\ell_r}+K_1^{\ell_r}C_L^4)\right),$$
$$C_{V,0}^8=C_V^0(K_1^0K_1^0+K_2^0),\qquad C_{V,-1}^8=0.$$

Proof. Recall that $\mathcal{L}^L$, $\bar{\mathcal{L}}^L$ and $\tilde{\mathcal{L}}^L$ are introduced in (3.18)–(3.20). Similarly to the beginning of the proof of Lemma 4.8, we apply relation (4.21) and replace the terms g_u by g_s. This leads to a decomposition of the form

$$\delta\left(\mathcal{L}^L V(y_\cdot^n)\right)_{su}-\bar{\mathcal{L}}^L\,(\partial VV)\,(y_s^n)\delta x_{su}-\tilde{\mathcal{L}}^L\,(\partial VV)\,(y_s^n)\delta b_{su}=\sum_{\ell=1}^4 J_{su}^\ell, \tag{4.28}$$

where J_{su}^1 and J_{su}^2 are defined in (4.23)–(4.24), and we also introduce the increments

$$J_{su}^3=-\sum_{i\in\mathcal{A}_L}c_{L,i}\langle(\partial^{\ell_1+1}VV)(y_s^n)\delta x_{su},\Xi_s^{\ell_2}\otimes\cdots\otimes\Xi_s^{\ell_{\alpha_i}}\rangle$$
$$J_{su}^4=-\sum_{i\in\mathcal{A}_L}\sum_{r=2}^{\alpha_i}c_{L,i}\langle\partial^{\ell_1}V(y_s^n),\Xi_s^{\ell_2}\otimes\cdots\otimes\left(\mathcal{L}^{\ell_r}V(y_s^n)\delta x_{su}+\mathcal{L}^{\ell_r-1}V(y_s^n)\delta b_{su}\right)\otimes\cdots\otimes\Xi_s^{\ell_{\alpha_i}}\rangle.$$

Notice that one can combine J^1, J^3 and J^4 into

$$\begin{aligned}&J_{su}^1+J_{su}^3+J_{su}^4\\&=\sum_{i\in\mathcal{A}_L}c_{L,i}\left\langle\left(\delta\left(\partial^{\ell_1}V(y_\cdot^n)\right)_{su}-(\partial^{\ell_1+1}VV)(y_s^n)\delta x_{su}\right),\Xi_s^{\ell_2}\otimes\cdots\otimes\Xi_s^{\ell_{\alpha_i}}\right\rangle\\&\quad+\sum_{i\in\mathcal{A}_L}\sum_{r=2}^{\alpha_i}c_{L,i}\left\langle\partial^{\ell_1}V(y_s^n),\Xi_s^{\ell_2}\otimes\cdots\otimes\left(\delta\Xi_{su}^{\ell_r}-\mathcal{L}^{\ell_r}V(y_s^n)\delta x_{su}-\mathcal{L}^{\ell_r-1}V(y_s^n)\delta b_{su}\right)\otimes\cdots\otimes\Xi_s^{\ell_{\alpha_i}}\right\rangle.\end{aligned}$$

We are now in a position to apply Lemma 4.6 and condition (4.27) in order to get:

$$\begin{aligned}|J_{su}^1+J_{su}^3+J_{su}^4|\leq&\sum_{i\in\mathcal{A}_L}c_{L,i}P_s^LC_V^0(K_1^0K_1^0+K_2^0)\omega(s,u)^{2/p}\\&+\sum_{i\in\mathcal{A}_L}\sum_{r=2}^{\alpha_i}c_{L,i}C_V^0K_2^{\ell_r}P_s^L\omega(s,u)^{2/p}.\end{aligned}$$

Combining this estimate and the relation (4.26) in (4.28), and taking into account the definition of the constant $C_{V,L}^8$ we obtain the desired relation. □

4.2. Upper-bound estimate of the derivatives

In this subsection, we derive a uniform upper-bound estimate for the Malliavin derivatives of y^n. For a given threshold $\alpha>0$, our estimates consist of three parts, which are estimates of the derivative over the steps of (1) small size ($\ll\alpha$); (2) medium size ($\approx\alpha$); (3) large size ($\gg\alpha$).

We now specify our threshold parameter α. Towards this aim, recall that K_1^L and K_2^L for $L=0,1,\ldots,N$, are introduced in (4.5) and K_μ is defined in (2.7). We also define:

$$K_3^L=K_\mu K_4^L\vee 1,\quad\text{where}\quad K_4^L=(C_{V,L}^8+C_{V,L-1}^8+4C_{V,L}^6+4C_{V,L}^7)\vee 1. \tag{4.29}$$

Then we shall resort to a positive constant α such that:

$$\alpha^{1/p}=\min\{1/2,1/K_2^L,1/K_3^L,L=0,1,\ldots,N\}. \tag{4.30}$$

Eventually we introduce some second chaos processes which play a prominent role in the analysis of Euler schemes (see [18]). Namely for $[s,t]\in[\![0,T]\!]$ we set

$$q_{st}^{ij}=\sum_{s\leq t_k<t}\left(x_{t_kt_{k+1}}^{2,ij}-\frac{1}{2}\Delta^{2H}\mathbf{1}_{\{i=j\}}\right),\quad\text{and}\quad q_{st}^{b,ij}=\sum_{s\leq t_k<t}\left(b_{t_kt_{k+1}}^{2,ij}-\frac{1}{2}\Delta^{2H}\mathbf{1}_{\{i=j\}}\right). \tag{4.31}$$

For convenience we also introduce a specific notation for the cross integrals between the independent fractional Brownian motions x and b. Namely we first introduce a Gaussian process w which encompasses the coordinates of both the driving nose x and the extra noise b. Specifically we define

$$\delta\mathrm{w}_{st}:=(\delta\mathrm{w}_{st}^1,\ldots,\delta\mathrm{w}_{st}^{2d}):=(\delta x_{st}^1,\ldots,\delta x_{st}^d,\delta b_{st}^1,\ldots,\delta b_{st}^d). \tag{4.32}$$

Then writing w^2 for the iterated integral of w (see Definition 2.1) we set

$$\tilde{\mathrm{w}}_{st} = \left(\mathrm{w}^{2,ij}_{st},\ i = d+1,\dots,2d,\ j = 1,\dots,d\right) \tag{4.33}$$

$$\tilde{q}_{st} = \left(\sum_{s\le t_k<t} \mathrm{w}^{2,ij}_{t_k t_{k+1}},\ i = d+1,\dots,2d,\ j = 1,\dots,d\right), \tag{4.34}$$

for $s,t \in \mathcal{S}_2([0,T])$. With this notation in hand, we now state our bound on derivatives of the Euler scheme.

Theorem 4.10. *Let y and y^n be the solution of the SDE (3.14) and the corresponding Euler scheme (3.15), respectively. Let Ξ be given in Definition 3.17. Suppose that $V \in C_b^{(L+2)\vee 3}$ for some integer $L \ge 0$. Let $p > 1/H$. Let $w = (x,b)$ be defined in (4.32) and let $\mathbf{w} := S_2(w)$ be the rough path lifted from w. We introduce a control ω by*

$$\omega(s,t) = \|\mathbf{w}\|^p_{p\text{-var};[\![s,t]\!]} + \|q\|^{p/2}_{p/2\text{-var};[\![s,t]\!]} + \|q^b\|^{p/2}_{p/2\text{-var};[\![s,t]\!]}, \qquad (s,t) \in \mathcal{S}_2([\![0,T]\!]), \tag{4.35}$$

where q is defined in (4.31). Denote $s_0 = 0$. Then given s_j, we define s_{j+1} recursively as

$$s_{j+1} = \begin{cases} s_j + \Delta, & \text{if } \omega(s_j, s_j+\Delta) > \alpha \\ \max\{u \in [\![0,T]\!] : u > s_j \text{ and } \omega(s_j,u) \le \alpha\}, & \text{if } \omega(s_j, s_j+\Delta) \le \alpha \end{cases} \tag{4.36}$$

Next we split the set of s_j's as

$$S_0 = \{s_j : \alpha/2 \le \omega(s_j,s_{j+1}) \le \alpha\}; \quad S_1 = \{s_j : \omega(s_j,s_{j+1}) < \alpha/2\}; \tag{4.37}$$

$$S_2 = \{s_j : \omega(s_j,s_{j+1}) > \alpha\}. \tag{4.38}$$

Then we have:

(a) *The following relation holds for all $(s,t) \in \mathcal{S}_2([\![0,T]\!])$:*

$$\|\Xi^L\|_{p\text{-var},[\![s,t]\!]} \le K \cdot \omega(s,t)^{1/p} |S_0 \cup S_1 \cup S_2| \cdot (\mathcal{M}_0 \cdot \mathcal{M}_1 \cdot \mathcal{M}_2)^L, \tag{4.39}$$

where we have set

$$\mathcal{M}_0 = \prod_{s_j\in S_0} \left(K\omega(s_j,s_{j+1})^{1/p} + 1\right), \qquad \mathcal{M}_1 = \prod_{s_j\in S_1} \left(K\omega(s_j,s_{j+1})^{1/p} + 1\right),$$

$$\mathcal{M}_2 = \prod_{s_j\in S_2} \left(K|\delta w_{s_j s_{j+1}}| + K\Delta^{2H} + 1\right), \tag{4.40}$$

and K is a constant independent of n.

(b) *For $(s,t) \in \mathcal{S}_2([\![s_j,s_{j+1}]\!])$ such that $s_j \in S_0 \cup S_1$ we have*

$$|\delta\Xi^L_{st} - \mathcal{L}^L V(y^n_s)\delta x_{st} - \mathcal{L}^{L-1}V(y^n_s)\delta b_{st}| \le K\omega(s,t)^{2/p} \cdot P^L_s.$$

Remark 4.11. The reader might argue that the right-hand side of (4.39) still depends on n. However, in our companion paper [17] we will show that this right-hand side is uniformly integrable in n. Thus Ξ^L_t is also uniformly integrable in n.

Remark 4.12. The fact that the quantities $\mathcal{M}_0$, $\mathcal{M}_1$, $\mathcal{M}_2$ are finite can be argued as follows: Note that S_0, S_1, S_2 are contained in the finite set $[\![0,T]\!]$, and therefore the number of components of the products in the definitions of $\mathcal{M}_0$, $\mathcal{M}_1$, $\mathcal{M}_2$ in (4.40) are finite. On the other hand, the control function ω is defined over the discrete interval $[\![0,T]\!]$, and thus it is also finite. We conclude that $\mathcal{M}_0$, $\mathcal{M}_1$, $\mathcal{M}_2$ are all finite.

Before proving Theorem 4.10, let us state a corollary giving the actual bound on the Malliavin derivatives of y^n. Recall that b is an independent copy of the fBm x as given in Definition 3.17 and we denote $\hat{D}$ the Malliavin derivative operator for b. We denote $\hat{\mathbb{E}}$ and $\hat{\mathbb{D}}^{L,p}$ the expectation and the Sobolev space corresponding to b, respectively.

Corollary 4.13. *Under the same conditions as for Theorem 4.10 and recalling our notation from Section 2.2, we have*

$$\sup_{n\in\mathbb{N}} \|\bar{D}^L y^n_t\|_{\bar{\mathcal{H}}^{\otimes L}} \le K \cdot \omega(0,T)^{1/p} |S_0 \cup S_1 \cup S_2| \cdot (\mathcal{M}_0 \cdot \mathcal{M}_1 \cdot \mathcal{M}_2)^L.$$

Proof. Because Ξ^L_t as a functional of b is in a finite chaos, we have $\|\Xi^L_t\|_{\hat{\mathbb{D}}^{L,p}} \le C(\hat{\mathbb{E}}|\Xi^L_t|^p)^{1/p}$. Our claim is thus an easy consequence of (4.39) combined with Lemma 3.18. ☐

Remark 4.14. A natural question raised by Corollary 4.13 is whether we have the convergence of those Malliavin derivatives $\bar{D}^L y^n_t$: $\bar{D}^L y^n_t \to \bar{D}^L y_t$ as $n \to \infty$. It has been shown in [12] that this convergence does hold when $H > 1/2$, and the convergence has been applied in the same paper to derive the asymptotic error of weak convergence of the Euler scheme. On the other hand, to our knowledge the convergence in the case $H < 1/2$ is still an open problem.

Proof of Theorem 4.10. The proof is divided into several steps.

Step 1. Representation of remainders. Recall the definition of Ξ^L in (3.21), that is

$$\delta\Xi^L_{t_kt_{k+1}} = \mathcal{L}^L V(y^n_{t_k})\delta x_{t_kt_{k+1}} + \mathcal{L}^{L-1}V(y^n_{t_k})\delta b_{t_kt_{k+1}} + \frac{1}{2}\sum_{j=1}^{d}\bar{\mathcal{L}}^L\left(\partial V_j\cdot V_j\right)(y^n_{t_k})\Delta^{2H} + \frac{1}{2}\sum_{j=1}^{d}\tilde{\mathcal{L}}^{L-1}\left(\partial V_j\cdot V_j\right)(y^n_{t_k})\Delta^{2H}. \tag{4.41}$$

Next observe that owing to our definition (4.31), (4.33) and (4.34) we have

$$\frac{1}{2}\Delta^{2H}\mathrm{Id}_d = x^2_{t_kt_{k+1}} - q_{t_kt_{k+1}}, \qquad \frac{1}{2}\Delta^{2H}\mathrm{Id}_d = b^2_{t_kt_{k+1}} - q^b_{t_kt_{k+1}} \quad \text{and} \quad \tilde{\mathrm{w}}^2_{t_kt_{k+1}} - \tilde{q}_{t_kt_{k+1}} = 0.$$

Hence recalling our notation (4.33)–(4.34), one can recast (4.41) as

$$\begin{aligned}\delta\Xi^L_{t_kt_{k+1}} = {} & \mathcal{L}^L V(y^n_{t_k})\delta x_{t_kt_{k+1}} + \mathcal{L}^{L-1}V(y^n_{t_k})\delta b_{t_kt_{k+1}} + \bar{\mathcal{L}}^L(\partial VV)(y^n_{t_k})(x^2_{t_kt_{k+1}} - q_{t_kt_{k+1}}) \\ & + \tilde{\mathcal{L}}^{L-1}(\partial VV)(y^n_{t_k})(b^2_{t_kt_{k+1}} - q^b_{t_kt_{k+1}}) + \tilde{\mathcal{L}}^L(\partial VV)(y^n_{t_k})(\tilde{\mathrm{w}}^2_{t_kt_{k+1}} - \tilde{q}_{t_kt_{k+1}}) \\ & + \bar{\mathcal{L}}^{L-1}(\partial VV)(y^n_{t_k})(\tilde{\mathrm{w}}^2_{t_kt_{k+1}} - \tilde{q}_{t_kt_{k+1}})^T.\end{aligned} \tag{4.42}$$

This suggests to define the following remainder process for $L = 0, 1, \ldots, N$ and $s, t \in [\![0, T]\!]$:

$$\begin{aligned}R^L_{st} = {} & -\delta\Xi^L_{st} + \mathcal{L}^L V(y^n_s)\delta x_{st} + \mathcal{L}^{L-1}V(y^n_s)\delta b_{st} \\ & + \bar{\mathcal{L}}^L(\partial VV)(y^n_s)(x^2_{st} - q_{st}) + \tilde{\mathcal{L}}^{L-1}(\partial VV)(y^n_s)(b^2_{st} - q^b_{st}) \\ & + \tilde{\mathcal{L}}^L(\partial VV)(y^n_s)(\tilde{\mathrm{w}}^2_{st} - \tilde{q}_{st}) + \bar{\mathcal{L}}^{L-1}(\partial VV)(y^n_s)(\tilde{\mathrm{w}}^2_{st} - \tilde{q}_{st})^T.\end{aligned} \tag{4.43}$$

Notice that as a straightforward consequence of (4.42) we have $R^L_{t_kt_{k+1}} = 0$ for all k.

Our aim is now to prove that R^L is indeed a small remainder by applying Lemma 2.3. Since $R^L_{t_kt_{k+1}} = 0$ it remains to analyze the increment δR as defined in (2.1). Now starting from (4.43), an elementary computation yields:

$$\delta R^L_{sut} = E^1_{sut} + \cdots + E^5_{sut},$$

where we define

$$\begin{aligned}E^1_{sut} &= -\delta\left(\mathcal{L}^L V(y^n_\cdot)\right)_{su}\delta x_{ut} \\ E^2_{sut} &= -\delta\left(\mathcal{L}^{L-1}V(y^n_\cdot)\right)_{su}\delta b_{ut} \\ E^3_{sut} &= \bar{\mathcal{L}}^L(\partial VV)(y^n_s)\delta x_{su}\otimes\delta x_{ut} + \tilde{\mathcal{L}}^L(\partial VV)(y^n_s)\delta b_{su}\otimes\delta x_{ut} \\ E^4_{sut} &= \tilde{\mathcal{L}}^{L-1}(\partial VV)(y^n_s)\delta b_{su}\otimes\delta b_{ut} + \bar{\mathcal{L}}^{L-1}(\partial VV)(y^n_s)\delta x_{su}\otimes\delta b_{ut} \\ E^5_{sut} &= -\delta\left(\bar{\mathcal{L}}^L(\partial VV)(y^n_\cdot)\right)_{su}(x^2_{ut} - q_{ut}) - \delta\left(\tilde{\mathcal{L}}^{L-1}(\partial VV)(y^n_\cdot)\right)_{su}(b^2_{ut} - q^b_{ut}) \\ & \quad - \delta\left(\tilde{\mathcal{L}}^L(\partial VV)(y^n_\cdot)\right)_{su}(\tilde{\mathrm{w}}^2_{ut} - \tilde{q}_{ut}) - \delta\left(\bar{\mathcal{L}}^{L-1}(\partial VV)(y^n_\cdot)\right)_{su}(\tilde{\mathrm{w}}^2_{ut} - \tilde{q}_{ut})^T.\end{aligned}$$

In the following steps we estimate the terms $E^1, \ldots, E^5$ differently on small and large steps.

Step 2. Estimate over small and medium size steps. We consider the intervals $[s_j, s_{j+1}]$ such that $s_j \in S_0 \cup S_1$, with S_0 and S_1 defined by (4.37). In the following, we show by induction that for $s, u, t \in [s_j, s_{j+1}]$ the following inequalities for $L = 0, 1, \ldots, N$ are satisfied:

$$|\delta\Xi^L_{st}| \le K^L_1 P^L_s\omega(s,t)^{1/p}, \quad |\delta\Xi^L_{st} - \mathcal{L}^L V(y^n_s)\delta x_{st} - \mathcal{L}^{L-1}V(y^n_s)\delta b_{st}| \le K^L_2 P^L_s\omega(s,t)^{2/p}, \tag{4.44}$$

$$|R^L_{st}| \le K^L_3 P^L_s\omega(s,t)^{3/p}, \qquad |\delta R^L_{sut}| \le K^L_4 P^L_s\omega(s,t)^{3/p}, \tag{4.45}$$

where we recall that $K^L_1, \ldots, K^L_4$ are defined in (4.5) and (4.29), and where P^L_s is introduced in Definition 4.1. Namely, suppose that (4.44)–(4.45) hold for $s, u, t \in [s_j, v]$. In the following, we are going to show that (4.44)–(4.45) also holds on $[s_j, v + \Delta]$.

To this aim, it is enough to estimate δR^L_{sut} for $s, u \in [s_j, v]$ and $t \in [v, v + \Delta]$. For such a tuple (s, u, t), we apply Lemma 4.8 to E_5 and Lemma 4.9 to $(E_1 + E_3)$ and $(E_2 + E_4)$. We obtain

$$\begin{aligned}|\delta R^L_{sut}| & \le (C^8_{V,L}P^L_s + C^8_{V,L-1}P^{L-1}_s)\omega(s,t)^{3/p} + 4(C^6_{L,V} + C^7_{L,V})P^L_s\omega(s,t)^{3/p} \\ & \le K^L_4 P^L_s\omega(s,t)^{3/p}.\end{aligned} \tag{4.46}$$

From (4.46), one can thus complete the proof of the second inequality in (4.45) by induction. The first inequality in (4.45) is then obtained from the second one by a direct application of Lemma 2.3.

We now turn our attention to the proof of (4.44). By applying relation (4.45), (4.16) and (4.17) to (4.43) and taking into account the condition $\omega(s,t)^{1/p} \le 1/K^L_3$, we obtain

$$\begin{aligned}& |\delta\Xi^L_{st} - \mathcal{L}^L V(y^n_s)\delta x_{st} - \mathcal{L}^{L-1}V(y^n_s)\delta b_{st}| \\ & \le (C^2_{L,V} + C^2_{L-1,V} + C^3_{L,V} + C^3_{L-1,V} + 1)P^L_s\omega(s,t)^{2/p}.\end{aligned} \tag{4.47}$$

This concludes the proof of the second relation in (4.44). In order to show the first relation we apply (4.15) to (4.47) and take into account the assumption that $\omega(s,t)^{1/p} \leq 1/K_2^L$. We get

$$|\delta \Xi_{st}^L| \leq (C_{L,V}^1 + C_{L-1,V}^1 + 1) P_s^L \omega(s,t)^{1/p}.$$

This completes the proof of (4.44)–(4.45) for $s,u,t \in [s_j, v+\Delta]$, under the hypothesis $s_j \in S_0 \cup S_1$. Our induction procedure is thus achieved.

Step 3. Estimate over large size steps. For large size steps, we will use a cruder estimate. Namely, when s_j sits in the set S_2 defined by (4.38), we have $s_{j+1} = s_j + \Delta$. It follows from Eq. (4.41) that

$$|\delta \Xi_{s_j s_{j+1}}^L| = |\delta \Xi_{s_j, s_j+\Delta}^L| \leq (C_{L,V}^1 + C_{L-1,V}^1 + C_{L,V}^2 + C_{L-1,V}^3) P_{s_j}^L \left(|\mathrm{w}_{s_j s_{j+1}}^1| + \Delta^{2H} \right), \tag{4.48}$$

where $\mathrm{w}_{st}^1 = \delta \mathrm{w}_{st}$, and thus with Definition 4.1 in mind we simply get

$$|\Xi_{s_{j+1}}^L| \leq P_{s_j}^L \left(K|\mathrm{w}_{s_j s_{j+1}}^1| + K\Delta^{2H} + 1 \right). \tag{4.49}$$

Step 4. Conclusion. We first derive the uniform upper-bound for Ξ^L. That is, for $t \in [\![s_j, s_{j+1}]\!]$ we write

$$|\Xi_t^L| \leq |\delta \Xi_{s_j t}^L| + |\Xi_{s_j}^L|. \tag{4.50}$$

Hence in the case $s_j \in S_0 \cup S_1$, applying (4.44) to (4.50) we have

$$|\Xi_t^L| \leq P_{s_j}^L \cdot \left(1 + K_1^L \omega(s_j, t)^{1/p}\right).$$

Moreover, in the case that $s_j \in S_2$, relation (4.49) implies that

$$|\Xi_t^L| \leq P_{s_j}^L (K|\mathrm{w}_{s_j s_{j+1}}^1| + K\Delta^{2H} + 1).$$

Iterating the above two estimates, we end up with

$$|\Xi_t^L| \leq K \cdot (\mathcal{M}_0 \cdot \mathcal{M}_1 \cdot \mathcal{M}_2)^L, \qquad \text{for all } t \in [\![0,T]\!], \tag{4.51}$$

which is our desired uniform bound.

We turn to the estimate of the increments of Ξ^L. We first write

$$|\delta \Xi_{st}^L| \leq \sum_{s \leq s_j < t} |\delta \Xi_{s \vee s_j, t \wedge s_{j+1}}^L|.$$

We apply the increment inequalities (4.44) for small sized steps and (4.48) for large sized steps. We also take into account the uniform estimate (4.51). We then obtain

$$|\delta \Xi_{st}^L| \leq K \left(\sum_{s_j \in S_0 \cup S_1} \omega(s_j, s_{j+1})^{1/p} + K \sum_{s_j \in S_2} \left(|\mathrm{w}_{s_j s_{j+1}}^1| + K\Delta^{2H} + 1 \right) \right) (\mathcal{M}_0 \cdot \mathcal{M}_1 \cdot \mathcal{M}_2)^L. \tag{4.52}$$

In the right-hand side of (4.52), bounding each $\omega(s_i, s_{i+1})$ by $\omega(s,t)$ for every $s_j \in S_0 \cup S_1$ we get

$$\sum_{s_j \in S_0 \cup S_1} \omega(s_j, s_{j+1})^{1/p} \leq \omega(s,t)^{1/p} |S_0 \cup S_1|. \tag{4.53}$$

In addition, recall that the control ω is defined by (4.35), which includes the term $\|\omega\|_{p\text{-var}}^p$. Hence if $s_j \in S_2$ (that is $\omega(s_j, s_{j+1}) > \alpha$) and Δ is small enough, we have

$$K \sum_{s_j \in S_2} |\mathrm{w}_{s_j s_{j+1}}^1| + K\Delta^{2H} \leq K\omega(s,t)^{1/p} |S_2|. \tag{4.54}$$

Plugging (4.53) and (4.54) into (4.52), we obtain

$$|\delta \Xi_{st}^L| \leq K \cdot \omega(s,t)^{1/p} |S_0 \cup S_1 \cup S_2| \cdot (\mathcal{M}_0 \cdot \mathcal{M}_1 \cdot \mathcal{M}_2)^L, \tag{4.55}$$

for all $(s,t) \in \mathcal{S}_2([\![0,T]\!])$. The upper-bound (4.39) for $|\delta \Xi_{st}^L|$ is exactly (4.55). □

4.3. Point-wise upper-bound estimate

Theorem 4.10 and Corollary 4.13 provide estimates in p-variation for the Malliavin derivatives of the Euler scheme y^n. In this subsection we use similar arguments in order to derive a pointwise estimate for the Malliavin derivatives of the form Dy_t (recall that those $\mathcal{H}$-valued derivatives have been defined in Section 2.2). Notice that we have stated and proved the theorem below for the first two derivatives of y^n. However, the extension of this result to higher order derivatives is just a matter of cumbersome notation.

Theorem 4.15. *Let the notations in Theorem 4.10 prevail. Suppose that $V \in C_b^4$. Take $r, r' \geq 0$ and let $k_0, k_0' \in \mathbb{N}$ be such that $r \in (t_{k_0}, t_{k_0+1}]$ and $r' \in (t_{k_0'}, t_{k_0'+1}]$. We define a Malliavin derivative vector ξ^n as*

$$\xi_t^n = (D_r y_t^n, D_r D_{r'} y_t^n) := (\xi_t^{n,1}, \xi_t^{n,2}) \tag{4.56}$$

Then $\xi^{n,L}$, $L = 1, 2$ satisfies the iterative Eq. (3.21) with $c_{L,i} = 1$ and $\tilde{c}_{L,i} = 0$. Furthermore, we have the estimate

$$\|\xi^{n,1}\|_{p\text{-var},[\![s,t]\!]} + \|\xi^{n,2}\|_{p\text{-var},[\![s,t]\!]} \leq K \cdot \omega(s,t)^{1/p} |S_0 \cup S_1 \cup S_2| \cdot (\mathcal{M}_0 \cdot \mathcal{M}_1 \cdot \mathcal{M}_2)^L , \tag{4.57}$$

for all $(s,t) \in \mathcal{S}_2[\![0,T]\!]$, where S_i, $\mathcal{M}_i$ are respectively defined for $i = 0, 1, 2$ in (4.37)–(4.38) and (4.40). Moreover, for $(s,t) \in \mathcal{S}_2([\![s_j, s_{j+1}]\!])$ such that $s_j \in S_0 \cup S_1$ and $L = 1, 2$ we have

$$|\delta\xi_{st}^{n,L} - \mathcal{L}^L V(y_s^n)\delta x_{st}| \leq K\omega(s,t)^{2/p} \cdot P_s^L . \tag{4.58}$$

In both estimates (4.57) and (4.58), K is a constant independent of r, r', j and n.

Proof. Recall that y^n is defined in (3.16). Let us first derive the iterative equation for the derivatives of y^n. Recall that $r \in (t_{k_0}, t_{k_0+1}]$. We can divide the differentiation of $\delta y_{t_k t_{k+1}}^n$ in three cases.

(i) If $k_0 > k$, then $r > t_{k+1}$. Therefore, since $y_{t_k} \in \mathcal{F}_{t_k}$ we get $D_r[\delta y_{t_k t_{k+1}}^n] = 0$.

(ii) If $k_0 = k$, then $t_k < r \leq t_{k+1}$. Moreover it is readily checked from Eq. (2.12) that $D_r[\delta x_{t_k t_{k+1}}] = \mathbf{1}_{[t_k, t_{k+1}]}(r)$. Hence we have $\mathbf{1}_{(t_k, t_{k+1}]}(r) = 1$ almost everywhere and differentiating (3.16) on both sides we obtain

$$D_r[\delta y_{t_k t_{k+1}}^n] = \sum_{j=1}^d V_j(y_{t_k}^n)\mathbf{1}_{[t_k, t_{k+1}]}(r) = \sum_{j=1}^d V_j(y_{t_{k_0}}^n) \equiv a_1 .$$

(iii) If $k_0 < k$, then we can differentiate both sides of (3.16). We obtain the equation:

$$\delta D_r y_{t_k t_{k+1}}^n = \langle \partial V(y_{t_k}^n), D_r y_{t_k}^n \rangle \delta x_{t_k t_{k+1}} + \frac{1}{2} \sum_{j=1}^d \langle \partial(\partial V_j V_j)(y_{t_k}^n), D_r y_{t_k}^n \rangle \Delta^{2H} .$$

Gathering item (i), (ii) and (iii) above, and recalling that we have set $a_1 = \sum_{j=1}^d V_j(y_{t_{k_0}}^n)$, we get the following expression for $t \in [\![0,T]\!]$, $r \leq t_k$ and $\xi_t^{n,1} = D_r y_t^n$:

$$\delta\xi_{t_k t_{k+1}}^{n,1} = \langle \partial V(y_{t_k}^n), \xi_{t_k}^{n,1} \rangle \delta x_{t_k t_{k+1}} + \frac{1}{2} \sum_{j=1}^d \langle \partial(\partial V_j V_j)(y_{t_k}^n), \xi_{t_k}^{n,1} \rangle \Delta^{2H} . \tag{4.59}$$

Also notice that we have obtained $\xi_t^{n,1} = 0$ if $r > t$. In particular, it is clear that $\xi^{n,1}$ satisfies the iteration equation in Definition 3.17 with $c_{L,i} = 1$ and $\tilde{c}_{L,i} = 0$, with $L = 1$ and initial time $t_0 = t_{k_0}$. Precisely, we have

$$\delta\xi_{t_k t_{k+1}}^{n,1} = \mathcal{L}_\xi^1 V(y_{t_k}^n)\delta x_{t_k t_{k+1}} + \frac{1}{2} \sum_{j=1}^d \bar{\mathcal{L}}_\xi^1 (\partial V_j V_j)(y_{t_k}^n) \Delta^{2H} . \tag{4.60}$$

Therefore, a direct application of Theorem 4.10 yields the estimate (4.57) for $\xi^{n,1}$.

In a similar way we can show that $\xi_t^{n,2} = D_r D_{r'} y_t^n$ satisfies the iterative equation in Definition 3.17 with $c_{L,i} = 1$ and $\tilde{c}_{L,i} = 0$, with $L = 2$ and initial time $t_0 = t_{k_0} \vee t_{k_0'}$. Indeed, a straightforward computation shows that

$$D_{r'}\xi_t^{n,1} = \xi_t^{n,2}, \qquad D_{r'}\mathcal{L}_\xi^1 V(y_{t_k}^n) = \mathcal{L}_\xi^2 V(y_{t_k}^n), \qquad D_{r'}\bar{\mathcal{L}}_\xi^1(\partial V_j V_j)(y_{t_k}^n) = \bar{\mathcal{L}}_\xi^2(\partial V_j V_j)(y_{t_k}^n).$$

Let $r \in [t_{k_0}, t_{k_0+1})$ and $r' \in [t_{k_0'}, t_{k_0'+1})$. Then, by differentiating both sides of (4.60) by $D_{r'}$ and taking into account the above three relations we get for all $t \geq t_0$

$$\xi_t^{n,2} = a_2 + \sum_{r \vee r' \leq t_k < t} \mathcal{L}_\xi^2 V(y_{t_k}^n)\delta x_{t_k t_{k+1}} + \frac{1}{2} \sum_{r \vee r' \leq t_k < t} \sum_{j=1}^d \bar{\mathcal{L}}_\xi^2 (\partial V_j V_j)(y_{t_k}^n)\Delta^{2H},$$

where a_2 is the initial value of the iterative equation defined as follows

$$a_2 = \mathbf{1}_{\{k_0 \geq k_0'\}} \cdot \sum_{j=1}^d \langle \partial V_j(y_{t_{k_0}}^n), D_{r'} y_{t_{k_0}}^n \rangle + \mathbf{1}_{\{k_0' \geq k_0\}} \cdot \sum_{j=1}^d \langle \partial V_j(y_{t_{k_0'}}^n), D_r y_{t_{k_0'}}^n \rangle .$$

We conclude that the estimate (4.57) also holds for $\xi^{n,2}$. The proof is complete. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

Yanghui Liu is supported by the PSC-CUNY, United States of America Award 66385-00 54.

References

[1] Gess B., C. Ouyang, S. Tindel, Density bounds for solutions to differential equations driven by Gaussian rough paths, J. Theoret. Probab. 33 (2) (2020) 611–648.
[2] F. Baudoin, E. Nualart, C. Ouyang, S. Tindel, On probability laws of solutions to differential systems driven by a fractional Brownian motion, Ann. Probab. 44 (4) (2016) 2554–2590.
[3] T. Cass, P. Friz, N. Victoir, Non-degeneracy of Wiener functionals arising from rough differential equations. (english summary), Trans. Amer. Math. Soc. 361 (6) (2009) 3359–3371.
[4] T. Cass, M. Hairer, C. Litterer, S. Tindel, Smoothness of the density for solutions to Gaussian rough differential equations, Ann. Probab. 43 (1) (2015) 188–239.
[5] T. Cass, C. Litterer, T. Lyons, Integrability and tail estimates for Gaussian rough differential equations, Ann. Probab. 41 (4) (2013) 3026–3050.
[6] G.M. Constantine, T.H. Savits, A multivariate Faa Di Bruno formula with applications, Trans. Amer. Math. Soc. 348 (2) (1996) 503–520.
[7] A.M. Davie, Differential equations driven by rough paths: An approach via discrete approximation, Appl. Math. Res. Express. 2008 (2008) 1–40.
[8] A. Deya, A. Neuenkirch, S. Tindel, A milstein-type scheme without Lévy area terms for SDEs driven by fractional Brownian motion, Ann. Inst. Henri Poincaré Probab. Stat. 48 (2) (2012) 518–550.
[9] P.K. Friz, S. Riedel, Convergence rates for the full Gaussian rough paths, Ann. Inst. Henri Poincaré Probab. Stat. 50 (1) (2014) 154–194.
[10] P.K. Friz, N.B. Victoir, Multidimensional Stochastic Processes As Rough Paths: Theory and Applications 120, Cambridge University Press, 2010.
[11] X. Geng, C. Ouyang, S. Tindel, Precise local estimates for hypoelliptic differential equations driven by fractional Brownian motion, 2019, arXiv preprint.
[12] Y. Hu, Y. Liu, D. Nualart, Rate of convergence and asymptotic error distribution of Euler approximation schemes for fractional diffusions, Ann. Appl. Probab. 26 (2) (2016) 1147–1207.
[13] Y. Hu, Y. Liu, D. Nualart, Taylor schemes for rough differential equations and fractional diffusions, Discrete Contin. Dyn. Syst. Ser. B 21 (9) (2016) 3115–3162.
[14] Y. Hu, Y. Liu, D. Nualart, Crank–Nicolson scheme for stochastic differential equations driven by fractional Brownian motions, Ann. Appl. Probab. 31 (1) (2021) 39–83.
[15] Y. Inahama, Malliavin differentiability of solutions of rough differential equations, J. Funct. Anal. 267 (5) (2014) 1566–1584.
[16] R.B. Leipnik, C.E.M. Pearce, The multivariate Faà Di Bruno formula and multivariate Taylor expansions with explicit integral remainder term, ANZIAM J. 48 (2007) 327–341.
[17] J.A. León, Y. Liu, S. Tindel, Weak convergence of the Euler scheme for SDEs driven by fractional Brownian motions, 2023, arXiv preprint.
[18] Y. Liu, S. Tindel, First-order Euler scheme for SDEs driven by fractional Brownian motions: the rough case, Ann. Appl. Probab. 29 (2) (2019) 758–826.
[19] D. Nualart, The Malliavin Calculus and Related Topics, second ed., Springer-Verlag, Berlin, 2006.
[20] D. Nualart, B. Saussereau, Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion, Stochastic Process. Appl. 119 (2) (2009) 391–409.