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EULER SCHEME FOR SDES DRIVEN BY FRACTIONAL BROWNIAN MOTIONS: INTEGRABILITY AND CONVERGENCE IN LAW

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We prove that the Euler scheme for stochastic differential equations driven by fractional Brownian motions (fBm) with Hurst parameter $H > 1/3$ and its Malliavin derivatives are integrable uniformly in step size n. Then we use the integrability results to derive the weak convergence rate $n^{1-4H+\varepsilon}$ for the Euler scheme. The proof for integrability is based on an application of the argument of (*Ann. Probab.* **41** (2013) 3026–3050) to a quadratic functional of the fBm. The proof of weak convergence applies Malliavin calculus and some upper-bound estimates for weighted random sums.

1. Introduction. This note is concerned with the following stochastic differential equation driven by a d-dimensional fractional Brownian motion (fBm in the sequel) x with Hurst parameter $\frac{1}{3} < H < \frac{1}{2}$:

$$dy_t = V_0(y_t)\,dt + V(y_t)\,dx_t, \quad y_0 = a, \tag{1.1}$$

where we assume that $a \in \mathbb{R}^m$, the collection of vector field $V_0 = (V_0^i)_{1\le i\le m}$ belongs to $C_b^2(\mathbb{R}^m, \mathbb{R}^m)$ and $V = (V_j^i)_{1\le i\le m, 1\le j\le d}$ sits in $C_b^3(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^d, \mathbb{R}^m))$. Under this setting the theory of rough paths gives a framework allowing to get existence and uniqueness results for equation (1.1), and the unique solution y in the rough paths sense has γ-Hölder continuity for all $\gamma < H$; see, for example, [15, 17].

One of the basic questions about systems like (1.1) concerns the existence of a proper numerical scheme approximating the solution y. In case of a Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$, the simplest possible solution to this problem is to use a Milstein type scheme. However Milstein type schemes involve second order expansions and iterated integrals of the fBm x, which should be morally thought of as objects of the form

$$x_{st}^2 = \int_s^t \int_s^u dx_v \otimes dx_u,$$

and are notoriously uneasy to simulate. Therefore several contributions aimed in the recent past at avoiding iterated integrals while still producing convergent numerical schemes for rough differential equations. The first article tackling this issue is [13], where the iterated integrals in x^2 were replaced by products of increments of x. The rate of convergence obtained in [13] was then pushed to its optimal limit in [14]. Let us also mention the article [35], which thoroughly explores Runge–Kutta methods based on the same idea of replacing iterated integrals by products of increments.

In this paper we will focus our attention on another numerical approximation, called first-order scheme in the sequel. The main idea behind this method is to simply replace the second order terms x^2 by their expected values. This yields simpler schemes than the aforementioned

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methods based on product of increments, and at the same time produces optimal convergence rates. Specifically, if x is a fBm with Hurst parameter H and one uses an approximating grid with mesh of order $1/n$, then the rate of convergence is of order $1/n^{2H-1/2}$. This method has first been introduced in [23] for a Hurst parameter $H > 1/2$, and has been extended to the rough path case in [29]. We also refer to [24, 25] for further extensions.

In order to describe our first-order numerical scheme, let us introduce some basic settings. For simplicity, we are considering a finite time interval $[0, T]$ and we take the uniform partition $\pi : 0 = t_0 < t_1 < \cdots < t_n = T$ on $[0, T]$. Specifically, for $k = 0, \dots, n$ we have $t_k = k\Delta$, where we denote $\Delta = \frac{T}{n}$. In the sequel, the quantity δx_{st} will stand for the vector $x_t - x_s$. Our generic approximation is called y^n, and it starts from the initial condition $y_0^n = y_0 = a$. With this notation in hand, we can now define our scheme recursively as follows (here and below we set $\delta x_{t_k t_{k+1}} = x_{t_{k+1}} - x_{t_k}$):

$$y_{t_{k+1}}^n = y_{t_k}^n + V_0(y_{t_k}^n)\Delta + V(y_{t_k}^n)\delta x_{t_k t_{k+1}} + \frac{1}{2}\sum_{j=1}^{d} \partial V_j V_j(y_{t_k}^n)\Delta^{2H}, \tag{1.2}$$

where the notation $\partial V_i V_j$ stands for a vector field of the form

$$\partial V_i V_j = \left(\sum_{l=1}^{m} \partial_l V_i^k V_j^l ; k = 1, \dots, m,\right) \tag{1.3}$$

and ∂_l stands for the partial derivative in the y_l direction: $\partial_l = \frac{\partial}{\partial y_l}$. As mentioned above, the rate of convergence of y^n to y is of order $1/n^{2H-1/2}$. One of the key results in [29] is a functional central limit theorem of the form

$$\lim_{n\to\infty} n^{2H-1/2}(y^n - y) \overset{(d)}{=} U,$$

where U is solution to a rough differential equation driven by x plus an additional Brownian term.

In the current contribution, we are mostly interested in the convergence in distribution of the approximation y^n defined by (1.2). This endeavor is motivated by three main reasons which can be summarized as follows:

(i) The weak convergence of a numerical scheme is directly related to the performance of simulation for stochastic models, which is a center issue in mathematical finance and engineering.

(ii) For diffusions processes, that is, stochastic differential equations driven by a Brownian motion, the convergence in distribution for numerical schemes is a classical problem. This is assessed e.g by the remarkable publications [2, 3]. As mentioned in those two references, a good knowledge about the weak convergence is useful in order to evaluate probabilities that y reaches a certain level, or to get some information about the moments of y.

(iii) In [4, 7, 19] we have started a long term program aiming at understanding the law of Gaussian rough differential systems. The current result might play an important role in this approach.

Let us now describe the main result contained in this paper.

THEOREM 1.1. *Suppose that $V \in C_b^4$ and x is a fBm with Hurst parameter $H > 1/3$. Let y be the solution of the rough differential equation (1.1) and let y^n be the corresponding Euler scheme (1.2). Then for any $\varepsilon > 0$, $f \in C_b^4(\mathbb{R}^m)$ and $t \in [0, T]$ there is a constant $C_{T,H,V,f} > 0$ independent of n such that*

$$\left|\mathbb{E}f(y_t^n) - \mathbb{E}f(y_t)\right| \le \frac{C_{T,H,V,f}}{n^{4H-1-\varepsilon}}. \tag{1.4}$$

To the best of our knowledge, Theorem 1.1 is the first weak convergence result for numerical schemes of differential equations driven by a fBm with $H < 1/2$. In order to get a broader perspective on weak convergence for stochastic differential systems, let us recall some of the rates obtained in previous contributions:

(a) It is well known that the weak convergence rate for an equation like (1.1) driven by a Brownian motion is n^{-1}, versus a rate $n^{-1/2}$ for the strong rate; This has first been established in the classical references [31, 36] (see also [2, 3]).

(b) In [24] the authors consider differential equations driven by a fBm for the range of Husrt parameter $H \in (1/2, 1)$. It is shown that the weak rate is n^{-1} like in the Brownian case, regardless of the value of H. This rate is sharp in the sense that the normalized error $n[\mathbb{E}f(y_t^n) - \mathbb{E}f(y_t)]$ converges to a nonzero limit for any test function $f \in C^4$.

(c) The recent articles [16, 18] consider the weak convergence of Euler schemes for a mixed stochastic integral model $\mathcal{I} = \int_0^T \varphi(B_t^H)\, dW_s$, where W is a Wiener process and B^H is a Liouville type fractional Brownian motion driven by W (with Hurst parameter $H \in (0, 1)$). It is proved that for a general choice of functions f, φ, the weak rate is $1/n^{(3H+1/2)\wedge 1}$ (to be contrasted with the strong rate of the Euler scheme for $\mathcal{I}$ above, which is $1/n^H$). Our result shows that this surprising behavior is probably due to some specific cancellations for mixed quantities like $\mathcal{I}$ (see further remarks about this fact in [16]).

Compared to this body of literature, our Theorem 1.1 shows that when $1/3 < H < 1/2$ the weak rate for equation (1.1) is n^{1-4H} (note that we believe that our rate is sharp for a generic test function). This generalizes in a very natural way the convergence rate n^{-1} obtained for $H \geq 1/2$, except for the slightly nonoptimal ε in relation (1.4). Notice that this small ε is due to the fact that our analysis of the scheme is mostly pathwise, in spite of dealing with a convergence in distribution. It is interesting to mention that Theorem 1.4 agrees with the rule of thumb in the martingale framework (see, e.g., Heston's model [1], Schrödinger's equation [11], reflected diffusions [5] or the stochastic heat equation [12]), namely that the weak rate n^{1-4H} is twice the strong rate $n^{1/2-2H}$ (see [29]).

REMARK 1.2. For sake of conciseness, we have not tried to quantify precisely the dependence $T \mapsto C_{T,H,V,f}$. Since our estimates depend on the behavior of Malliavin derivatives, we expect this dependence to be of exponential type.

At the core of our methodology for the proof of Theorem 1.1 lies a combination of rough paths and Malliavin techniques, plus some specific tools for discrete rough paths that have been developed by two of the authors in [29, 30]. Those elements are summarized in Section 2 and Sections 4.1-4.2. Specifically, our first main step of the proof is a generalization of the duality approach developed in [9] to the fBm case. In particular, we will show that the Malliavin integration by parts can be applied to the error $\mathbb{E}f(y_t^n) - \mathbb{E}f(y_t)$ in the left-hand side of (1.4), and that the estimate of the error can be transformed to the study of some weighted sums involving the Malliavin derivatives of the Euler scheme (1.2) and related processes. With the transformation established, a main ingredient in the current contribution is to prove the integrability of Malliavin derivatives for the Euler scheme, uniformly in our approximation parameter n. A key observation in this direction is that the Euler scheme (1.2) is a discrete-time equation driven by the mix of a rough path (i.e., the process x) and a quadratic *Young* path (that is, a path which is a quadratic functional of x and has a Hölder component greater than $1/2$; see (2.20) for the precise definition). This representation enables us to adapt the very fruitful idea of *greedy sequence* put forward in [8], in order to achieve exponential integrability in a rough paths context. A new situation for the Euler scheme is that now we have a greedy sequence corresponding not only to x but also to the quadratic path q introduced in (2.20). One of our main efforts will then consist in showing a tail estimate for the greedy sequence via Borell's inequality. Furthermore, due to the discrete feature of equation (1.2), a separate estimate will involve the *big* steps related to our partition of $[0, T]$ (namely the steps for which the increments $\delta x_{t_k t_{k+1}}$ are very large) separately. These delicate estimates will be developed in Section 3.

The paper is structured as follows. Section 2 contains the preliminary results on rough paths, Malliavin calculus, and the Euler scheme. In Section 3 we show that the Malliavin derivatives of the Euler scheme has moments of any order. After some preparations in Section 4.1-4.5, we prove the weak convergence of the Euler scheme in Section 4.6.

NOTATION 1.3. In what follows, we take $n \in \mathbb{N}$ and $\Delta = T/n$, and consider the uniform partition: $0 = t_0 < t_1 < \cdots < t_n = T$ on $[0, T]$, where $t_k = k\Delta$. We denote by $[\![s, t]\!]$ the discrete interval: $[\![s, t]\!] = \{t_k \in [s, t] : k = 0, \ldots, n\}$. For $u \in [t_k, t_{k+1})$, we denote $\eta(u) = t_k$. For an interval $[s, t] \subset [0, T]$ we define the continuous- and discrete-time simplexes $\mathcal{S}_2([s, t]) = \{(u, v) : s \le u \le v \le t\}$ and $\mathcal{S}_2([\![s, t]\!]) = \mathcal{S}_2([s, t]) \cap [\![s, t]\!]^2$. We use the letters C and K to denote generic constant which can change from line to line.

2. Preliminary results. In this section we recall some basic notions of rough paths theory and their application to fractional Brownian motion, which allow a proper definition of equation (1.1). We also give the necessary elements of Malliavin calculus in order to quantify the weak convergence rate. Eventually we recall the pathwise estimates obtained in [28] for the Malliavin derivatives of our Euler scheme. Notice that this basic presentation can be found in a very similar way in our companion paper [28].

2.1. *Elements of rough paths theory.* This subsection is devoted to introduce some basic concepts of rough paths theory. We are going to restrict our analysis to a generic p-variation regularity of the driving path of order $1 \le p < 3$, in order to keep expansions to a reasonable size. We also fix a finite time horizon $T > 0$. The following notation will prevail until the end of the paper: for a finite-dimensional vector space $\mathcal{V}$ and two functions $f \in C([0, T], \mathcal{V})$ and $g \in C(\mathcal{S}_2([0, T]), \mathcal{V})$ we set

$$\delta f_{st} = f_t - f_s, \quad \text{and} \quad \delta g_{sut} = g_{st} - g_{su} - g_{ut}. \tag{2.1}$$

Let us introduce the analytic requirements in terms of p-variation regularity which will be used in the sequel. Namely consider two paths $x \in C([0, T], \mathbb{R}^d)$ and $x^2 \in C(\mathcal{S}_2([0, T]), (\mathbb{R}^d)^{\otimes 2})$. Then we denote

$$\begin{aligned} \|x\|_{p\text{-var},[s,t]} &:= \left(\sup_{\mathcal{P}} \sum_{(u,v)\in\mathcal{P}} |\delta x_{uv}|^p\right)^{1/p}, \\ \|x^2\|_{p/2\text{-var},[s,t]} &:= \left(\sup_{\mathcal{P}} \sum_{(u,v)\in\mathcal{P}} |x^2_{uv}|^{p/2}\right)^{2/p}, \end{aligned} \tag{2.2}$$

where the supremum is taken among all partitions of the time interval $[s, t]$, and for a partition $\mathcal{P}$ of $[s, t]$ we write $(u, v) \in \mathcal{P}$ if u and v are two consecutive partition points of $\mathcal{P}$. When the semi-norms in (2.2) are finite we say that x and x^2 are respectively in $C^{p\text{-var}}([s, t], \mathbb{R}^d)$ and $C^{p/2\text{-var}}(\mathcal{S}_2([s, t]), (\mathbb{R}^d)^{\otimes 2})$. For convenience, we denote $\|x\|_{p\text{-var}} := \|x\|_{p\text{-var},[0,T]}$ and $\|x^2\|_{p/2\text{-var}} := \|x^2\|_{p/2\text{-var},[0,T]}$. With this preliminary notation in hand, we can now turn to the definition of rough path.

DEFINITION 2.1. Let $x \in C([0, T], \mathbb{R}^d)$, $x^2 \in C(\mathcal{S}_2([0, T]), (\mathbb{R}^d)^{\otimes 2})$, and $1 \le p < 3$. Denote $x^1_{st} = \delta x_{st}$. We call $\mathbf{x} := S_2(x) := (x^1, x^2)$ a (second-order) p-rough path if $\|x\|_{p\text{-var}} < \infty$ and $\|x^2\|_{p/2\text{-var}} < \infty$, and if the following algebraic relation holds true:

$$\delta x^2_{sut} = x^2_{st} - x^2_{su} - x^2_{ut} = \delta x_{su} \otimes \delta x_{ut}, \tag{2.3}$$

where we have invoked (2.1) for the definition of δx and δx^2. For a p-rough path $S_2(x)$, we define a p-variation semi-norm as follows:

$$\|S_2(x)\|_{p\text{-var}} := \|x\|_{p\text{-var}} + \|x^2\|_{p/2\text{-var}}^{1/2}. \tag{2.4}$$

An important subclass of rough paths are the so-called *geometric p-variation rough paths*. A geometric p-variation rough path is a p-rough path (x, x^2) such that there exists a sequence of smooth $\mathbb{R}^d$-valued paths $(x^n, x^{2,n})$ verifying

$$\lim_{n\to\infty}(\|x - x^n\|_{p\text{-var}} + \|x^2 - x^{2,n}\|_{p/2\text{-var}}) = 0. \tag{2.5}$$

We will mainly consider geometric rough paths in the remainder of the article.

In relation to (2.5), notice that when x is a smooth $\mathbb{R}^d$-valued path, we can choose x^2 defined as the following iterated Riemann type integral:

$$x^2_{st} = \int_s^t \int_s^u dx_v \otimes dx_u. \tag{2.6}$$

It is then easily verified that $S_2(x) = (x^1, x^2)$, with x^2 defined in (2.6), is a p-rough path with $p = 1$. In fact, this is also the unique way to lift a smooth path to a p-rough path for some $p \geq 1$.

Recall now that we interpret equation (1.1) in the rough paths sense. That is, we shall consider the following general rough differential equation: (RDE):

$$y_t = a + \int_0^t V_0(y_s)\, ds + \int_0^t V(y_s)\, dx_s, \quad t \in [0, T], \tag{2.7}$$

where V_0 and V are smooth enough coefficients and x is a rough path as given in Definition 2.1. We shall interpret equation (2.7) in a way introduced by Davie in [10], which is conveniently compatible with numerical approximations.

DEFINITION 2.2. Let (x, x^2) be a p-rough path with $p < 3$. We say that y is a solution of (2.7) on $[0, T]$ if $y_0 = a$ and there exists a control function ω on $[0, T]$ (i.e., ω is a two variable function on $\mathcal{S}_2([0, T])$ which satisfies the super-additivity condition $\omega(s, t) \geq \omega(s, u) + \omega(u, t)$ for $s, u, t \in [0, T] : s < u < t$), a constant $K > 0$ and $\mu > 1$ such that

$$\left|\delta y_{st} - \int_s^t V_0(y_u)\, du - V(y_s)\delta x_{st} - \sum_{i,j=1}^d \partial V_i V_j(y_s) x^{2,ij}_{st}\right| \leq K\omega(s, t)^\mu \tag{2.8}$$

for all $(s, t) \in \mathcal{S}_2([0, T])$, where we recall that δy is defined by (2.1) and $\partial V_i V_j$ is defined as in (1.3).

Notice that if y solves (2.7) according to Definition 2.2, then it is also a controlled process as defined in [15, 21]. Namely, if y satisfies relation (2.8), then we also have

$$\delta y_{st} = V(y_s)\delta x_{st} + r^y_{st},$$

where $r^y \in C^{p/2\text{-var}}(\mathcal{S}_2([0, T]))$. We can thus define iterated integrals of y with respect to itself thanks to the sewing map; see Proposition 1 in [21]. This yields the following decomposition:

$$\left|\int_s^t y^i_u\, dy^j_u - y^i_s \delta y^j_{st} - \sum_{i',j'=1}^d V^i_{i'} V^j_{j'}(y_s) x^{2,i'j'}_{st}\right| \leq K\omega(s, t)^{3/p},$$

for all $(s,t) \in \mathcal{S}_2([0,T])$ and $i,j = 1,\dots,m$. In other words, the signature type path $S_2(\mathbf{y}) = (y^1, y^2)$ defines a rough path according to Definition 2.1, where y^2 denotes the iterated integral of y.

We can now state an existence and uniqueness result for rough differential equations. The reader is referred to, for example, [17], Theorem 10.36, for further details.

THEOREM 2.3. *Assume that $V = (V_j)_{1\le j\le d}$ is a collection of C_b^3-vector fields on $\mathbb{R}^m$. Then there exists a unique RDE solution to equation* (2.7), *understood as in Definition* 2.2. *In addition, there exists a constant $K > 0$ such that the unique solution y satisfies the following estimate*:

$$|S_2(y)_{st}| \le K\big(1 \vee \|S_2(x)\|^p_{p\text{-var},[s,t]}\big).$$

Whenever $V = (V_j)_{1\le j\le d}$ is a collection of linear vector fields, existence and uniqueness still hold for equation (2.7). *Furthermore, there exist constants $K_1, K_2 > 0$ such that we have the estimate*

$$|S_2(y)_{st}| \le K_1 \|S_2(x)\|_{p\text{-var},[s,t]} \exp\big(K_2 \|S_2(x)\|^p_{p\text{-var}}\big).$$

We close this section by recalling a sewing map lemma with respect to discrete control functions. It is a generalization of the sewing lemma [15], Lemma 4.2, to a discrete setting. It should also be seen as an elaboration of [29], Lemma 2.5, and proves to be useful in the analysis of the numerical scheme. Let $\pi : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ be a generic partition of the interval $[0,T]$ for $n \in \mathbb{N}$. We denote by $[\![s,t]\!]$ the discrete interval $\{t_k : s \le t_k \le t\}$ for $0 \le s < t \le T$.

LEMMA 2.4. *Suppose that ω is a control on $[\![0,T]\!]$. In other words, ω is a two variable function on $\mathcal{S}_2([\![0,T]\!])$ which satisfies a super-additivity condition*: $\omega(s,t) \ge \omega(s,u) + \omega(u,t)$ *for $s,u,t \in [\![0,T]\!] : s < u < t$. Consider a Banach space $\mathcal{B}$ with norm $|\cdot|$ and $R : \mathcal{S}_2([\![0,T]\!]) \to \mathcal{B}$, and denote $\delta R_{sut} = R_{st} - R_{su} - R_{ut}$. Suppose that $|R_{t_k t_{k+1}}| \le \omega(t_k, t_{k+1})^\mu$ for all $t_k \in [\![0,T]\!]$, and that $|\delta R_{sut}| \le \omega(s,t)^\mu$ with the exponent $\mu > 1$. Then the following relation holds*:

$$|R_{st}| \le K_\mu \omega(s,t)^\mu, \quad \textit{where } K_\mu = 2^\mu \sum_{l=1}^{\infty} l^{-\mu}. \tag{2.9}$$

The lemma allows to bound discrete sums which are crucial in our numerical scheme context. As a first application along those lines we present a probabilistic result below, which combines Proposition 4.1 and Remark 4.2 in [29].

LEMMA 2.5. *Consider two processes f and g such that for all $s,t \in [\![0,T]\!]$ we have*

$$\|\delta f_{st}\|_{L^{2p}} \lesssim |t-s|^\alpha, \quad \textit{and} \quad \|\delta g_{st}\|_{L^{2p}} \lesssim |t-s|^\beta,$$

for a given $p \ge 1$ and α, β such that $\alpha + \beta > 1$. Let J_{st} be the discrete sum given by

$$J_{st} = \sum_{s \le t_k < t} \delta f_{s t_k} \delta g_{t_k t_{k+1}}. \tag{2.10}$$

Then we have

$$\|J_{st}\|_{L^p} \lesssim (t-s)^{\alpha+\beta}.$$

2.2. *Rough path above fractional Brownian motion.* We now specialize our setting to a path $x = (x^1, \ldots, x^d)$ defined as a standard d-dimensional fBm on $[0, T]$ with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$. This fBm is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and we assume that the σ-algebra $\mathcal{F}$ is generated by x. In this situation, recall that the covariance function of each coordinate of x is defined on $\mathcal{S}_2([0, T])$ by

$$R(s,t) = \frac{1}{2}\left[s^{2H} + t^{2H} - |t-s|^{2H}\right], \tag{2.11}$$

where recall that the simplex $\mathcal{S}_2([0, T])$ is introduced in Notation 1.3. We start by reviewing some properties of the covariance function of x considered as a function on $(\mathcal{S}_2([0, T]))^2$. Namely, take (u, v, s, t) in $(\mathcal{S}_2([0, T]))^2$ and set

$$R([u,v],[s,t]) = \mathbb{E}\left[\delta x^j_{uv} \delta x^j_{st}\right], \quad j = 1, \ldots, d. \tag{2.12}$$

Then, whenever $H > 1/4$, it can be shown that the integral $\int R\, dR$ is well defined as a Young integral in the plane (see, e.g., [17], Section 6.4). Furthermore, if the intervals $[u, v]$ and $[s, t]$ are disjoint, we have

$$R([u,v],[s,t]) = \int_u^v \int_s^t \mu(dr'\, dr). \tag{2.13}$$

Here and in the following, the signed measure μ is defined as

$$\mu(dr'\, dr) = -H(1-2H)|r-r'|^{2H-2}\, dr'\, dr. \tag{2.14}$$

Using the elementary properties above, it is shown in [17], Chapter 15, that for any piecewise linear or mollifier approximation x^n to x, the smooth rough path $S_2(x^n)$ defined by (2.6) converges in the p-variation semi-norm (2.4) to a p-geometric rough path $S_2(x) := (x^1, x^2)$ (given as in Definition 2.1) for $3 > p > 1/H$. In addition, for $i \neq j$ the covariance of $x^{2,ij}$ can be expressed in terms of a two-dimensional Young integral:

$$\mathbb{E}\left[x^{2,ij}_{uv} x^{2,ij}_{st}\right] = \int_u^v \int_s^t R([u,r],[s,r'])\, dR(r',r). \tag{2.15}$$

It is also established in [17], Chapter 15, that $S_2(x)$ enjoys the following integrability property.

PROPOSITION 2.6. *Let $x = (x^1, \ldots, x^d)$ be a fBm with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $S_2(x) := (x^1, x^2)$ be the geometric rough path above x as given in Definition 2.1, and $p \in (1/H, 3)$. Then there exists a random variable $G_p \in \bigcap_{i \geq 1} L^i(\Omega)$ such that $\|S_2(x)\|_{p\text{-var}} \leq G_p$ $\mathbb{P}$-almost surely, where $\|\cdot\|_{p\text{-var}}$ is defined by* (2.4).

According to Theorem 2.3, given that the vector fields $V \in C^3_b$, equation (2.7) driven by a d-dimensional fBm x with Hurst parameter $H > 1/3$ admits a unique solution.

2.3. *Malliavin calculus for* **x**. As mentioned in the Introduction, we will analyze the convergence of distribution for our numerical approximations thanks to Malliavin calculus tools. We proceed to recall the main concepts which will be used later in the paper and refer to [33] for further details. We start by labeling a definition for the Cameron–Martin type space $\mathcal{H}$ related to our fractional Brownian motion x.

DEFINITION 2.7. Denote by $\mathcal{E}_{[a,b]}$ the set of step functions on an interval $[a,b] \subset [0,T]$. We call $\mathcal{H}_{[a,b]}$ the Hilbert space defined as the closure of $\mathcal{E}_{[a,b]}$ with respect to the scalar product

$$\langle \mathbf{1}_{[u,v]}, \mathbf{1}_{[s,t]} \rangle_{\mathcal{H}_{[a,b]}} = R([u,v],[s,t]).$$

In order to alleviate notation, we will write $\mathcal{H} = \mathcal{H}_{[a,b]}$ when $[a,b] = [0,T]$. Notice that the mapping $\mathbf{1}_{[s,t]} \to \delta x_{st}$ can be extended to an isometry between $\mathcal{H}_{[a,b]}$ and the Gaussian space associated with $\{x_t, t \in [a,b]\}$. We denote this isometry by $h \to \int_a^b h \delta^\diamond x$. The random variable $\int_a^b h \delta^\diamond x$ is called the (first-order) Wiener integral and is also denoted by $I_1(h)$.

The space $\mathcal{H}$ is very useful in order to define Wiener integrals with respect to x. In this paper we also need to introduce another Cameron–Martin type space $\bar{\mathcal{H}}$. The space $\bar{\mathcal{H}}$ allows to identify pathwise derivatives with respect to x and the Malliavin derivatives. In order to construct $\bar{\mathcal{H}}$, let $\mathcal{R}$ be the linear operator such that $\mathcal{R} : h \in \mathcal{H} \to \langle h, \mathbf{1}_{[0,t]} \rangle_{\mathcal{H}}$. Then the space $\bar{\mathcal{H}}$ is defined as the Hilbert space $\bar{\mathcal{H}} = \mathcal{R}(\mathcal{H})$ equipped with the inner product

$$\langle \mathcal{R}(g), \mathcal{R}(h) \rangle_{\bar{\mathcal{H}}} = \langle g, h \rangle_{\mathcal{H}}.$$

We refer to [20, 34] for more details about the spaces $\mathcal{H}$ and $\bar{\mathcal{H}}$.

For the sake of conciseness, we refer to [33] for a proper definition of Malliavin derivatives and related Sobolev spaces in Gaussian analysis. Let us just mention that we will denote the Malliavin derivative by DF, the Sobolev spaces by $\mathbb{D}^{k,p}$ and the corresponding norms by $\|F\|_{k,p}$. We denote by $D^k F$ the kth iteration of the Malliavin derivative D applied on F. The nth order chaos of x is denoted by $\mathcal{K}_n^x$. Also notice that we are considering a d-dimensional fBm $x = (x^1, \dots, x^d)$. Therefore, we shall consider partial Malliavin derivatives with respect to each coordinate x^i in the sequel. Those partial derivatives will be denoted by $D^{(i)}$. Then for $h = (h^1, \dots, h^d) \in \mathcal{H}^d$ we write $D_h F = \sum_{i=1}^d \langle D^{(i)} F, h^i \rangle_{\mathcal{H}}$. For $L \geq 2$ we also denote by D_h^L the iterated versions of D_h. Namely we set

$$D_h^L F = D_h \circ \cdots \circ D_h F. \tag{2.16}$$

The Sobolev spaces related to the Malliavin derivatives are denoted by $\mathbb{D}^{k,p}$ and the corresponding norms are written $\|\cdot\|_{k,p}$. The dual of the Malliavin derivative is the Skorohod integral, for which we use the notation $\delta^\diamond$. Its domain includes the space $\mathbb{D}^{1,2}(\mathcal{H}^d)$, and the integration by parts formula can be read as

$$\mathbb{E}[F \delta^\diamond(u)] = \mathbb{E}[\langle DF, u \rangle_{\mathcal{H}^d}], \tag{2.17}$$

valid for $F \in \mathbb{D}^{1,2}$ and $u \in \mathbb{D}^{1,2}(\mathcal{H}^d)$.

2.3.1. *Differentiability*. As we will see below, under the condition that $V \in C_b^{\lfloor 1/\gamma \rfloor + 1}$ the solution y to (2.7) is differentiable in the Malliavin calculus sense. We shall express its Malliavin derivative in terms of the Jacobian Φ of the equation, which is defined by the relation $\Phi_t^{ij} = \partial_{a_j} y_t^{(i)}$, where recall that $a = (a_1, \dots, a_m)$ is the initial value of the system (2.7). Setting ∂V_j for the Jacobian of V_j seen as a function from $\mathbb{R}^m$ to $\mathbb{R}^m$, let us recall that Φ is the unique solution to the linear equation

$$\Phi_t = \mathrm{Id}_m + \int_0^t \partial V_0(y_s) \Phi_s \, ds + \sum_{j=1}^d \int_0^t \partial V_j(y_s) \Phi_s \, dx_s^j. \tag{2.18}$$

Moreover, the following results hold true.

PROPOSITION 2.8. *Let y be the solution to equation* (2.7) *and suppose* $(V_0, V_1, \dots, V_d)$ *is a collection of vector fields in* C_b^3. *Then for every* $i = 1, \dots, m$, $t > 0$, *and* $a \in \mathbb{R}^m$, *we have* $y_t^{(i)} \in \mathbb{D}^{2,p}(\mathcal{H})$ *for* $p \geq 1$ *and*

$$D_s^{(j)} y_t = \Phi_{s,t} V_j(y_s), \quad j = 1, \dots, d, 0 \leq s \leq t,$$

where $D_s^{(j)} y_t^{(i)}$ *is the* j*th component of* $D_s y_t^{(i)}$, $\Phi_t = \partial_a y_t$ *solves equation* (2.18) *and* $\Phi_{s,t} = \Phi_t \Phi_s^{-1}$.

Let us now quote the result [8], which gives a useful estimate for moments of the Jacobian of rough differential equations driven by Gaussian processes. Note that this result is expressed in terms of p-variations, for which we refer to [17].

PROPOSITION 2.9. *Consider a fractional Brownian motion* x *with Hurst parameter* $H \in (1/4, 1/2]$ *and* $p > 1/H$. *Then for any* $\eta \geq 1$, *there exists a finite constant* c_η *such that the Jacobian* Φ *defined by* (2.18) *satisfies*

$$\mathbb{E}\big[\|\Phi\|_{p\text{-var};[0,1]}^{\eta}\big] = c_\eta. \tag{2.19}$$

2.4. *Pathwise estimate of Euler scheme and its derivatives.* Recall that the Euler scheme y^n is defined in (1.2). In this subsection we state a pathwise upper-bound estimate of the Malliavin derivatives of y^n obtained in our companion paper [28]. We first introduce some notation.

Let b be a fBm independent of x. Recall that the rough paths above x and b are denoted by (x^1, x^2) and (b^1, b^2), respectively (see Definition 2.1). We introduce some second chaos processes which play a prominent role in the analysis of Euler schemes (see [29]). Namely for $[s,t] \in [\![0, T]\!]$ we set

$$\begin{aligned} q_{st}^{ij} &= \sum_{s \leq t_k < t} \left(x_{t_k t_{k+1}}^{2,ij} - \frac{1}{2} \Delta^{2H} \mathbf{1}_{\{i=j\}} \right), \\ q_{st}^{b,ij} &= \sum_{s \leq t_k < t} \left(b_{t_k t_{k+1}}^{2,ij} - \frac{1}{2} \Delta^{2H} \mathbf{1}_{\{i=j\}} \right). \end{aligned} \tag{2.20}$$

REMARK 2.10. The reader might wonder about the introduction of an additional fBm b. As we will see in Theorem 2.12, this additional fBm is crucial to bound Malliavin derivatives. The proof of Theorem 2.12 (see our companion paper [28]) is based on a technique borrowed from [26].

A main observation of the technique in [26] is that, given that y is the solution of a differential equation driven by a fBm x, the estimate of the Malliavin derivatives of y can be reduced to the estimate of a related differential equation driven by the joint process (x, b). This makes the Malliavin calculus bounds much more readable (the use of Cameron–Martin spaces is mostly avoided) and allows Hilbert–Schmidt norms estimates of the Malliavin derivatives.

In the current paper, our study of weak convergence for Euler scheme (1.2) requires the estimate of first and second Malliavin derivatives of the Euler scheme. For this purpose we have reproduced the techniques of [26] for the Euler scheme (1.2) in [28]. The introduction of the fBm b in the statement of Theorem 2.12 is thus a consequence of the application of this technique.

We recall a basic inequality taken from [29], Lemma 3.4: for $(s,t) \in \mathcal{S}_2([\![0, T]\!])$ we have

$$\big(\mathbb{E}\big[\|q_{st}\|^2\big]\big)^{1/2} \lesssim \frac{(t-s)^{1/2}}{n^{2H-1/2}}. \tag{2.21}$$

We also introduce a Gaussian process w which encompasses the coordinates of both the driving noise x and the extra noise b. Specifically we define w such that

$$\delta \mathrm{w}_{st} := (\delta \mathrm{w}^1_{st}, \dots, \delta \mathrm{w}^{2d}_{st}) := (\delta x^1_{st}, \dots, \delta x^d_{st}, \delta b^1_{st}, \dots, \delta b^d_{st}). \tag{2.22}$$

Let $\mathbf{w} = S_2(\mathrm{w}) = (w^1, w^2)$ be the p-rough path above w (see Definition 2.1). Denote by $\| \cdot \|_{p\text{-var}; [\![s,t]\!]}$ the p-variation norm on $[\![s, t]\!]$:

$$\|\mathbf{w}\|_{p\text{-var}; [\![s,t]\!]} = \left(\sup_{\mathcal{P}} \sum_{(u,v)\in\mathcal{P}} |\mathbf{w}_{uv}|^p \right)^{1/p},$$

where $|\mathbf{w}_{uv}| := |w^1_{uv}| + |w^2_{uv}|^{1/2}$, and the supremum is taken among all discrete intervals $\mathcal{P}$ such that $\mathcal{P} \subset [\![s, t]\!]$ and we write $(u, v) \in \mathcal{P}$ if u and v are two consecutive points of $\mathcal{P}$. We define a control ω by

$$\omega(s, t) = \|\mathbf{w}\|^p_{p\text{-var}; [\![s,t]\!]} + \|q\|^{p/2}_{p/2\text{-var}; [\![s,t]\!]} + \|q^b\|^{p/2}_{p/2\text{-var}; [\![s,t]\!]}, \quad (s, t) \in \mathcal{S}_2([\![0, T]\!]), \tag{2.23}$$

where q is defined in (2.20).

Our pathwise upper-bound estimate is achieved by considering small, medium, and big steps of the Euler scheme separately. Let $\alpha > 0$ be a positive constant. The small and medium steps are those such that $\omega(t_k, t_{k+1}) \le \alpha$, and the big steps otherwise. Precisely, let $s_0 = 0$ and define s_{j+1} recursively as

$$s_{j+1} = \begin{cases} s_j + \Delta, & \text{if } \omega(s_j, s_j + \Delta) > \alpha, \\ \max\{u \in [\![0, T]\!] : u > s_j \text{ and } \omega(s_j, u) \le \alpha\}, & \text{if } \omega(s_j, s_j + \Delta) \le \alpha. \end{cases} \tag{2.24}$$

Then we split the set of s_j's as

$$S_0 = \{s_j : \alpha/2 \le \omega(s_j, s_{j+1}) \le \alpha\}; \qquad S_1 = \{s_j : \omega(s_j, s_{j+1}) < \alpha/2\}; \tag{2.25}$$

$$S_2 = \{s_j : \omega(s_j, s_{j+1}) > \alpha\}. \tag{2.26}$$

We set

$$\begin{aligned} \mathcal{M}_0 &= \prod_{s_j \in S_0} (K\omega(s_j, s_{j+1})^{1/p} + 1), \\ \mathcal{M}_1 &= \prod_{s_j \in S_1} (K\omega(s_j, s_{j+1})^{1/p} + 1), \\ \mathcal{M}_2 &= \prod_{s_j \in S_2} (K|\delta \mathrm{w}_{s_j s_{j+1}}| + K\Delta^{2H} + 1), \end{aligned} \tag{2.27}$$

and K is a constant independent of n.

REMARK 2.11. The reason why we are considering small, medium and big steps separately in our upper-bound estimate is the following: when the increments are small or medium, the estimate of the Euler scheme can be derived in a similar way as for the continuous-time differential equations (see, e.g., the proof of [8], Theorem 8.4, for the continuous-time case). More specifically, we can find a proper constant $\alpha > 0$ depending on the coefficient function V and the Hurst parameter H only (α is defined explicitly in (4.27) in Section 4.2 in our companion paper [28]) such that the estimate of Euler scheme is obtained on $[s_j, s_{j+1}]$ for $s_j \in [0, T]$ such that $\omega(s_j, s_{j+1}) \le \alpha$. (Recall that $\omega(s, t)$ is defined in equation (2.23).) By iterating this estimate we can obtain a global upper bound for the Euler scheme.

On the other hand, when the increments are large, that is, when $\omega(s_j, s_j + T/n) > \alpha$, the above argument will require to take $s_{j+1} \in (s_j, s_j + T/n)$. Therefore, the iteration of the argument used in the small/medium increment case would not allow to get out of the time interval $[s_j, s_j + T/n]$. We thus have to proceed differently for our estimates in that case.

We now recall the following pathwise estimates for the Euler scheme in [28], Theorem 4.13.

THEOREM 2.12. *Take $r, r' \geq 0$. One can find integers k_0, k_0' such that $r \in (t_{k_0}, t_{k_0+1}]$ and $r' \in (t_{k_0'}, t_{k_0'+1}]$. Suppose that $V \in C_b^4$ and $p > 1/H$. Define a Malliavin derivative vector ξ^n as*

$$\xi_t^n = (y_t^n, D_r y_t^n, D_r D_{r'} y_t^n) := (\xi_t^{n,0}, \xi_t^{n,1}, \xi_t^{n,2}). \tag{2.28}$$

Then we can find a constant $\alpha > 0$ depending on V and H only such that for $L = 0, 1, 2$ and all $(s,t) \in \mathcal{S}_2[\![0,T]\!]$ we have the estimate

$$\|\xi^{n,L}\|_{p\text{-var},[\![s,t]\!]} \leq K \cdot \omega(s,t)^{1/p} \cdot \mathcal{G}, \tag{2.29}$$

where the random variable $\mathcal{G}$ is defined by

$$\mathcal{G} = |S_0 \cup S_1 \cup S_2| \cdot (\mathcal{M}_0 \cdot \mathcal{M}_1 \cdot \mathcal{M}_2)^L, \tag{2.30}$$

and where the quantities S_i, $\mathcal{M}_i$ are respectively defined for $i = 0, 1, 2$ in (2.25)–(2.26) and (2.27). Moreover, for $(s,t) \in \mathcal{S}_2([\![s_j, s_{j+1}]\!])$ such that $s_j \in S_0 \cup S_1$ we have

$$\begin{aligned} |\delta y_{st}^n - V(y_s^n)\delta x_{st}| &\leq K\omega(s,t)^{2/p}, \quad \text{and} \\ |\delta \xi_{st}^{n,L} - \mathcal{L}^L V(y_s^n)\delta x_{st}| &\leq K\omega(s,t)^{2/p} \cdot \mathcal{G}^2, \quad L = 1, 2, \end{aligned} \tag{2.31}$$

where we have set

$$\mathcal{L}^0 V(y_s^n) = V(y_s^n), \qquad \mathcal{L}^1 V(y_s^n) = \partial V(y_s^n) D_r y_s^n,$$

and where $\mathcal{L}^2 V(y_s^n)$ is defined by

$$\mathcal{L}^2 V(y_s^n) = \partial^2 V(y_s^n) D_r y_s^n D_{r'} y_s^n + \partial V(y_s^n) D_{r'} D_r y_s^n.$$

(*The reader is referred to* [28], *equations* (3.17) *and* (3.27), *for the general definition of the operator $\mathcal{L}^L$ and the explicit expression of the constant α, respectively.*) *In both estimates* (2.29) *and* (2.31), *K is a constant independent of r, r', j and n.*

REMARK 2.13. Our bound (2.31) involves the control ω defined by (2.23). This control depends on the fBm b and the quadratic process q^b. As mentioned in Remark 2.10, the fBm b thus plays a prominent role in Theorem 2.12.

REMARK 2.14. Theorem 2.12 is an upper-bound estimate for the first and second-order Malliavin derivatives of the Euler scheme. The reader is referred to [28], Theorem 4.13, for a general result for Malliavin derivatives of all orders. On the other hand, Theorem 2.12 will be sufficient for our purpose in this paper. Specifically, we will conduct Malliavin integration by parts twice in our study of the Euler scheme, and therefore only first and second-order Malliavin derivatives will appear. For this reason we have used the operators $\mathcal{L}^L$ for $L = 0, 1, 2$ only.

2.5. *Sharpness of the weak convergence rate*. In Theorem 1.1 we claim a rate of convergence $(1/n)^{4H-1-\varepsilon}$ for the Euler scheme. Before getting into the details of the proof for this statement, we now analyze a simple example showing that this rate is sharp.

Specifically, consider the following simple stochastic differential equation:

$$dy_t^1 = dx_t^1, \qquad dy_t^2 = y_t^1\, dx_t^2, \qquad y_0^1 = y_0^2 = 0, \quad t \in [0,1]. \tag{2.32}$$

We denote $y=(y^1,y^2)$ the solution. The driving process $x:=(x^1,x^2)$ is a two-dimensional fBm. In the setting of equation (1.1), this corresponds to the situation

$$m=d=2,\qquad T=1,\qquad V_0\equiv 0,\qquad a=0,\qquad V_1(y)=\begin{pmatrix}1\\0\end{pmatrix},\qquad V_2(y)=\begin{pmatrix}0\\y_1\end{pmatrix}.$$

Moreover, equation (2.32) admits the explicit solution

$$y_t^1=x_t^1,\qquad y_t^2=\int_0^t x_s^1\,dx_s^2,$$

where the integral is understood in the Stratonovich sense. Also notice that y^1 and y^2 are both components of the signature $S_2(x)$ in Definition 2.1.

In the specific case of equation (2.32), the numerical scheme (1.2) becomes

$$y_t^{1,n}=x_t^1,\qquad y_t^{2,n}=\sum_{0\le t_k<t} x_{t_k}^1\delta x_{t_kt_{k+1}}^2,\qquad t\in[\![0,1]\!],\tag{2.33}$$

where $t_k=k/n$, $k=0,\dots,n$. It has been shown in [32] (and also as a particular case of [29]) that a sharp strong rate of convergence of y^n to y is of order $(1/n)^{2H-1/2}$. As far as the weak rate of convergence is concerned, let us consider the test function

$$f:\mathbb{R}^2\to\mathbb{R}\quad\text{such that } f(z^1,z^2)=|z^2|^2.$$

Notice that f is not bounded as requested in Theorem 1.1. It is still a valid test function for a quick validation of our convergence rate. More specifically, a direct computation gives

$$\begin{aligned}\mathbb{E}f(y_1)&=\mathbb{E}\left(\left|\int_0^1 x_s^1\,dx_s^2\right|^2\right)\\&=\int_{[0,1]^2}R(u,v)\,dR(u,v)\\&=\sum_{0\le t_k,t_{k'}<1}\int_{[t_k,t_{k+1}]\times[t_{k'},t_{k'+1}]}R(u,v)\,dR(u,v).\end{aligned}\tag{2.34}$$

On the other hand, for the numerical approximation y_1^n we have

$$\begin{aligned}\mathbb{E}f(y_1^n)&=\mathbb{E}\left(\left|\sum_{0\le t_k<1}x_{t_k}^1\delta x_{t_kt_{k+1}}^2\right|^2\right)\\&=\sum_{0\le t_k,t_{k'}<1}R(t_k,t_{k'})R\big([t_k,t_{k+1}],[t_{k'},t_{k'+1}]\big)\\&=\sum_{0\le t_k,t_{k'}<1}\int_{[t_k,t_{k+1}]\times[t_{k'},t_{k'+1}]}R(t_k,t_{k'})\,dR(u,v).\end{aligned}\tag{2.35}$$

Taking the difference between (2.34) and (2.35) we obtain

$$\mathbb{E}f(y_1)-\mathbb{E}f\big(y_1^n\big)=\sum_{0\le t_k,t_{k'}<1}\int_{[t_k,t_{k+1}]\times[t_{k'},t_{k'+1}]}\big(R(u,v)-R(t_k,t_{k'})\big)\,dR(u,v).\tag{2.36}$$

Now applying the self-similarity property $n^{2H}R(s,t)=R(ns,nt)$ and then the change of variable $(nu,nv)\to(u,v)$ to (2.36) we obtain

$$\mathbb{E}f(y_1)-\mathbb{E}f\big(y_1^n\big)=(1/n)^{4H-1}\lambda(n),\tag{2.37}$$

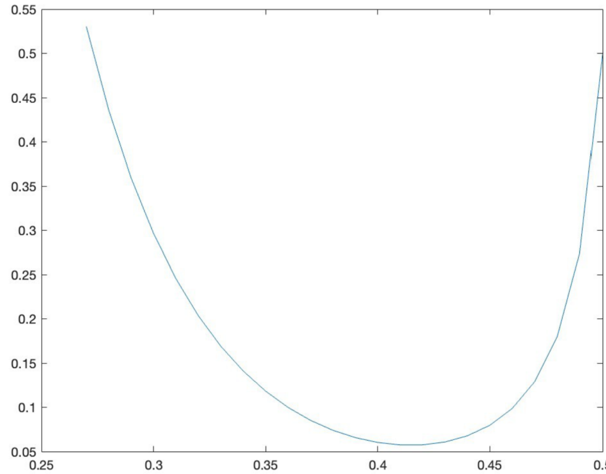

FIG. 1. *The value of* $\lim_{n\to\infty}\lambda(n)$ *versus* H

where we denote

$$\lambda(n) = \frac{1}{n}\sum_{k,k'=0}^{n-1}\int_{[k,k+1]\times[k',k'+1]} \big(R(u,v) - R(k,k')\big)\, dR(u,v).$$

Figure 1 is a plot of the numerical values of the limit $\lim_{n\to\infty}\lambda(n)$ for various H, carried out using MATLAB.

The plot shows that $\lambda(n)$ has a nonzero limit for all $H \in (1/3, 1/2)$. According to relation (2.37), this implies that $(1/n)^{4H-1}$ is the exact rate of (2.33). The sharpness of this rate then follows.

2.6. *Outline of proof for Theorem* 1.1. We conclude this section by providing an outline of the proof for our main result, Theorem 1.1. For convenience we consider the one-dimensional setting $m = d = 1$ only.

Our first step of the proof is to derive the following representation of the error process $y - y^n$:

$$y_t - y_t^n = \sum_{0\le t_k<t} F_{t_k} q_{t_k t_{k+1}} + \text{higher-order multiple integrals}, \tag{2.38}$$

where q is the one-dimensional version of the quadratic functional q^{ij} defined in (2.20). Namely, in dimension $d = 1$ we have $q_{t_k t_{k+1}} = \frac{1}{2}((\delta x_{t_k t_{k+1}})^2 - \Delta^{2H})$. The reader is referred to relation (4.63) in Section 4.5 for the precise representation of the error $y_t^n - y_t$. Specifically, the weighted sum $\sum_{0\le t_k<t} F_{t_k} q_{t_k t_{k+1}}$ is corresponding to $J_t^1 + J_t^5$. Notice that the representation (2.38) and (4.63) are obtained via the "fundamental" solutions, denoted by Γ and Λ (see Definition 4.12), of a linear equation satisfied by the error process $y - y^n$. For $t \in [0, T]$, F_t is a random variable involving the processes Γ, Λ, y and y^n.

Let f be a smooth function. With relation (2.38) in hand and applying an elementary interpolation, we have

$$\mathbb{E}f(y_t) - \mathbb{E}f\big(y_t^n\big) = \sum_{0\le t_k<t} \mathbb{E}(\tilde{F}_{t_k} q_{t_k t_{k+1}}) + \mathbb{E}(\text{higher-order multiple integrals}), \tag{2.39}$$

where $\tilde{F}$ is a process similar to F. Note that if one applies deterministic type estimates to the right-hand side of (2.39), the convergence rate can be shown to be $(1/n)^{2H-1/2}$; see [29]. Our goal is to show that using Malliavin calculus we obtain an order $(1/n)^{4H-1}$.

To this aim, our second main step is to prove that processes Γ, Λ, y and y^n and their Malliavin derivatives are integrable uniformly in n. This is the content of Section 3 and Section 4.3-4.4. More specifically, we show that $\mathcal{M}_0$, $\mathcal{M}_1$ and $\mathcal{M}_2$ in (2.27) are integrable; see Section 3. Then, according to Theorem 2.12, this implies that the Malliavin derivatives of y^n are also integrable. In Section 4.3-4.4 we derive an upper-bound estimate for Γ, Λ and their Malliavin derivatives similar to those of Theorem 2.12; see Lemma 4.13. Then by applying the integrability of $\mathcal{M}_0$, $\mathcal{M}_1$ and $\mathcal{M}_2$ again we obtain the uniform integrability for Γ, Λ and their Malliavin derivatives.

With the uniform integrability in hand, our next main step is to apply the integration by parts to the expected value $\mathbb{E}[F_{t_k} q_{t_k t_{k+1}}]$, which gives:

$$\mathbb{E}[F_{t_k} q_{t_k t_{k+1}}] = \mathbb{E}\langle D^2 F_{t_k}, \beta_{t_k t_{k+1}} \rangle_{\mathcal{H}^{\otimes 2}},$$

where β stands for the indicate function $\beta_{st} = \mathbf{1}\{(u,v) : s \le u \le v \le t\}$. Then some nice fBm scaling and monotonicity properties observed in Section 4.1 allow to show that the quantity $\mathbb{E}\langle D^2 F_{t_k}, \beta_{t_k t_{k+1}} \rangle_{\mathcal{H}^{\otimes 2}}$ is bounded by $(1/n)^{4H}$.

Finally, we show that the higher-order multiple integrals in (2.39) can also be bounded by $(1/n)^{4H-1}$, with the help of the results developed in Section 4.2.

3. Uniform integrability for Malliavin derivatives of the Euler scheme. In this section we tackle the integrability issue for the Malliavin derivatives of the Euler scheme. Before proceeding to our main considerations, some remarks about our global strategy are in order. Recall that $\mathcal{M}_0$, $\mathcal{M}_1$, $\mathcal{M}_2$ are defined in (2.27), b is an independent copy of x, and q^b are defined in (2.20). Recall that the signature $\mathbf{x} = S_2(x) = (x^1, x^2)$ of x is defined in Definition 2.1.

REMARK 3.1. Due to the bound (2.29), the integrability of $\mathcal{M}_0$, $\mathcal{M}_1$, $\mathcal{M}_2$ is our main task towards a uniform bound for the Malliavin derivatives as a function of n. We mostly focus on this problem in the sequel.

REMARK 3.2. Observe that b (resp. q^b) in (2.23) is a mere copy of the fBm x (resp. the quadratic sum q). Furthermore, expression (2.23) involves both quantities $\|q\|_{p/2\text{-var}}$ and $\|q^b\|_{p/2\text{-var}}$. Duplicates like this one would be ubiquitous in our computations. Therefore, in order to avoid lengthy expressions, we will simply omit the b-terms in the sequel. With this convention, (2.23) would become

$$\omega(s,t) = |\mathbf{x}|^p_{p\text{-var};[\![s,t]\!]} + |q|^{p/2}_{p/2\text{-var};[\![s,t]\!]}. \tag{3.1}$$

To be completely clear, let us highlight the fact that we do not assume $b = 0$ here. We have just chosen to drop the b-terms from our expressions involving ω in (2.23), for notational sake. We hope this does not lead to confusions.

We now turn our attention to the integrability of the random variables $\mathcal{M}_i$.

3.1. *Uniform integrability of $\mathcal{M}_1$ and $\mathcal{M}_2$.* In this subsection, we consider the uniform integrability of $\mathcal{M}_1$ and $\mathcal{M}_2$ in (2.27). The proof is achieved thanks to a tail analysis of the cardinality of the large size steps, that is, steps with size $> \alpha$.

THEOREM 3.3. *Let $\mathcal{M}_2 = \mathcal{M}_2(n)$ be the random variable defined by* (2.27). *Suppose that $\mathcal{M}_2$ is given by*

$$\mathcal{M}_2 = \prod_{s_j \in S_2} \left(K|\delta x_{s_i s_{i+1}}| + K\Delta^{2H} + 1 \right), \tag{3.2}$$

where S_2 is the subset of $[\![0,T]\!]$ displayed in (2.26). *Suppose that x is a fBm with Hurst parameter $H > 1/3$. Then we have $\sup_{n\in\mathbb{N}} \mathbb{E}[|\mathcal{M}_2|^\nu] < \infty$ for all $\nu \ge 1$.*

PROOF. Recall that ω is defined by (2.23). With our convention (3.1), since b and q^b are omitted, the control ω is reduced to

$$\omega(t_k, t_{k+1}) = |\mathbf{x}_{t_k t_{k+1}}|^p + |q_{t_k t_{k+1}}|^{p/2},$$

where $|\mathbf{x}_{t_k t_{k+1}}|$ stands for $|x^1_{t_k t_{k+1}}| + |x^2_{t_k t_{k+1}}|^{1/2}$, so that definition (2.26) yields the following relation:

$$S_2 = \{t_k : \omega(t_k, t_{k+1}) > \alpha\} = \{t_k : |\mathbf{x}_{t_k t_{k+1}}|^p + |q_{t_k t_{k+1}}|^{p/2} > \alpha\}. \tag{3.3}$$

Moreover, it is readily checked that

$$|\mathbf{x}_{t_k t_{k+1}}|^p + |q_{t_k t_{k+1}}|^{p/2} \lesssim |\delta x_{t_k t_{k+1}}|^p + |x^2_{t_k t_{k+1}}|^{p/2} + |\Delta|^{Hp} \lesssim |\mathbf{x}_{t_k t_{k+1}}|^p + |\Delta|^{Hp}.$$

If we choose n so that $|\Delta|^{Hp} \le \alpha/2$, that is, $n \ge T \cdot (\alpha/2)^{-\frac{1}{Hp}} =: K_{\alpha,p,T}$, from the expression (3.3) we get

$$S_2 \subset \{t_k : |\mathbf{x}_{t_k t_{k+1}}|^2 > (\alpha/2)^{2/p}\} =: S_3.$$

This inclusion implies that

$$\mathcal{M}_2 = \prod_{s_j \in S_2} (K|\mathbf{x}_{s_j s_{j+1}}| + 1) \le \prod_{s_j \in S_3} (K|\mathbf{x}_{s_j s_{j+1}}| + 1) \le C_\alpha \prod_{s_j \in S_3} K|\mathbf{x}_{s_j s_{j+1}}|, \tag{3.4}$$

where we have invoked the fact that $|\mathbf{x}_{s_j s_{j+1}}| > (\alpha/2)^{1/p}$ whenever $s_j \in S_3$ for the last inequality. In the following, we show that the ν-moment of the right-hand side of (3.4) is bounded uniformly in n. This implies that $\sup_{n \ge K_{\alpha,p,T}} \mathbb{E}[|\mathcal{M}_2|^\nu] < \infty$. Recall that α is a constant depending on V and H only (see Theorem 2.12). It will thus follow that

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|\mathcal{M}_2|^\nu] < \infty.$$

We now divide the proof in several steps.

Step 1: Some pathwise bounds. Recall again that x is a fBm with Hurst parameter $H > 1/3$. Pick $\beta < H$. Denote by $\|\delta x\|_{[0,T],\beta}$ and $\|x^2\|_{[0,T],2\beta}$ the Hölder norms of δx and x^2, respectively. Namely,

$$\|\delta x\|_{[0,T],\beta} = \sup_{(s,t) \in \mathcal{S}_2([0,T])} \frac{|\delta x_{st}|}{|t-s|^\beta} \quad \text{and} \quad \|x^2\|_{[0,T],2\beta} = \sup_{(s,t) \in \mathcal{S}_2([0,T])} \frac{|x^2_{st}|}{|t-s|^{2\beta}}.$$

Let

$$\mathcal{G} \equiv \|\mathbf{x}\|_{[0,T],\beta} := \|\delta x\|_{[0,T],\beta} + \|x^2\|^{1/2}_{[0,T],2\beta}. \tag{3.5}$$

It is well known that $\mathcal{G}$ is almost surely finite. This implies that $|\mathbf{x}_{s_j s_{j+1}}| \le \mathcal{G} n^{-\beta}$, and therefore

$$\mathcal{M}_2 \le \prod_{t_k \in S_3} K(\mathcal{G} n^{-\beta}) \le (K\mathcal{G} n^{-\beta})^{|S_3|}. \tag{3.6}$$

In addition, according to (3.5) we have

$$S_3 \subset \bigcup_{i,j} (S^i_{31} \cup S^{ij}_{32} \cup S^{ij}_{33}),$$

where the sets S_{31}, S_{32}, S_{33} are defined by

$$\begin{aligned} S^i_{31} &= \{t_k : (\delta x^i_{t_k t_{k+1}})^2 > \alpha_p\}, \qquad S^{ij}_{32} = \{t_k : x^{2,i,j}_{t_k t_{k+1}} > \alpha_p\}, \\ S^{ij}_{33} &= \{t_k : -x^{2,i,j}_{t_k t_{k+1}} > \alpha_p\}, \end{aligned} \tag{3.7}$$

and where taking into account the dimension of $\mathbf{x}$, we take α_p defined by

$$\alpha_p = \left(\frac{\alpha}{2(d^2+d+1)}\right)^{2/p}. \tag{3.8}$$

Therefore recalling that K designates a generic constant, we have obtained the following upper bound for the random variable $\mathcal{M}_2$:

$$\mathcal{M}_2 \le \left(\prod_{i=1}^{d} (K\mathcal{G}n^{-\beta})^{|S_{31}^i|}\right) \cdot \left(\prod_{i,j=1}^{d} (K\mathcal{G}n^{-\beta})^{|S_{32}^{ij}|}\right) \cdot \left(\prod_{i,j=1}^{d} (K\mathcal{G}n^{-\beta})^{|S_{33}^{ij}|}\right). \tag{3.9}$$

In the following we consider the integrability of the random variable $(K\mathcal{G}n^{-\beta})^{|S_{31}^i|}$ for all $i = 1, \dots, d$ which appear in the right-hand side of (3.9). The other terms in (3.9) can be handled very similarly.

Step 2: Tail estimates for $|S_{31}|$. For each subset $\mathbf{u} \equiv \{u_j;\, j = 1, \dots, n'\}$, $n' \le n$ of the set of discrete instants $[\![0, T]\!] = \{t_k, k = 0, 1, \dots, n\}$ we introduce the quantity:

$$\mathcal{X}^i(\mathbf{u}) = \sum_{t_k \in \{u_j;\, j=1,\dots,n'\}} \left(|\delta x^i_{t_k t_{k+1}}|^2 - \Delta^{2H}\right).$$

Suppose that $|S_{31}^i| = n'$ and denote $S_{31}^i = \{u_j;\, j = 1, \dots, n'\} =: \mathbf{u} \subset [\![0, T]\!]$. Then by the definition of S_{31}^i in (3.7) we have $|\delta x^i_{u_j u_{j+1}}|^2 > \alpha_p$ for all $u_j \in S_{31}^i$, where α_p is defined by (3.8). It follows that for n such that $\Delta^{2H} < \alpha_p/2$ we have

$$\mathcal{X}^i(\mathbf{u}) > n'(\alpha_p - \alpha_p/2) = n'\alpha_p/2.$$

We have thus proved that

$$\{|S_{31}^i| = n'\} \subset \bigcup_{\mathbf{u} \subset \{t_k\}} \{\mathcal{X}^i(\mathbf{u}) > n'\alpha_p/2\}. \tag{3.10}$$

As a consequence of the above relation, we trivially get

$$\mathbb{P}\{|S_{31}^i| = n'\} \le \sum_{\mathbf{u} \subset \{t_k\}} \mathbb{P}\{\mathcal{X}^i(\mathbf{u}) > n'\alpha_p/2\}. \tag{3.11}$$

Next set $\sigma_{\mathbf{u}}^2 = (\mathbb{E}|\mathcal{X}^i(\mathbf{u})|^2)$. Owing to a slight variation of (2.21) we have $\sigma_{\mathbf{u}}^2 \lesssim 1/n^{4H-1}$. Therefore starting from the right-hand side of (3.11) we get

$$\begin{aligned} \mathbb{P}(|S_{31}^i| = n') &\le \sum_{\{u_j;\, j=1,\dots,n'\} \subset \{t_k\}} \mathbb{P}\left\{\frac{\mathcal{X}^i(\mathbf{u})}{\sigma_{\mathbf{u}}} > \frac{K_{p,\alpha} \cdot n'}{\sigma_{\mathbf{u}}}\right\} \\ &\le \sum_{\{u_j;\, j=1,\dots,n'\} \subset \{t_k\}} \mathbb{P}\left\{\frac{\mathcal{X}^i(\mathbf{u})}{\sigma_{\mathbf{u}}} > \frac{K_{p,\alpha} \cdot n'}{1/n^{2H-1/2}}\right\}. \end{aligned} \tag{3.12}$$

The right-hand side of (3.12) is handled in the following way: taking into account the fact that $\mathcal{X}^i(\mathbf{u})/\sigma_{\mathbf{u}}$ is a normalized random variable in the second chaos of x, we apply Borell's inequality (see, e.g., [22], Theorem 5.12). In addition the number of sets of the form $\mathbf{u} = \{u_j;\, j = 1, \dots, n'\}$ is $\binom{n}{n'}$. Hence we end up with

$$\begin{aligned} \mathbb{P}(|S_{31}^i| = n') &\le \frac{n!}{n'!(n-n')!} \exp(-n^{2H-1/2} \cdot K_{p,\alpha} \cdot n') \\ &\le n^{n'} \exp(-n^{2H-1/2} \cdot K_{p,\alpha} \cdot n'). \end{aligned} \tag{3.13}$$

Step 3: Computations involving $\mathcal{G}$. Let us now turn our attention to the term $\mathcal{G}$ in (3.5). Since our fBm x is a Gaussian process, Fernique's lemma asserts that $\mathbb{P}(\mathcal{G} > a) \le e^{-Ka^2}$ for a given constant K and any $a \ge 1$. This sub-Gaussian bound is sufficient to claim that for all $n' \ge 1$ we have

$$\mathbb{E}\mathcal{G}^{n'} \le K(n')^{n'} = Ke^{n' \ln n'}. \tag{3.14}$$

We are now ready to go back to the study of the random variable $(K\mathcal{G}n^{-\beta})^{\nu|S_{31}^i|}$, $K > 0$. Namely we apply Hölder's inequality with two conjugates $p, q > 1$, and we combine this with (3.13) and (3.14). We get

$$\begin{aligned} \mathbb{E}\big((K\mathcal{G}n^{-\beta})^{\nu n'}\mathbf{1}_{\{|S_{31}^i|=n'\}}\big) &= K^{\nu n'} n^{-\beta\nu n'} \big(\mathbb{E}[\mathcal{G}^{p\nu n'}]\big)^{1/p} \cdot \mathbb{P}\big(|S_{31}^i| = n'\big)^{1/q} \\ &\le e^{\nu n' \ln n'} n^{-\beta \nu n'} \cdot \mathbb{P}\big(|S_{31}^i| = n'\big)^{1/q} \\ &\le \exp\big(f(n')\big), \end{aligned} \tag{3.15}$$

where the function f is defined by

$$f(n') := K_1 n' \ln n' - \beta K n' \ln n + K_2 n' \ln n - K_3 n^{2H-1/2} n', \tag{3.16}$$

for three positive constants K_1, K_2, K_3 whose exact value is irrelevant.

We now compute the maximum of the function f thanks to elementary considerations. First we calculate

$$f''(n') = K/n' \ge 0.$$

Therefore f is upward convex and

$$\sup_{2 \le n' \le n} f(n') \le f(2) \vee f(n) \le f(2) + f(n). \tag{3.17}$$

Moreover one can explicitly compute $f(2)$ and $f(n)$ thanks to the expression (3.16). We obtain

$$f(n) = (K_1 + K_2 - \beta K) n \ln n - K n^{2H+1/2}, \qquad f(2) = K - K \ln n - K n^{2H-1/2}.$$

Reporting this expression into (3.15), we discover that

$$\begin{aligned} \mathbb{E}(K\mathcal{G}n^{-\beta})^{\nu|S_{31}^i|} &= \sum_{n'=0}^{n} \mathbb{E}\big((K\mathcal{G}n^{-\beta})^{\nu n'} \mathbf{1}_{\{|S_{31}|=n'\}}\big) \\ &\le \sum_{n'=2}^{n} \exp\big(f(n')\big) \\ &\le n \exp\big(f(n) + f(2)\big) \le n \exp\big(C_1 n \ln n - C_2 n^{2H+1/2}\big). \end{aligned} \tag{3.18}$$

Step 4: Conclusion. Since we have assumed $H > 1/3$, it is readily checked that the right-hand side of (3.18) is dominated by a constant. Therefore, we end up with the inequality $\sup_{n\ge1} \mathbb{E}[(K\mathcal{G}n^{-\beta})^{\nu|S_{31}^i|}] \equiv M < \infty$. This concludes the uniform (in n) integrability of $(K\mathcal{G}n^{-\beta})^{\nu|S_{31}^i|}$, for all indices $i = 1, \dots, d$. The integrability of the other two quantities $(K\mathcal{G}n^{-\beta})^{\nu|S_{32}^{ij}|}$ and $(\mathcal{G}n^{-\beta})^{\nu|S_{33}^{ij}|}$ can be shown in a similar way. Combining these integrability results with relation (3.9) and with Hölder's inequality, we obtain the uniform integrability of $\mathcal{M}_2^\nu$. Our proof is complete. $\square$

REMARK 3.4. As the reader can see, we have used the assumption $H > 1/3$ in order to bound the right-hand side of (3.18). This hypothesis is ubiquitous in our considerations. As far as extensions to lower values of H, let us mention the following two points: (1) We have not proved yet that the our numerical scheme (1.2) is convergent when $H \in (1/4, 1/3]$. Now for the case $H > 1/3$, rough paths computations are based on second order expansions. Therefore we only need to deal with tail estimate of quadratic functionals in (3.10). We are not sure if our arguments in Theorem 3.3 still apply for higher-order rough path expansions or numerical methods. (2) To the best of our knowledge, a proper construction of a geometric rough path above a fBm x with $H \leq 1/4$ is still an open question. Therefore solving equation (1.1) for $H \leq 1/4$ is also an open problem.

Once the bound of $\mathcal{M}_2$ is established, we can link the expected value of $\mathcal{M}_1$ to that of $\mathcal{M}_2$ by the observation that there are less steps with a small size ($< \alpha$) than with a large size ($> \alpha$). This is the content of the following result.

COROLLARY 3.5. *Let $\mathcal{M}_1$ be defined in* (2.27) *and we are still working with a fBm x with Hurst parameter $H > 1/3$. Then $\sup_{n\in\mathbb{N}} \mathbb{E}[|\mathcal{M}_1|^\nu] < \infty$ for all $\nu \geq 1$.*

PROOF. In order to consider the integrability of $\mathcal{M}_1$ we observe that by the definition of S_1, for $s_j \in S_1$ we must have $\omega(s_j, s_{j+1}) \leq \alpha/2$ and $\omega(s_{j+1}, s_{j+1} + \Delta) \geq \alpha/2$. This implies that the cardinality of S_1 is less than that of the set $S_3' = \{t_k : \omega(t_k, t_{k+1}) > \alpha/2\}$, namely, $|S_1| \leq |S_3'|$, and so

$$\mathcal{M}_1 = \prod_{s_j \in S_1} \big(K_1 \omega(s_j, s_{j+1})^{1/p} + 1\big) \leq (K_1\alpha + 1)^{|S_1|} \leq K^{|S_3'|}.$$

Observe that $K^{|S_3'|}$ is in the form similar to (3.6) for $\mathcal{M}_2$. So in a similar way as in Theorem 3.3, we can show that $K^{\nu|S_3'|}$ and thus $\mathcal{M}_1^\nu$ is uniformly integrable. □

3.2. *Integrability of $\mathcal{M}_0$.* In this section, we will take care of the products in (2.27) involving small steps of ω. Now recall that those steps, defined by (2.23), involve the Gaussian process w and the second chaos process q. The presence of q will require a specific translation procedure on the Wiener space, which is carried out in Section 3.2.1. Then a weighted sum argument is invoked in Section 3.2.2.

3.2.1. *Translation of the fBm and some functionals.* Let us recall that x is a fBm with $H > 1/3$ and q is defined in (2.20). In this subsection, we consider an upper-bound estimate for the translation of the fBm x and the process q. Notice that in the sequel our generic random element in the space Ω will be denoted by ϕ. In the following result, we deal with the Cameron–Martin space (or equivalently, the reproducing kernel Hilbert space) related to our fBm; the reader is referred to our companion paper [28], Section 2.2, for an introduction of this space.

LEMMA 3.6. *Take $3 > p > 1/H$ and $p' > 1$ such that $1/p + 1/p' > 1$. Let h be a path in $C^{p'\text{-var}}([0,T], \mathbb{R}^m)$, and let T_h denote the translation operator: $T_h\phi = \phi + h$ on the Cameron–Martin space associated with our fBm. Then the following translation inequality holds:*

$$\begin{aligned} &\|T_h q\|^{p/2}_{p/2\text{-var},[\![s,t]\!]} + \|T_h \mathbf{x}\|^p_{p\text{-var},[\![s,t]\!]} \\ &\quad \leq K_p\big(\|q\|^{p/2}_{p/2\text{-var},[\![s,t]\!]} + \|\mathbf{x}\|^p_{p\text{-var},[s,t]} + \|h\|^p_{p'\text{-var},[s,t]}\big), \end{aligned} \tag{3.19}$$

where K_p is a constant depending only on p.

PROOF. The estimate of $\|T_h\mathbf{x}\|^p_{p\text{-var},[\![s,t]\!]}$ is shown in Lemma 3.1 [8]. In the following we consider the estimate of $\|T_h q\|^{p/2}_{p/2\text{-var},[\![s,t]\!]}$. Specifically, consider an element $u,v\in\mathcal{S}_2([\![s,t]\!])$. By definition we can write

$$T_h q_{uv} = q_{uv} + A^1_{uv} + A^2_{uv} + A^3_{uv}, \tag{3.20}$$

where

$$A^1_{uv} = \sum_{u\le t_k<v}\int_{t_k}^{t_{k+1}} \delta h_{t_k r}\otimes dx_r, \qquad A^2_{uv} = \sum_{u\le t_k<v}\int_{t_k}^{t_{k+1}} \delta x_{t_k r}\otimes dh_r, \tag{3.21}$$

$$A^3_{uv} = \sum_{u\le t_k<v}\int_{t_k}^{t_{k+1}} \delta h_{t_k r}\otimes dh_r. \tag{3.22}$$

Next we further decompose the term A^1 into

$$A^1_{uv} = A^{11}_{uv} + A^{12}_{uv}, \tag{3.23}$$

where A^{11} and A^{12} are respectively defined by

$$A^{11}_{uv} = \int_u^v \delta h_{ur}\otimes dx_r, \quad \text{and} \quad A^{12}_{uv} = -\mathcal{J}^v_u(h,x), \tag{3.24}$$

and where the term $\mathcal{J}$ above is given as

$$\mathcal{J}^v_u(h,x) = \sum_{u\le t_k<v} \delta h_{ut_k}\otimes \delta x_{t_k t_{k+1}}. \tag{3.25}$$

In the following, we bound the terms on the right-hand side of (3.20).

First, by a direct computation for all $(u,r,v)\in\mathcal{S}_3([\![s,t]\!])$ we have

$$\delta A^{12}_{urv} = \delta h_{ur}\otimes \delta x_{rv}. \tag{3.26}$$

In order to bound δA^{12}, we consider the function

$$\omega(u,v) =: \|h\|_{p'\text{-var},[u,v]}\|\mathbf{x}\|_{p\text{-var},[u,v]}. \tag{3.27}$$

It is well known that since $1/p+1/p'>1$, ω is a control function. In fact, it is easy to show that $\omega_1 =: \omega^{1/\mu}$ is a control function for μ such that $1/p+1/p'>\mu>1$. It follows from (3.26) and the definition of ω_1 that

$$|\delta A^{12}_{urv}| \le \omega_1(u,v)^\mu.$$

In addition, it is readily checked from our definition (3.25) that $A^{12}_{t_k t_{k+1}}=0$ for all $t_k\in[\![s,t]\!]$. Therefore a direct application of Lemma 2.4 yields:

$$|A^{12}_{uv}| \le K_\mu \omega_1(u,v)^\mu = K_\mu\omega(u,v). \tag{3.28}$$

Let us turn to the estimate of A^{11} defined by (3.24). In that case, due to the fact that A^{11} can be interpreted as a Young integral, some elementary estimates (see, e.g., [37]) reveal that

$$|A^{11}_{uv}| \le \omega(u,v). \tag{3.29}$$

Hence reporting (3.28) and (3.29) into (3.23) we end up with

$$|A^1_{uv}| \le (K_\mu+1)\omega(u,v) \le (K_\mu+1)\big(\|h\|^2_{p'\text{-var},[u,v]} + \|\mathbf{x}\|^2_{p\text{-var},[u,v]}\big), \tag{3.30}$$

where we recall that the control ω is given by (3.27). The term A^2 in (3.21) can be bounded in a similar way as for A^1, and we obtain the same estimate as in (3.30). The details are thus omitted.

In order to bound A^3 defined by (3.22), we apply Young's inequality again and also the super-additivity of the control $\omega_2(u,v) =: \|h\|^2_{p'\text{-var},[u,v]}$ (Notice that ω_2 is a control owing to the fact that $p' < 2$). We get

$$\left|A^3_{uv}\right| \le \sum_{u \le t_k < v} \|h\|^2_{p'\text{-var},[t_k,t_{k+1}]} \le \|h\|^2_{p'\text{-var},[u,v]}.$$

Putting together the estimates of A^1, A^2 and A^3 and equation (3.20), we obtain

$$|T_h q_{uv}| \le |q_{uv}| + 2(K_\mu + 1)\|\mathbf{x}\|^2_{p\text{-var},[u,v]} + (2K_\mu + 3)\|h\|^2_{p'\text{-var},[u,v]}. \tag{3.31}$$

Now consider a generic partition $\pi = \{u_j\}$ of $[\![s,t]\!]$. Thanks to (3.31) and super-additivity properties we have

$$\begin{aligned}
&\sum_{\{u_j\}} |T_h q_{u_j u_{j+1}}|^{p/2} \\
&\quad \le K_p\Bigg(\sum_{\{u_j\}} |q_{u_j u_{j+1}}|^{p/2} + \sum_{\{u_j\}} \|\mathbf{x}\|^p_{p\text{-var},[u_j,u_{j+1}]} + \sum_{\{u_j\}} \|h\|^p_{p'\text{-var},[u_j,u_{j+1}]}\Bigg) \\
&\quad \le K_p\big(\|q\|^{p/2}_{p/2\text{-var},[s,t]} + \|\mathbf{x}\|^p_{p\text{-var},[s,t]} + \|h\|^p_{p'\text{-var},[s,t]}\big).
\end{aligned}$$

Finally, taking the sup over all partitions of $[s,t]$ on the left side we obtain the desired estimate (3.19). □

3.2.2. *Integrability of $\mathcal{M}_0$.* This section is devoted to a study of the intermediate sized increments of ω. Otherwise stated, we are ready to show the uniform integrability of $\mathcal{M}_0$.

THEOREM 3.7. *Let S_0 and $\mathcal{M}_0$ be defined in* (2.25) *and* (2.27), *respectively, for a fBm x with Hurst parameter $H > 1/3$ and a threshold $\alpha > 0$. Then for any given $\gamma < 2H + 1$ there exists $K = K_\gamma$ such that for all $a \ge 1$ we have*

$$\mathbb{P}\big(|S_0| > a\big) \le K e^{-K a^\gamma}. \tag{3.32}$$

In particular, $\sup_{n\in\mathbb{N}} \mathbb{E}[\mathcal{M}_0^\nu] < \infty$ *for all $\nu \ge 1$.*

PROOF. The proof will be done in several steps.

Step 1. Preparations. Recall that S_0 is given by

$$S_0 = \{s_j : \alpha/2 \le \omega(s_j, s_{j+1}) \le \alpha\}, \tag{3.33}$$

where ω is defined in (2.23). As mentioned in Remark 3.2, we will drop the b-terms in ω for sake of conciseness. Precisely, we will prove Theorem 3.7 for S_0 given in (3.33) but with ω defined in (3.1), instead of in (2.23). Note that with this consideration the set $\{|S_0| \le a\}$ is now a collection of sample paths of the fBm x, instead of that of the extended fBm (x,b). We would like to insist again on the fact that we do not assume $b = 0$ here. We just omit the b-terms in our computations for notational sake.

Let us go back to inequality (3.19). Remember that $p > 1/H$ therein. Since $H > 1/4$, it is easily checked that one can pick $p' > (H + 1/2)^{-1}$ such that p, p' still satisfy $\frac{1}{p} + \frac{1}{p'} > 1$. This pair of p, p' will be fixed for the remainder of the proof. Recalling the constant K_p featuring in (3.19) and our threshold α, we also choose $\beta > 0$ small enough so that $\alpha/2 - K_p\beta > 0$. Since $p > 1/H$, according to [29], Remark 3.6, there exists an almost surely finite random variable G_p such that $\sup_{n\in\mathbb{N}} \|q\|^{p/2}_{p/2\text{-var},[\![s,t]\!]} \le G_p$. Related to those quantities, we define the following two sets:

$$A^n = \{\phi \in \Omega : \|q\|^{p/2}_{p/2\text{-var},[\![s,t]\!]} + \|\mathbf{x}\|^p_{p\text{-var},[s,t]} < \beta\},$$
$$A = \{\phi \in \Omega : G_p + \|\mathbf{x}\|^p_{p\text{-var},[s,t]} < \beta\},$$

where we recall that the typical element of $(\Omega, \mathcal{F}, \mathbb{P})$ is denoted by ϕ. It is clear that $A \subset A^n$.

Step 2. Tail inclusion relations. Let $a \ge 1$ be our generic threshold. Having the notation of Step 1 in mind we define a constant κ as follows:

$$\kappa = \left(\frac{\alpha/2 - K_p\beta}{K_p}\right)^{1/p} a^{1/p'}. \tag{3.34}$$

Recall that the Cameron–Martin type space $\bar{\mathcal{H}}$ is defined in Section 2.3. Let us call $B_{\bar{\mathcal{H}}}$ the unit ball in $\bar{\mathcal{H}}$, namely: $B_{\bar{\mathcal{H}}} = \{h \in \bar{\mathcal{H}}; \|h\|_{\bar{\mathcal{H}}} \le 1\}$. Our first aim is to show that

$$A + \kappa B_{\bar{\mathcal{H}}} \subset A^n + \kappa B_{\bar{\mathcal{H}}} \subset \{|S_0| \le a\}. \tag{3.35}$$

Suppose that $\phi \in A^n + \kappa B_{\bar{\mathcal{H}}}$. In the following, we show that $|S_0| \le a$ for such ϕ, which then implies the relation (3.35). First, for $\phi \in A^n + \kappa B_{\bar{\mathcal{H}}}$ we have $\phi - h \in A^n$ for some $h \in \kappa B_{\bar{\mathcal{H}}}$, and thus

$$\|q(\phi - h)\|^{p/2}_{p/2\text{-var},[\![s,t]\!]} + \|\mathbf{x}(\phi - h)\|^p_{p\text{-var},[s,t]} < \beta.$$

Recall that $T_h\mathbf{x}(\phi) = \mathbf{x}(\phi + h)$ for any $h \in \bar{\mathcal{H}}$ almost surely. Hence the above relation becomes

$$\|T_{-h}q(\phi)\|^{p/2}_{p/2\text{-var},[\![s,t]\!]} + \|T_{-h}\mathbf{x}(\phi)\|^p_{p\text{-var},[s,t]} < \beta. \tag{3.36}$$

We now consider the control ω defined by $\omega(s,t) = \|q\|^{p/2}_{p/2\text{-var},[\![s,t]\!]} + \|\mathbf{x}\|^p_{p\text{-var},[\![s,t]\!]}$. For a generic element $\phi \in A^n + \kappa B_{\bar{\mathcal{H}}}$ we have

$$\omega(s,t)(\phi) = \|T_h T_{-h} q(\phi)\|^{p/2}_{p/2\text{-var},[\![s,t]\!]} + \|T_h T_{-h}\mathbf{x}(\phi)\|^p_{p\text{-var},[\![s,t]\!]}.$$

Hence invoking Lemma 3.6 we get

$$\omega(s,t)(\phi) \le K_p\big(\|T_{-h}q(\phi)\|^{p/2}_{p/2\text{-var},[\![s,t]\!]} + \|T_{-h}\mathbf{x}(\phi)\|^p_{p\text{-var},[s,t]} + \|h\|^p_{p'\text{-var},[s,t]}\big),$$

and owing to (3.36) one ends up with the following relation valid for all $\phi \in A^n + \kappa B_{\bar{\mathcal{H}}}$:

$$\omega(s,t)(\phi) \le K_p\beta + K_p\|h\|^p_{p'\text{-var},[s,t]}.$$

In particular, when $s = s_j$ and $t = s_{j+1}$ for $s_j \in S_0$ we obtain

$$\alpha/2 \le \omega(s_j, s_{j+1}) \le K_p\beta + K_p\|h\|^p_{p'\text{-var},[s_j,s_{j+1}]},$$

and thus

$$\|h\|^{p'}_{p'\text{-var},[s_j,s_{j+1}]} \ge \left(\frac{\alpha/2 - K_p\beta}{K_p}\right)^{p'/p}. \tag{3.37}$$

Since $\omega_1(s,t) \equiv \|h\|^{p'}_{p',[s,t]}$ is a control it follows from (3.37) that

$$\|h\|^{p'}_{p'\text{-var},[0,T]} \ge \sum_{s_j\in S_0} \|h\|^{p'}_{p'\text{-var},[s_j,s_{j+1}]} \ge \left(\frac{\alpha/2 - K_p\beta}{K_p}\right)^{p'/p} |S_0|. \tag{3.38}$$

We now specify the left-hand side of (3.38). First since we have chosen $p' > (H+1/2)^{-1}$, the reference [7], Page 14, asserts that $|h|_{\bar{\mathcal{H}}} \geq \|h\|_{p'\text{-var},[0,T]}$. Moreover we have assumed that $h \in \kappa B_{\bar{\mathcal{H}}}$. We thus obtain

$$\kappa^{p'} \geq |h|_{\bar{\mathcal{H}}}^{p'} \geq \|h\|_{p'\text{-var},[0,T]}^{p'}.$$

Plugging this inequality into (3.38), we obtain that if $\phi \in A^n + \kappa B_{\bar{\mathcal{H}}}$ then

$$|S_0| \leq \kappa^{p'} \left(\frac{\alpha/2 - K_p \beta}{K_p} \right)^{-p'/p} = a,$$

where the last identity stems from the definition (3.34) of κ. We have thus proved that if $\phi \in A^n + \kappa B_{\bar{\mathcal{H}}}$, then $|S_0| \leq a$. This concludes the proof of (3.35).

Step 3. Tail estimates. Let us introduce some extra bits of notation. Namely we write Φ for the standard Gaussian CDF. For a set $A \subset \Omega$ we also define $a_A \in \mathbb{R}$ as the number such that $\Phi(a_A) = \mathbb{P}(A)$. Then the isoperimetric type inequality in [27], Theorem 4.3, together with (3.35), yield

$$\mathbb{P}(|S_0| > a) \leq \mathbb{P}\big((A + \kappa B_{\bar{\mathcal{H}}})^c\big) \leq e^{-K(a_A+\kappa)^2} = e^{K\frac{\kappa^2}{2} - K(a_A+\kappa)^2} e^{-K\frac{\kappa^2}{2}}.$$

Let $K_A > 0$ be an upper bound of the quadratic function $f(\kappa) = K\frac{\kappa^2}{2} - K(a_A + \kappa)^2$ on $\mathbb{R}$. Then considering a constant K which can change from line to line and recalling the definition (3.34) of κ, we get

$$\mathbb{P}(|S_0| > a) \leq K_A e^{-K\frac{\kappa^2}{2}} = K_A e^{-K a^{2/p'}}. \tag{3.39}$$

Recall again that p' can be chosen arbitrarily close to $(H+1/2)^{-1}$. Hence $2/p'$ is of the form $2H + 1 - \varepsilon$ for a small $\varepsilon > 0$. This conclude the tail estimate (3.32). It follows immediately from (3.32) that $|S_0|^\nu$ and thus $\mathcal{M}_0^\nu$ is uniformly integrable for any $\nu \geq 1$. □

3.3. *Integrability of Malliavin derivatives.* With the preliminary results of Sections 3.1 and 3.2 in hand, we can now turn to the integrability result for the Malliavin derivatives of the Euler scheme. Notice that we restrict our analysis here to the first two Malliavin derivatives of y^n. However, it is clear that our estimates could be extended to arbitrary Malliavin derivatives.

THEOREM 3.8. *Let y^n be the Euler scheme defined by* (1.2). *The first and second Malliavin derivatives of y^n are contained in the vector ξ^n introduced in* (2.28). *We assume that the vector field V is C_b^4 and that x is a fBm with Hurst parameter $H > 1/3$. Then for all $\nu \geq 1$ we have*

$$\mathbb{E}\big[\|\xi^n\|_{p\text{-var}}^\nu\big] < \infty. \tag{3.40}$$

In particular, the following sup-norm inequality holds true:

$$\mathbb{E}\Big[\sup_{n\in\mathbb{N}, r, r', t\in[0,T]} |\xi_t^n|^\nu\Big] < \infty. \tag{3.41}$$

PROOF. Inequality (3.40) follows by showing that all terms in the right-hand side of (2.29) have moments of all orders. Applying Theorem 3.7, Corollary 3.5 and Theorem 3.3 respectively we obtain the integrability of $\mathcal{M}_0$, $\mathcal{M}_1$, $\mathcal{M}_2$. The integrability of $|S_0|$ follows from (3.32). The integrability of $|S_1|$ and $|S_2|$ are implied by the relation $|S_i| \lesssim \mathcal{M}_i$, $i = 1, 2$, respectively. The upper bound (3.41) is an easy consequence of (3.40). □

4. Weak convergence. With our Malliavin derivative and integrability estimates in hand, in this section we consider the weak convergence of the Euler scheme. The first sections are preparations of the main result.

4.1. *Estimation of an inner product in $\mathcal{H}^{\otimes 2}$.* In this subsection, we derive a useful upper-bound estimate for an inner product of the form $\langle \varphi, \mathbf{1}_{[u,v]} \otimes \mathbf{1}_{[s,t]} \rangle_{\mathcal{H}^{\otimes 2}}$, involving some indicator functions. We first need a positivity result for the rectangular increment function R of the fBm.

LEMMA 4.1. *Recall that the covariance R is defined in (2.11), with rectangular increments $R([u, v], [s, t])$ introduced in (2.12). Then for any $u, v, s, t \in \mathbb{R}$ such that $s \le u \le v \le t$ we have*

$$R\big([u, v], [s, t]\big) \ge 0.$$

PROOF. We first write

$$R\big([u, v], [s, t]\big) = R\big([u, v], [u, v]\big) + R\big([u, v], [u, v]^C\big),$$

where we denoted $[u, v]^C = [s, t] \setminus [u, v]$. Since $R([u, v], [u, v]) = (v - u)^{2H}$ it suffices to show that $R([u, v], [u, v]^C) \ge -(v - u)^{2H}$.

By definition (2.11)–(2.12) of R we can write

$$\begin{aligned} R\big([u, v], [u, v]^C\big) &= R\big([u, v], [s, u]\big) + R\big([u, v], [v, t]\big) \\ &= \frac{1}{2}\big(|v - s|^{2H} - |u - s|^{2H} - |v - u|^{2H}\big) \\ &\quad + \frac{1}{2}\big(|t - u|^{2H} - |t - v|^{2H} - |v - u|^{2H}\big). \end{aligned} \tag{4.1}$$

Note that $|v - s|^{2H} - |u - s|^{2H}$ and $|t - u|^{2H} - |t - v|^{2H}$ are nonnegative. We thus obtain

$$R\big([u, v], [u, v]^C\big) \ge -|v - u|^{2H}. \tag{4.2}$$

The proof is complete. □

The above positivity result leads to a surprisingly easy bound on products in $\mathcal{H}^{\otimes 2}$.

LEMMA 4.2. *Let $\varphi \in \mathcal{H}^{\otimes 2}$ and $p > 0$ be such that $2H + 1/p > 1$. Assume that $\varphi \in C^{p\text{-var}}([0, T], C^{p\text{-var}}([0, T], \mathbb{R}))$. For $s, t, u, v \in [0, T]$: $s < t$, $u < v$ we define $\alpha(\eta, \zeta) = \mathbf{1}_{[u,v]}(\eta)\mathbf{1}_{[s,t]}(\zeta)$ for $\eta, \zeta \in [0, T]$. Then the following relation holds*:

$$\big|\langle \varphi, \alpha \rangle_{\mathcal{H}^{\otimes 2}}\big| \le 4(t - s)^{2H}(v - u)^{2H}\|\varphi\|_\infty. \tag{4.3}$$

PROOF. Starting from Definition 2.7 and taking limits on indicator functions of rectangles (following the arguments in the proof of [17], Lemma 15.39), one can prove that the inner product between φ and α in $\mathcal{H}^{\otimes 2}$ can be expressed as a double 2D Young integral of the form

$$\begin{aligned} \langle \varphi, \alpha \rangle_{\mathcal{H}^{\otimes 2}} &= \langle \varphi, \mathbf{1}_{[u,v]} \otimes \mathbf{1}_{[s,t]} \rangle_{\mathcal{H}^{\otimes 2}} \\ &= \int_{[0,T]^4} \varphi(\eta, \zeta)\mathbf{1}_{[u,v]}(\eta')\mathbf{1}_{[s,t]}(\zeta')\, dR(\eta, \eta')\, dR(\zeta, \zeta'). \end{aligned} \tag{4.4}$$

We note that the condition $\varphi \in C^{p\text{-var}}([0,T], C^{p\text{-var}}([0,T],\mathbb{R}))$ is needed here while applying the arguments in [17], Lemma 15.39. One can then integrate out the η' and ζ' variables in the right-hand side of (4.4) in order to get

$$\langle \varphi, \alpha\rangle_{\mathcal{H}^{\otimes 2}} = \int_{[0,T]^2} \varphi(\eta,\zeta)\, dR\big(\eta,[u,v]\big)\, dR\big(\zeta,[s,t]\big). \tag{4.5}$$

We further decompose the inner product $\langle \varphi, \alpha\rangle_{\mathcal{H}^{\otimes 2}}$ using the identity:

$$\mathbf{1} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,$$

where the functions $\alpha_1, \dots, \alpha_4$ are given by

$$\alpha_1(\eta,\zeta) = \mathbf{1}_{[u,v]}(\eta)\mathbf{1}_{[s,t]}(\zeta), \qquad \alpha_2(\eta,\zeta) = \mathbf{1}_{[u,v]^C}(\eta)\mathbf{1}_{[s,t]}(\zeta),$$
$$\alpha_3(\eta,\zeta) = \mathbf{1}_{[u,v]}(\eta)\mathbf{1}_{[s,t]^C}(\zeta), \qquad \alpha_4(\eta,\zeta) = \mathbf{1}_{[u,v]^C}(\eta)\mathbf{1}_{[s,t]^C}(\zeta),$$

and where similar to what we wrote in Lemma 4.1, we have set $[s,t]^C = [0,T]\setminus[s,t]$ and $[u,v]^C = [0,T]\setminus[u,v]$. Otherwise stated, we recast (4.5) as

$$\langle \varphi, \alpha\rangle_{\mathcal{H}^{\otimes 2}} = \int_{[0,T]^2} \varphi(\eta,\zeta)\, dR\big(\eta,[u,v]\big)\, dR\big(\zeta,[s,t]\big) = \sum_{i=1}^{4} J_T^i, \tag{4.6}$$

where the terms J_T^i are respectively defined by

$$J_T^i = \int_{[0,T]^2} \varphi(\eta,\zeta)\alpha_i(\eta,\zeta)\, dR\big(\eta,[u,v]\big)\, dR\big(\zeta,[s,t]\big).$$

Those four terms will be handled with slightly different arguments. That is, for J_T^1, owing to Lemma 4.1 we have that both $dR(\zeta,[s,t])$ and $dR(\eta,[u,v])$ are positive when $\eta \in [u,v]$ and $\zeta \in [s,t]$. Therefore, we have

$$\big|J_T^1\big| \le \|\varphi\|_\infty R\big([s,t],[s,t]\big)R\big([u,v],[u,v]\big) \le (t-s)^{2H}(v-u)^{2H}\|\varphi\|_\infty. \tag{4.7}$$

For the second term J_T^2 in (4.6) we observe that $dR(\zeta,[s,t])$ is positive and $dR(\eta,[u,v])$ is negative. Therefore the product $dR(\eta,[s,t]) \cdot dR(\eta,[u,v])$ does not change sign and we get

$$\big|J_T^2\big| \le \|\varphi\|_\infty \big|R\big([u,v],[u,v]^C\big)R\big([s,t],[s,t]\big)\big|.$$

Hence thanks to an elementary computation similar to (4.1)–(4.2) we discover that

$$\big|J_T^2\big| \le (t-s)^{2H}(v-u)^{2H}\|\varphi\|_\infty. \tag{4.8}$$

In conclusion, gathering (4.7), (4.8) and similar bounds for J_T^3, J_T^4 into (4.6), we get the desired estimate (4.3). This concludes the proof. □

We now extend the previous lemma to the indicator of a simplex in $[0,T]^2$.

LEMMA 4.3. *Let $\varphi \in \mathcal{H}^{\otimes 2}$ be as in Lemma 4.2. Let $\beta \in \mathcal{H}^{\otimes 2}$ be of the form*

$$\beta_{st}(u,v) = \mathbf{1}_{\mathcal{S}_2([s,t])}(u,v), \tag{4.9}$$

where we recall that the simplex $\mathcal{S}_2([s,t])$ is defined in Notation 1.3. Then there exists a constant C_H such that the following relation holds:

$$\big|\langle \varphi, \beta_{st}\rangle_{\mathcal{H}^{\otimes 2}}\big| \le C_H (t-s)^{4H}\|\varphi\|_\infty. \tag{4.10}$$

PROOF. We will use a dyadic partition of the function β. Namely for $n \geq 0$ and $0 \leq i \leq 2^n$ we set $u_{i,n} = s + 2^{-n}(t-s)i$. Next for $\ell \geq 1$ we define

$$\beta_{st}^{\ell} = \sum_{n=1}^{\ell} \sum_{i=0}^{2^{n-1}-1} \mathbf{1}_{[u_{2i,n}, u_{2i+1,n}] \times [u_{2i+1,n}, u_{2i+2,n}]}.$$

Then it can be shown that $\|\beta_{st}^{\ell} - \beta_{st}\|_{\mathcal{H}^{\otimes 2}} \to 0$. In order to prove the lemma it thus suffices to show that for all $\ell \geq 1$ we have

$$|\langle \varphi, \beta_{st}^{\ell} \rangle_{\mathcal{H}^{\otimes 2}}| \leq (2^{4H} - 2)^{-1} (t-s)^{4H} \|\varphi\|_{\infty}.$$

In the following we prove this relation with the help of Lemma 4.2. We first observe that by the definition of β^{ℓ}

$$|\langle \varphi, \beta_{st}^{\ell} \rangle_{\mathcal{H}^{\otimes 2}}| \leq \sum_{n=1}^{\ell} \sum_{i=0}^{2^{n-1}-1} |\langle \varphi, \mathbf{1}_{[u_{2i,n}, u_{2i+1,n}] \times [u_{2i+1,n}, u_{2i+2,n}]} \rangle_{\mathcal{H}^{\otimes 2}}|.$$

Applying Lemma 4.2 with $(s, u, v, t) = (u_{2i,n}, u_{2i+1,n}, u_{2i+1,n}, u_{2i+2,n})$, we obtain

$$\begin{aligned} |\langle \varphi, \beta_{st}^{\ell} \rangle_{\mathcal{H}^{\otimes 2}}| &\leq \sum_{n=1}^{\ell} \sum_{i=0}^{2^{n-1}-1} (2^{-n}(t-s))^{2H} (2^{-n}(t-s))^{2H} \|\varphi\|_{\infty} \\ &= \frac{1}{2} (t-s)^{4H} \|\varphi\|_{\infty} \sum_{n=1}^{\ell} (2^n)^{1-4H} \leq \frac{1}{2^{4H} - 2} (t-s)^{4H} \|\varphi\|_{\infty}. \end{aligned}$$

This completes the proof of our claim (4.10). $\square$

In the sequel we will also need an inequality for products in $\mathcal{H}$. Its proof is similar to the proof of Lemma 4.3 and is omitted for sake of conciseness.

LEMMA 4.4. *Let $\varphi \in \mathcal{H}$ be a function in $C^{p\text{-var}}([0, T])$. Then the following relation holds*:

$$|\langle \varphi, \mathbf{1}_{[s,t]} \rangle_{\mathcal{H}}| \leq (t-s)^{2H} \|\varphi\|_{\infty}$$

for all $(s, t) \in \mathcal{S}_2(0, T)$.

4.2. *An extension of the sewing lemma.* In this section we extend Lemma 2.4 to the integral of two controlled processes. Our findings are summarized in the following lemma.

LEMMA 4.5. *Let $S_2(x) := (x^1, x^2)$ be the geometric rough path above x as given in Definition 2.1, and $p < 3$. We consider two couples of paths (z, z') and $(\tilde{z}, \tilde{z}')$ with $z, \tilde{z} \in C([s, t], \mathbb{R}^m)$ and $z', \tilde{z}' \in C([s, t], \mathbb{R}^{m \times d})$. Let $\omega^x(u, v) = \|x\|_{p\text{-var},[u,v]}^p$ for $(u, v) \in \mathcal{S}_2([s, t])$. We assume the existence of two controlled functions ω^z, ω_1^z and $\omega^{z'}$ on $[\![s, t]\!]$ such that for all $(u, v) \in \mathcal{S}_2([\![s, t]\!])$ we have*

$$\begin{aligned} &|\delta z_{uv} - z'_u x^1_{uv}| \leq \omega^z(u, v)^{2/p}, \qquad |\delta z_{uv}| \leq \omega_1^z(u, v)^{1/p}, \\ &|\delta z'_{uv}| \leq \omega^{z'}(u, v)^{1/p}. \end{aligned} \tag{4.11}$$

We also assume that the relations in (4.11) hold for $\tilde{z}$, with related increments $\tilde{z}'$, $\omega^{z'}$. Next we introduce some new control functions:

$$\omega^{x,z,z'} = \omega^x + \omega^z + \omega^{z'}, \quad \textit{and} \quad \omega_1^{x,z,z'} = \omega^{x,z,z'} + \omega_1^z, \tag{4.12}$$

and similarly for $(\tilde{z}, \tilde{z}')$. *We now define some remainder terms in the integrals of* z *with respect to* $\tilde{z}$ *or* x. *Namely for* $(u, v) \in \mathcal{S}_2([\![s, t]\!])$ *we set*

$$R^{z\tilde{z}}_{uv} = \int_u^v (\delta z_{ur} - z'_u x^1_{ur}) \otimes d\tilde{z}_r \quad \text{and} \quad R^{\tilde{z}x}_{uv} = \int_u^v (\delta \tilde{z}_{ur} - \tilde{z}'_u x^1_{ur}) \otimes dx_r, \tag{4.13}$$

where the above integrals are understood in the rough path sense. We suppose that the increments R *are such that for any point* $t_k \in [\![s, t]\!]$ *we have*

$$|R^{z\tilde{z}}_{t_k t_{k+1}}| \leq \omega^x(t_k, t_{k+1})^{3/p} \quad \text{and} \quad |R^{\tilde{z}x}_{t_k t_{k+1}}| \leq \omega^x(t_k, t_{k+1})^{3/p}. \tag{4.14}$$

Then the following relation holds for all $(u, v) \in \mathcal{S}_2([\![s, t]\!])$:

$$|R^{z\tilde{z}}_{uv}| \leq K_p[\omega^R(u, v)]^{\mu}, \tag{4.15}$$

where $\mu > 1$ *is a given constant and where* $K_p > 0$ *is a constant depending on* p. *In* (4.15), *we shall see that the control* ω^R *is defined by the relation*

$$\begin{aligned} [\omega^R(u, v)]^{\mu} &:= (\omega^z(u, v)^{2/p} + \omega^{z'}(u, v)^{1/p}\omega^x(u, v)^{1/p})\omega_1^{x,\tilde{z},\tilde{z}'}(u, v)^{1/p} \\ &\quad \times (\omega^{x,\tilde{z},\tilde{z}'}(u, v)^{1/p} + \|\tilde{z}'\|_{\infty,[u,v]} + 1), \end{aligned} \tag{4.16}$$

in which $\mu > 1$ *is taken such that* $\omega^R(u, v)$ *is a control* (*note that this can be done thanks to the relation* $3/p > 1$).

PROOF. The proof of the lemma is an application of Lemma 2.4. Namely the existence of R as a rough integral is ensured by general rough paths considerations (see, e.g., [21]). Then some elementary manipulations starting from the definition (4.13) of R show that for $(u, s, v) \in \mathcal{S}_3([\![0, T]\!])$ we have

$$\delta R^{z\tilde{z}}_{usv} = (\delta z_{us} - z'_u x^1_{us}) \otimes \delta\tilde{z}_{sv} + z'_{us} \int_s^v x^1_{sr} \otimes d\tilde{z}_r, \tag{4.17}$$

where we recall that δ is defined as in (2.1).

We first consider the case when $(\tilde{z}, x)$ is in the place of $(z, \tilde{z})$, that is, the remainder $R^{\tilde{z}x}$ defined in (4.13). In this case one can recast (4.17) as

$$\delta R^{\tilde{z}x}_{usv} = (\delta\tilde{z}_{us} - \tilde{z}'_u x^1_{us}) \otimes x^1_{sv} + \tilde{z}'_{us} x^2_{sv}. \tag{4.18}$$

Applying the conditions in (4.11) we thus get

$$|\delta R^{\tilde{z}x}_{usv}| \leq \omega^{\tilde{z}}(u, v)^{2/p} \cdot \omega^x(u, v)^{1/p} + \omega^{\tilde{z}'}(u, v)^{1/p} \cdot \omega^x(u, v)^{2/p}. \tag{4.19}$$

Moreover, we have assumed that (4.14) holds true for the increments $R^{\tilde{z}x}_{t_k t_{k+1}}$. Hence a direct application of Lemma 2.4 implies that the upper bound in (4.19) for $\delta R^{\tilde{z}x}_{usv}$ also holds for $R^{\tilde{z}x}_{uv}$, that is, the relation (4.15) holds in the special case when $(\tilde{z}, x)$ is in the place of $(z, \tilde{z})$.

In order to prove (4.15) for a general $\tilde{z}$, let us first bound the integral $\int_s^v x^1_{sr} \otimes d\tilde{z}_r$ in (4.17). To this aim, we observe that a simple integration by parts (valid for integrals driven by the geometric rough path (x^1, x^2) thanks to a limiting procedure on smooth approximations) yields the relation

$$\int_s^v x^1_{sr} \otimes d\tilde{z}_r = x^1_{sv} \otimes \delta\tilde{z}_{sv} - \int_s^v \delta\tilde{z}_{sr} \otimes dx_r. \tag{4.20}$$

Note that the integrals in (4.20) are well defined in the controlled rough path sense; see, for example, [15]. Next owing to our conditions (4.11) for $\tilde{z}$, the first term in the right hand side of (4.20) is bounded by

$$|x^1_{sv} \otimes \delta\tilde{z}_{sv}| \leq \omega^x(s, v)^{1/p}\omega_1^{\tilde{z}}(s, v)^{1/p}. \tag{4.21}$$

For the second term in the right-hand side of (4.20), let us write

$$\int_s^v \delta\tilde{z}_{sr} \otimes dx_r = \tilde{z}'_s \otimes x^2_{sv} + R^{\tilde{z}x}_{sv}.$$

Since we have obtained that (4.15) holds for $R^{\tilde{z}x}$, for every $s \le u < v \le t$ we end up with

$$\begin{aligned}\left|\int_u^v \delta\tilde{z}_{ur} \otimes dx_r\right| &\le |R^{\tilde{z}x}_{uv}| + |\tilde{z}'_u|\omega^x(u,v)^{2/p} \\ &\le \omega^{\tilde{z}}(u,v)^{2/p} \cdot \omega^x(u,v)^{1/p} + \omega^{\tilde{z}'}(u,v)^{1/p} \cdot \omega^x(u,v)^{2/p} \\ &\quad + \|\tilde{z}'\|_{\infty,[u,v]}\omega^x(u,v)^{2/p}.\end{aligned} \tag{4.22}$$

We can now safely plug (4.21) and (4.22) (with u replaced by s) into relation (4.20). This yields the estimate

$$\begin{aligned}\left|\int_s^v x^1_{sr} \otimes d\tilde{z}_r\right| &\le \omega^{\tilde{z}}(u,v)^{2/p} \cdot \omega^x(u,v)^{1/p} + \omega^{\tilde{z}'}(u,v)^{1/p} \cdot \omega^x(u,v)^{2/p} \\ &\quad + \|\tilde{z}'\|_{\infty,[u,v]} \cdot \omega^x(u,v)^{2/p} + \omega^x(s,v)^{1/p}\omega^{\tilde{z}}_1(s,v)^{1/p}.\end{aligned} \tag{4.23}$$

Let us now return to relation (4.17). By a simple application of (4.11) one discovers that

$$|\delta R^{z\tilde{z}}_{usv}| \le \omega^z(u,v)^{2/p} \cdot \omega^{\tilde{z}}_1(u,v)^{1/p} + \omega^{z'}(u,v)^{1/p} \cdot \left|\int_s^v x^1_{sr} \otimes d\tilde{z}_r\right|.$$

Inserting (4.23) into this relation, we thus obtain

$$\begin{aligned}|\delta R^{z\tilde{z}}_{usv}| &\le K_p\big(\omega^z(u,v)^{2/p} + \omega^{z'}(u,v)^{1/p}\omega^x(u,v)^{1/p}\big)\omega_1^{x,\tilde{z},\tilde{z}'}(u,v)^{1/p} \\ &\quad \times \big(\omega^{x,\tilde{z},\tilde{z}'}(u,v)^{1/p} + \|\tilde{z}'\|_{\infty,[u,v]} + 1\big).\end{aligned} \tag{4.24}$$

Note that the right-hand side of (4.24) is equal to $K_p\omega^R(u,v)$ defined in (4.16). Taking (4.14) into account, another use of Lemma 2.4 (together with an application of [17], Exercise 1.9 (iii), to show that ω^R is a control), proves our claim (4.15). □

4.3. *Interpolation of the Euler method.* Recall that the Euler scheme y^n is defined in (1.2). We also refer the reader to Section 2.4-2.5 for an upper-bound estimate and some preliminary discussions of the Euler scheme. In this section we shall extend our Euler scheme to a process in continuous time and obtain some uniform bounds. Specifically, recall that the Euler approximation y^n is defined on $[\![0,T]\!]$ by (1.2) and for convenience we will take $V_0 \equiv 0$. For t in the continuous interval $[0,T]$ we shall use the following interpolation:

$$\delta y^n_{t_k t} = V(y^n_{t_k})\delta x_{t_k t} + \frac{1}{2}\sum_{j=1}^d \partial V_j V_j(y^n_{t_k})(t-t_k)^{2H}, \quad t \in [t_k, t_{k+1}]. \tag{4.25}$$

Recall that $\mathbf{x} = S_2(x) = (x^1, x^2)$ denotes the rough path above x (as given in Definition 2.1).

In the sequel we will also need some continuous interpolations of the processes q and q^b, which had been defined on the grid in (2.20). Namely for $(s,t) \in \mathcal{S}_2([0,T])$ such that $t_{k_1} \le s < t_{k_1+1} < \cdots < t_{k_2-1} \le t < t_{k_2}$, we define

$$q^{ij}_{st} = \sum_{k_1 \le k < k_2}\left(x^{2,ij}_{t_k\vee s, t_{k+1}\wedge t} - \frac{1}{2}(t_{k+1}\wedge t - t_k \vee s)^{2H}\mathbf{1}_{\{i=j\}}\right), \tag{4.26}$$

$$q^{b,ij}_{st} = \sum_{k_1 \le k < k_2}\left(b^{2,ij}_{t_k\vee s, t_{k+1}\wedge t} - \frac{1}{2}(t_{k+1}\wedge t - t_k \vee s)^{2H}\mathbf{1}_{\{i=j\}}\right). \tag{4.27}$$

With the above definition (4.26) in hand, we will also extend the definition of ω from the grid to $[0,T]$. This extension will be useful in the estimate of y^n and its Malliavin derivatives on a continuous interval; see Lemma 4.8. Recall that ω has been defined on $\mathcal{S}_2([\![0,T]\!])$ by (2.23) for a fixed partition length parameter n, and that we are considering $p<3$. The next result extend ω to $\mathcal{S}([0,T])$.

LEMMA 4.6. *For $(s,t)\in\mathcal{S}_2([0,T])$ define*

$$\omega(s,t)=\|\mathbf{w}\|^p_{p\text{-var};[s,t]}+\|q\|^{p/2}_{p/2\text{-var};[s,t]}+\|q^b\|^{p/2}_{p/2\text{-var};[s,t]}+|t-s|. \tag{4.28}$$

Then ω is a control on $[0,T]$. In other words, ω is super-additive, continuous and vanishes on the diagonal.

REMARK 4.7. Note that we have included an additional component $|t-s|$ in the definition (4.28). This is needed for the estimate of y^n_{st} when $|t-s|$ is small. Specifically, take a continuous interval $[s,t]$ such that $[s,t]\le[t_k,t_{k+1}]$. Then according to (4.25) we have

$$\delta y^n_{st}=V(y^n_{t_k})\delta x_{st}+\frac{1}{2}\sum_{j=1}^d\partial V_jV_j(y^n_{t_k})[(t-t_k)^{2H}-(s-t_k)^{2H}].$$

It follows that we have

$$\begin{aligned}|\delta y^n_{st}|&\lesssim|\delta x_{st}|+|(t-t_k)^{2H}-(s-t_k)^{2H}|\\&\lesssim|\delta x_{st}|+(t-s)^{2H}.\end{aligned} \tag{4.29}$$

We can now bound $|\delta x_{st}|$ by $\|\mathbf{w}\|_{p\text{-var};[s,t]}$ and $(t-s)^{2H}$ by $|t-s|^{1/p}$. Raising those terms to a power p, this gives the estimate $|\delta y^n_{st}|^p\lesssim\omega(s,t)$, with ω as in (4.28).

PROOF OF LEMMA 4.6. We first note that

$$\|q\|^{p/2}_{p/2\text{-var};[s,t]}\le\|q\|^{p/2}_{p/2\text{-var};[\![s,t]\!]}+\|\mathbf{w}\|^p_{p\text{-var};[s,t]}+|t-s|, \tag{4.30}$$

and the same kind of inequality holds true for q^b. Therefore $\omega(s,t)$ is finite almost surely. Thanks to the definition (4.28) of ω it is also readily checked that the superadditivity and zero on the diagonal properties hold true.

It remains to show the continuity of ω. To this aim, taking into account the definition (4.28) of ω, it is easily seen that we only have to focus on the increments q and q^b in (4.26)–(4.27). Moreover q and q^b are handled exactly in the same way. Hence we will just focus our attention on $\tilde{\omega}$ given by

$$\tilde{\omega}(s,t):=\|q\|^{p/2}_{p/2\text{-var};[s,t]}. \tag{4.31}$$

Take $s<u<t<\eta(u)+\Delta$, where we recall from Notation 1.3 that $\eta(u)$ is the largest $t_k\in[\![0,T]\!]$ such that $t_k\le u$. In the following we show that $\tilde{\omega}(s,t)-\tilde{\omega}(s,u)\to0$ as $t-u\to0$, which is one of the main steps towards the continuity of $\tilde{\omega}$.

Owing to the definition (4.31) of $\tilde{\omega}$, for any $\varepsilon>0$, we can find a partition of $[s,t]$, denoted by $s=v_0<\cdots<v_N=t$ such that

$$\tilde{\omega}(s,t)\le\sum_{i=0}^{N-1}|q_{v_iv_{i+1}}|^{p/2}+\varepsilon. \tag{4.32}$$

Suppose that $u \in [v_{i_0}, v_{i_0+1}] := [v, v']$. Then we can bound the summation in (4.32) by the following:

$$\begin{aligned}\tilde{\omega}(s,t) &\leq \sum_{i=0}^{i_0-1} |q_{v_i v_{i+1}}|^{p/2} + |q_{vv'}|^{p/2} + \sum_{i=i_0+1}^{N-1} |q_{v_i v_{i+1}}|^{p/2} + \varepsilon \\ &\leq \tilde{\omega}(s,v) + |q_{vv'}|^{p/2} + \tilde{\omega}(v',t) + \varepsilon.\end{aligned} \tag{4.33}$$

Now go back to (4.30) for v' and t and pick u close enough to t so that $v' \in [u,t]$ satisfies $\|q\|_{p/2\text{-var};[\![v',t]\!]} = 0$ (since our grid $[\![u',t]\!]$ has fixed mesh T/n, this is easily seen when $u \to t$, owing to our expression (4.26)). One can thus recast (4.30) as

$$\tilde{\omega}(v',t) := \|q\|^{p/2}_{p/2\text{-var};[v',t]} \leq \|\mathbf{w}\|^{p}_{p\text{-var};[v',t]} + |t - v'|. \tag{4.34}$$

It is then easily seen from (4.34) that $\lim_{v'\to t} \tilde{\omega}(v',t) = 0$. Since $u < v' < t$, we will pick u close enough to t so that $\tilde{\omega}(v',t) \leq \varepsilon$. Plugging this information into (4.33), we obtain

$$\tilde{\omega}(s,t) \leq \tilde{\omega}(s,v) + |q_{vv'}|^{p/2} + 2\varepsilon. \tag{4.35}$$

In addition, if $u \to t$ we also have $|v' - u| \to 0$. Therefore, basic continuity properties of q ensure that $||q_{vv'}|^{p/2} - |q_{vu}|^{p/2}| \leq \varepsilon$ if u is close enough to t. Hence (4.35) becomes

$$\tilde{\omega}(s,t) \leq \tilde{\omega}(s,v) + |q_{vu}|^{p/2} + 3\varepsilon \leq \tilde{\omega}(s,u) + 3\varepsilon,$$

where we have used the super-additivity property of $\tilde{\omega}$ for the second inequality. Since $\tilde{\omega}(s,t) \geq \tilde{\omega}(s,u)$ by monotonicity properties, we have obtained

$$|\tilde{\omega}(s,t) - \tilde{\omega}(s,u)| \leq 3\varepsilon, \tag{4.36}$$

for all $(s,u,t) \in \mathcal{S}_3([0,T])$ such that $|t-u|$ is sufficiently small. Since ε in (4.36) can be arbitrarily small, this proves that $\lim_{u\to t} \tilde{\omega}(s,u) = \tilde{\omega}(s,t)$. The same kind of arguments also show that $\lim_{u\to s} \tilde{\omega}(u,t) = \tilde{\omega}(s,t)$, which completes our proof. □

We now go back to the interpolated version of our Euler scheme y^n. In the following we show that (y^n, x, b) is a rough path, which is an important step in the convergence analysis.

LEMMA 4.8. *Consider the interpolated Euler scheme introduced in* (4.25). *Recall that x is our driving fBm and b is another fBm with parameter $H > 1/3$, independent of x. Also recall that the augmented process $\mathrm{w} = (x,b)$ has been introduced in* (2.22). *We assume that the vector field V sits in C_b^4.*

Denote by Z the couple $Z = (y^n, \mathrm{w})$. Then Z admits a lift $S_2(Z)$ according to Definition 2.1. *Moreover, recalling the sets S_0, S_1 in* (2.25), *consider $s_j \in S_0 \cup S_1$. Then for all $(s,t) \subset \mathcal{S}_2([s_j, s_{j+1}])$ we have the following uniform bound in n*:

$$\|S_2(Z)\|_{p\text{-var},[s,t]} \leq K \cdot \omega(s,t)^{1/p}, \tag{4.37}$$

where $p > 1/H$ and K is a constant depending on V only, and where the control ω is defined in (4.28).

PROOF. Take $(s,t) \subset (s_j, s_{j+1})$ such that $s_j \in S_0 \cup S_1$. Theorem 2.12, applied for $L = 0$, shows that for $(s,t) \in \mathcal{S}_2([\![s_j, s_{j+1}]\!])$ we have

$$|\delta y^n_{st}| \leq K\omega(s,t)^{1/p} \quad \text{and} \quad |\delta y^n_{st} - V(y^n_s)\delta x_{st}| \leq K\omega(s,t)^{2/p}, \tag{4.38}$$

where ω is the control given in (4.28). Using standard interpolation methods, it can be shown in a straightforward way that the relation for $|\delta y^n_{st}|$ in (4.38) still holds if $(s,t) \in$

$\mathcal{S}_2([s_j, s_{j+1}])$. So in order to prove (4.37) it remains to show that $\int_s^t y_{su}^n \otimes d\mathrm{w}_u$ and $\int_s^t y_{su}^n \otimes dy_u^n$ are bounded by $\omega(s,t)^{2/p}$.

Consider the following remainder process for $(s,t) \subset (s_j, s_{j+1})$ such that $s_j \in S_0 \cup S_1$:

$$R_{st} = \int_s^t (\delta y_{su}^n - V(y_s^n)\delta x_{su}) \otimes d\mathrm{w}_u. \tag{4.39}$$

Note that R is a remainder of the form $R^{y^n w}$, defined as in (4.13). In order to apply Lemma 4.5 to this remainder, we need to check that $R_{t_k t_{k+1}} \leq \omega(t_k, t_{k+1})^{3/p}$ as in (4.14). Now according to (4.39) we have

$$R_{t_k t_{k+1}} = \int_{t_k}^{t_{k+1}} (\delta y_{t_k u}^n - V(y_{t_k}^n)\delta x_{t_k u}) \otimes d\mathrm{w}_u. \tag{4.40}$$

Furthermore, owing to our interpolation formula (4.25), for all $u \in [t_k, t_{k+1}]$ we have

$$\delta y_{t_k u}^n - V(y_{t_k}^n)\delta x_{t_k u} = \frac{1}{2}\sum_{j=1}^d \partial V_j V_j(y_{t_k}^n)(u - t_k)^{2H}.$$

Reporting this identity into (4.40), we end up with

$$R_{t_k t_{k+1}} = \frac{1}{2}\sum_{j=1}^d \partial V_j V_j(y_{t_k}^n) \otimes \int_{t_k}^{t_{k+1}} (u - t_k)^{2H}\, d\mathrm{w}_u. \tag{4.41}$$

The stochastic integral in the right-hand side of (4.41) can be interpreted in the Young sense, and it is easy to see that $|R_{t_k t_{k+1}}| \lesssim \omega(t_k, t_{k+1})^{3/p}$, where ω is still the control introduced in (4.28). This proves (4.14) for the remainder R, and therefore one can safely apply Lemma 4.5 in order to get that, provided $s_j \in S_0 \cup S_1$,

$$|R_{st}| \leq K\omega(s,t)^{3/p} \quad \text{for } (s,t) \in \mathcal{S}_2(⟦s_j, s_{j+1}⟧). \tag{4.42}$$

Going back to (4.39), observe that we have

$$\int_s^t \delta y_{su}^n \otimes d\mathrm{w}_u = V(y_s^n)\int_s^t \delta x_{su} \otimes d\mathrm{w}_u + R_{st}.$$

Invoking (4.42) and since x is part of the rough path w, we easily get

$$\left|\int_s^t \delta y_{su}^n \otimes d\mathrm{w}_u\right| \leq K\omega(s,t)^{2/p}, \quad \text{for } (s,t) \in \mathcal{S}_2(⟦s_j, s_{j+1}⟧). \tag{4.43}$$

In the following we extend the above estimate for $\int_s^t \delta y_{su}^n\, d\mathrm{w}_u$ to any $(s,t) \in \mathcal{S}_2([s_j, s_{j+1}])$. That is, we take (s,t) such that $s \in [t_k, t_{k+1}]$ and $t \in [t_{k'}, t_{k'+1}]$. We write

$$\begin{aligned}
\int_s^t \delta y_{su}^n \otimes d\mathrm{w}_u &= \int_s^{t_{k+1}} \delta y_{su}^n \otimes d\mathrm{w}_u + \int_{t_{k+1}}^{t_{k'}} \delta y_{su}^n \otimes d\mathrm{w}_u + \int_{t_{k'}}^t \delta y_{su}^n \otimes d\mathrm{w}_u \\
&= \int_s^{t_{k+1}} \delta y_{su}^n \otimes d\mathrm{w}_u + \delta y_{st_{k+1}}^n \otimes \delta \mathrm{w}_{t_{k+1}t_{k'}} + \int_{t_{k+1}}^{t_{k'}} \delta y_{t_{k+1}u}^n \otimes d\mathrm{w}_u \\
&\quad + \delta y_{st_{k'}}^n \otimes \delta \mathrm{w}_{t_{k'}t} + \int_{t_{k'}}^t \delta y_{t_{k'}u}^n \otimes d\mathrm{w}_u =: \sum_{i=1}^5 I_i.
\end{aligned} \tag{4.44}$$

Let us bound the terms $I_1, \ldots, I_5$ above. First it follows from (4.43) that $|I_3| \leq K\omega(s,t)^{2/p}$. It is also clear that $|I_2|$ and $|I_4|$ are bounded by the same estimate $\omega(s,t)^{2/p}$. In order to bound I_1, observe that according to our interpolation formula (4.25) we have

$$I_1 = V(y_{t_k}^n) \int_s^{t_{k+1}} \delta x_{su} \otimes d\mathrm{w}_u \tag{4.45}$$
$$+ \frac{1}{2} \sum_{j=1}^{d} \partial V_j V_j(y_{t_k}^n) \otimes \int_s^{t_{k+1}} [(u - t_k)^{2H} - (s - t_k)^{2H}] d\mathrm{w}_u.$$

It is clear that the first integral in (4.45) is bounded by $\omega(s, t_{k+1})^{2/p}$. Note also that $(u - t_k)^{2H} - (s - t_k)^{2H} \le (u - s)^{2H}$. So applying Young's inequality (see, e.g., [17], Theorem 6.8) we obtain that the second integral in (4.45) is bounded by $(t_{k+1} - s)^{2H} \omega(s, t_{k+1})^{1/p}$. Combining these two estimates we obtain that $|I_1| \lesssim \omega(s, t_{k+1})^{2/p}$. The term I_5 is bounded in the similar way. Putting together our upper bounds on $I_1, \dots, I_5$, we have thus obtained that (4.43) holds for any $(s, t) \in \mathcal{S}_2([s_j, s_{j+1}])$. Summarizing our considerations so far, we have proved that

$$\|M^1\|_{p\text{-var};[s,t]} \le K\omega(s, t) \quad \text{with } M_{uv}^1 := \int_u^v \delta y_{ur}^n \otimes d\mathrm{w}_r. \tag{4.46}$$

The proof is now finished along the same arguments. Namely the increment $M_{uv}^2 := \int_u^v \delta y_{ur}^n \otimes dy_{ur}^n$ can be bounded similar to M^1, leading to the same inequality as (4.46). More precisely, we consider the remainder:

$$\tilde{R}_{st} = \int_s^t (\delta y_{su}^n - V(y_s^n)\delta x_{su}) \otimes dy_u^n.$$

Then, as for R, we can show that relation (4.14) holds for the remainder $\tilde{R}$, and therefore Lemma 4.5 can be applied to obtain the estimate $M_{st}^2 \lesssim \omega(s, t)^{2/p}$ for $(s, t) \in \mathcal{S}_2([\![s_j, s_{j+1}]\!])$ and $s_j \in S_0 \cup S_1$. Now by considering a decomposition for M_{st}^2 similar to (4.44) we can extend this estimate to $(s, t) \in \mathcal{S}_2([s_j, s_{j+1}])$. This proves our claim (4.37). □

LEMMA 4.9. *Let the assumptions in Lemma 4.8 prevail. Under the setting of Theorem 2.12, we consider the vector-valued stochastic process* $\tilde{Z} = (y^n, D_r y^n, D_{r'} D_r y^n, \mathrm{w})$ *for* $r, r' \in [0, T]$. *Then for* $(s, t) \subset (s_j, s_{j+1})$ *with* $s_j \in S_0 \cup S_1$ *we have (uniformly in* n, r *and* r'*):*

$$\|S_2(\tilde{Z})\|_{p\text{-var},[s,t]} \le K \cdot \omega(s, t)^{1/p} \cdot \mathcal{G}^2, \tag{4.47}$$

where the quantities $\mathcal{G}$ *have been introduced in* (2.30).

REMARK 4.10. Note that in Lemma 4.9, we are bounding the Malliavin derivatives $D_r y^n, D_r D_{r'} y^n$ uniformly in r, r' (instead of looking at their $\mathcal{H}$-norms). These are the bounds we need for our future estimates in the proof of the sharp rate $(1/n)^{4H-1}$ (see, e.g., relation (4.79)). In addition, in our study we also found that $\mathcal{H}$-norms for Malliavin derivatives would not yield the desired rate.

PROOF OF LEMMA 4.9. The estimate (4.47) can be obtained along the same lines as in Lemma 4.8. We apply Theorem 2.12 with $L = 1, 2$ to get:

$$|\delta(D_r y^n)_{st}| \le K\omega(s, t)^{1/p} \cdot \mathcal{G}, \qquad |\delta(D_r y^n)_{st} - \mathcal{L}V(y_s^n)\delta x_{st}| \le K\omega(s, t)^{2/p} \cdot \mathcal{G}^2,$$
$$|\delta(D_{rr'}^2 y^n)_{st}| \le K\omega(s, t)^{1/p} \cdot \mathcal{G}, \qquad |\delta(D_{rr'}^2 y^n)_{st} - \mathcal{L}^2 V(y_s^n)\delta x_{st}| \le K\omega(s, t)^{2/p} \cdot \mathcal{G}^2.$$

Then we replace relations in (4.38) by these relations, and replace $\omega(s, t)$ by $\omega(s, t) \cdot \mathcal{G}^p$ in the inequalities for the remainders. This completes our proof. □

4.4. *Integrability of some linear equations.* Our convergence estimates are based on linearization procedures. In this section we bound some related linear differential equations. We start by defining the objects we wish to study.

DEFINITION 4.11. Recall that every V^i has to be seen, for $i = 1, \dots, m$, as a smooth vector field on $\mathbb{R}^m$. Let y^n be the interpolated scheme (4.25). Then for $i = 1, \dots, m$ we define an averaged $\mathbb{R}^{d\times m}$-valued process $\tilde{V}^i(t) = \{\tilde{V}^i_{ji'}(t);\, j = 1, \dots, d, i' = 1, \dots, m\}$. This process is indexed by $t \in [0, T]$ and is given by

$$\tilde{V}^i_{ji'}(t) = \int_0^1 \partial_{i'} V^i_j\big(\theta y_t + (1-\theta) y^n_t\big)\, d\theta.$$

We also define a $(\mathbb{R}^{d\times m})^m$-valued process as $\tilde{V}(t) = (\tilde{V}^1(t), \dots, \tilde{V}^m(t))$.

We are now ready to define the linear equation we wish to analyze.

DEFINITION 4.12. Let $\tilde{V}$ be the $(\mathbb{R}^{d\times m})^m$-valued process introduced in Definition 4.11. We will call Γ the $\mathbb{R}^{m\times m}$-valued solution to the following systems of equations on $[0, T]$:

$$\Gamma^{ii'}_t = \mathrm{Id}_{ii'} + \sum_{j=1}^{d} \sum_{i''=1}^{m} \int_0^t \tilde{V}^i_{ji''}(s) \Gamma^{i''i'}_s \, dx^j_s, \quad \text{for } i, i' \in \{1, \dots, m\}. \tag{4.48}$$

For conciseness we will simply write (4.48) as $\Gamma_t = \mathrm{Id} + \int_0^t \tilde{V}(s)\Gamma_s\, dx_s$. We also denote by Λ the inverse of Γ, namely Λ is defined by the relation $\Lambda_t\Gamma_t \equiv \mathrm{Id}$.

Our next lemma presents an important estimate for the processes Γ and Λ defined above.

LEMMA 4.13. *Let the assumption be as in Theorem 2.12. Let $p > \frac{1}{H}$ and q be such that $\frac{1}{p} + \frac{1}{q} > 1$. Let Γ and Λ be defined in (4.48). Then*

(a) *For all $(s,t) \in \mathcal{S}_2([0,T])$ we have*

$$\begin{aligned} &|\delta\Gamma_{st}| + \sup_{r\in[0,T]} \big|D_r(\delta\Gamma_{st})\big| + \sup_{r,r'\in[0,T]} \big|D^2_{rr'}(\delta\Gamma_{st})\big| \\ &\quad \le K \cdot \omega(s,t)^{1/p} \cdot |S_0 \cup S_1 \cup S_2| \cdot \mathcal{M}_0 \cdot \mathcal{M}_1 \cdot \exp(\mathcal{N}_T), \end{aligned} \tag{4.49}$$

where ω is defined in (4.28) and $\mathcal{N}_T$ is some random variable such that $\mathcal{N}_T^{1/q}$ has a Gaussian tail. The relation still holds when Γ is replaced by Λ.

(b) *Both processes Γ and Λ and their Malliavin derivatives are uniformly integrable. Precisely, for all $p \ge 1$ we have*

$$\mathbb{E}\Big[\sup_{n\in\mathbb{N}, r, r', t\in[0,T]} \big(|\Gamma_t|^p + \big|D_r(\Gamma_t)\big|^p + \big|D^2_{rr'}(\Gamma_t)\big|^p\big)\Big] < \infty. \tag{4.50}$$

PROOF. Applying Corollary 3.5 and Theorem 3.7 to the right-hand side of (4.49) we conclude the integrability relation in (4.50). It thus remains to prove relation (4.49).

In the following we prove the estimate (4.49) for Γ. Note that due to the fact that the initial condition Id in (4.48) is nondegenerate, the process Λ is well defined and satisfies a differential equation which is very similar to Γ (see, e.g., [7, 23] for more details). Therefore the estimate of Λ in (4.49) can be obtained by following the same steps as for Γ. The proof for Λ and its derivatives is thus omitted.

With our Definition 4.11 in mind, let us first introduce an auxiliary process ξ given, for $i, i'' \in \{1, \dots, m\}$ and $t \in [0, T]$, by

$$\xi_t^{ii''} = \sum_{j=1}^{d} \int_0^t \tilde{V}_{ji''}^i(s)\, dx_s^j.$$

We now separate the estimates for ξ into two different cases

(i) *Case* $(s,t) \subset (s_j, s_{j+1})$ *such that* $s_j \in S_0 \cup S_1$. In this situation, since (y^n, x, b) can be lifted to a rough path (see Lemma 4.8(i)) and y is a process controlled by x, it is readily checked that for all $(s,t) \subset (s_j, s_{j+1})$ such that $s_j \in S_0 \cup S_1$ we have

$$\|S_2(\xi)\|_{p\text{-var},[s,t]} \le K \cdot \omega(s,t)^{1/p}. \tag{4.51}$$

Furthermore, note that one can recast equation (4.48) as a linear system of the form

$$d\Gamma_t^{ii'} = \sum_{i''=1}^{m} \Gamma_t^{i''i'}\, d\xi_t^{ii''}. \tag{4.52}$$

Observe that the path ξ is a functional of the process Z introduced in Lemma 4.8. Now we recall from [17], Theorem 10.53, that for a linear equation like (4.52), there exist two constants C_1, C_2 such that

$$\begin{aligned} |S_2(\Gamma, Z)_{st}| &\le C_1 |\Gamma_s| \cdot \|S_2(\xi)\|_{p\text{-var},[s,t]} \cdot \exp\bigl(C_2 \|S_2(\xi)\|_{p\text{-var},[s,t]}\bigr) \\ &\le C_1 |\Gamma_s| \cdot \omega(s,t)^{1/p} \exp\bigl(C_2 \omega(s,t)^{1/p}\bigr), \end{aligned} \tag{4.53}$$

where the second relation stems from (4.51). In addition, we have chosen $s_j \in S_0 \cup S_1$. Therefore, one can simplify (4.53) and obtain that for any $(s,t) \in \mathcal{S}_2([s_j, s_{j+1}])$,

$$\|S_2(\Gamma, Z)\|_{p\text{-var},[s,t]} \le K \omega(s,t)^{1/p} \cdot |\Gamma_s|. \tag{4.54}$$

(ii) *Case* $(s,t) \subset (s_j, s_{j+1})$ *such that* $s_j \in S_2$. For $s_j \in S_2$ equation (4.48) is a linear equation driven by x and so we can apply the integrability result [8], equation (4.10) and Theorem 6.3, to get

$$\|S_2(\Gamma)\|_{p\text{-var},[s,t]} \le K |\Gamma_s| \cdot \|\mathbf{x}\|_{p\text{-var};[s,t]} \cdot \exp(\delta \mathcal{N}_{st}) \le K |\Gamma_s| \cdot \exp(\delta \mathcal{N}_{st}), \tag{4.55}$$

where $\mathcal{N}_t$, $t \ge 0$ is a process such that $\mathcal{N}_0 = 0$ and the random variable $\exp(K \cdot \mathcal{N}_t)$ is integrable for any constant $K > 0$ and $t \ge 0$.

We are ready to show the estimate (4.49) for $\delta\Gamma_{st}$. We first take $s = s_j$ and $t \in [s_j, s_{j+1}]$ for some j and apply the inequality $|\Gamma_t| - |\Gamma_s| \le |\delta\Gamma_{st}| \le \|S_2(\Gamma)\|_{p\text{-var}}$ to (4.54)–(4.55). This gives respectively

$$|\Gamma_t| - |\Gamma_s| \le K |\Gamma_s| \cdot \omega(s,t)^{1/p} \quad \text{and} \quad |\Gamma_t| - |\Gamma_s| \le K |\Gamma_s| \exp(\delta \mathcal{N}_{st}).$$

It follows that

$$|\Gamma_t| \le \bigl(K \cdot \omega(s,t)^{1/p} + 1\bigr) |\Gamma_s| \quad \text{and} \quad |\Gamma_t| \le K \exp(\delta \mathcal{N}_{st}) |\Gamma_s|, \tag{4.56}$$

respectively. Iterating (4.56) and recalling that $|\Gamma_0| = |\mathrm{Id}| = 1$, we obtain

$$\begin{aligned} |\Gamma_t| &\le \prod_{s_j \in S_0 \cup S_1} \bigl(K \cdot \omega(s_j, s_{j+1})^{1/p} + 1\bigr) \exp(K \mathcal{N}_t) \\ &\le \mathcal{M}_0 \cdot \mathcal{M}_1 \cdot \exp(K \mathcal{N}_T). \end{aligned} \tag{4.57}$$

Substituting (4.57) into both (4.54)–(4.55) we obtain respectively

(4.58)
$$|\delta\Gamma_{st}| \le \omega(s,t)^{1/p}\mathcal{M}_0 \cdot \mathcal{M}_1 \cdot \exp(K\mathcal{N}_T) \quad \text{and}$$
$$|\delta\Gamma_{st}| \le \|\mathbf{x}\|_{p\text{-var};[s,t]}\mathcal{M}_0 \cdot \mathcal{M}_1 \cdot \exp(K\mathcal{N}_T).$$

Now take $(s,t) \in \mathcal{S}_2([0,T])$ such that $s_j \le s < s_{j+1} < \cdots < s_{j'} \le t < s_{j'+1}$. Since $|\delta\Gamma_{st}| \le |\delta\Gamma_{ss_{j+1}}| + \cdots + |\delta\Gamma_{s_{j'}t}|$, applying (4.58) and the fact that $\|\mathbf{x}\|_{p\text{-var};[s,t]} \le \omega(s,t)^{1/p}$ we obtain

$$|\delta\Gamma_{st}| \le K\big(\omega(s,s_{j+1})^{1/p} + \cdots + \omega(s_{j'},t)^{1/p}\big) \cdot \mathcal{M}_0 \cdot \mathcal{M}_1 \cdot \exp(K\mathcal{N}_T).$$

Finally, note that

$$\begin{aligned}\omega(s,s_{j+1})^{1/p} + \cdots + \omega(s_{j'},t)^{1/p} &\le \omega(s,t)^{1/p} + \cdots + \omega(s,t)^{1/p}\\ &\le \omega(s,t)^{1/p} \cdot |S_0 \cup S_1 \cup S_2|.\end{aligned}$$

It follows that

$$|\delta\Gamma_{st}| \le K\omega(s,t)^{1/p} \cdot |S_0 \cup S_1 \cup S_2| \cdot \mathcal{M}_0 \cdot \mathcal{M}_1 \cdot \exp(\mathcal{N}_T).$$

Namely we have proved (4.49) for Γ. It remains to upper bound the Malliavin derivatives of Γ.

Recall that Γ satisfies equation (4.48), with $\tilde{V}$ given in Definition 4.11. For sake of clarity, the remainder of our computations will be done assuming that all our quantities are real-valued (we will therefore drop the indices from our next equations). Moreover, according to our standing assumptions, the process $\tilde{V}$ is Malliavin differentiable. Hence using standard arguments for the differentiation of rough differential equations (see [6, 7, 26, 34]) we get that $D_r\Gamma_t$ satisfies the linear equation:

$$\begin{aligned}D_r\Gamma_t &= D_r\int_0^t \tilde{V}(s)\Gamma_s\,dx_s\\ &= \tilde{V}(r)\Gamma_r + \int_r^t D_r\tilde{V}(s)\cdot\Gamma_s\,dx_s + \int_r^t \tilde{V}(s)D_r\Gamma_s\,dx_s,\end{aligned}$$

where note that a direct differentiation gives:

$$D_r\tilde{V}(s) = \int_0^1 \partial^2 V\big(\theta y_s + (1-\theta)y_s^n\big)\big(\theta D_r y_s + (1-\theta)D_r y_s^n\big)\,d\theta.$$

Therefore one can use the variation of constant method, similar to [26], equation (2.7), in order to get the following representation for $D_r\Gamma_t$:

(4.59)
$$D_r\Gamma_t = \Gamma_t^r\tilde{V}(r)\Gamma_r + \Gamma_t^r\int_r^t \Lambda_s^r D_r\tilde{V}(s)\cdot\Gamma_s\,dx_s,$$

where $\{\Gamma_t^r; t \in [r,T]\}$ is the solution of equation (4.52) such that $\Gamma_r^r = \mathrm{Id}$ and Λ_t^r is the inverse of Γ_t^r. Note that because Γ and Γ^r satisfy the same equation with different initials, the estimate of Γ in (4.55) also holds for Γ^r. In order to estimate $D_r\Gamma_t$, it thus remains to get the estimate (4.49) for the integral

(4.60)
$$\int_r^t \Lambda_s^r D_r\tilde{V}(s)\cdot\Gamma_s\,dx_s$$

in (4.59). Recall that Γ is the solution of the linear system (4.52) driven by Z, where we recall that $Z = (y^n, \mathrm{w})$. According to (4.47) $(D_r y, D_r y^n, Z)$ can be lifted to a rough path. So Γ can also be considered as the solution of a linear system driven by $(D_r y, D_r y^n, Z)$. Hence along the same line as for (4.53) we can estimate the quantity (4.60), and thus we obtain the bound (4.49) for $D_r\Gamma_t$.

We turn to the equation satisfied by $D^2_{rr'}\Gamma$. Differentiating (4.59), we let the patient reader check that the second derivative verifies a linear equation of the form

$$D^2_{rr'}\Gamma_t = D_{r'}\big[\tilde V(r)\Gamma_r\big] + D_r\big[\tilde V(r')\Gamma_{r'}\big] + \mathcal{E}_{rr'}(t) + \int_{r\vee r'}^t \tilde V(s) D^2_{rr'}\Gamma_s\, dx_s, \tag{4.61}$$

where the term $\mathcal{E}_{rr'}(t)$ is defined by

$$\mathcal{E}_{rr'}(t) = \int_{r\vee r'}^t D_{r'}\tilde V(s)\cdot D_r\Gamma_s\, dx_s + \int_{r\vee r'}^t D_r\tilde V(s)\cdot D_{r'}\Gamma_s\, dx_s + \int_{r\vee r'}^t D_{rr'}\tilde V(s)\cdot \Gamma_s\, dx_s.$$

It is clear that the process $D^2_{rr'}\Gamma$ satisfies a linear equation system analogous to (4.59). The estimate can thus be obtained by following the same arguments as above, invoking again [26]. This completes the proof of (4.49). □

4.5. *A decomposition of the error process.* In [29], equations (6.14) and (7.6), we have decomposed the error process $y_t - y_t^n$ according to the Jacobian of the equation and some remainder terms. In the following proposition we get a similar decomposition, adapted to our needs for the weak convergence estimates. Notice that similar to what we did in Section 4.4, we will drop the indices from our formulae below for sake of readability.

LEMMA 4.14. *We work under the conditions of Lemma 4.8. Recall that x is a standard d-dimensional fBm with Hurst parameter H. Let y and y^n be the solutions of equation (2.7) and the Euler scheme (4.25), respectively. Let Γ and Λ be respectively the solution of equation (4.48) and its inverse $\Lambda = \Gamma^{-1}$. We set $\eta(s) = t_k$ for $s \in [t_k, t_{k+1})$. For $t \in [0, T]$ we also define*

$$\begin{aligned} I_t &= \frac{1}{2}\int_0^t \partial V\partial V V\big(y^n_{\eta(s)}\big)\cdot\big(s-\eta(s)\big)^{2H}\, dx_s \\ &\quad + \int_0^t \bigg(\int_{\eta(s)}^s \int_{\eta(s)}^u \partial^2 V\big(y^n_v\big)\, dy^n_v\, dy^n_u\bigg)\, dx_s \\ &=: I_t^1 + I_t^2. \end{aligned} \tag{4.62}$$

Then the difference $y_t^n - y_t$ can be decomposed as

$$y_t - y_t^n = \sum_{e=1}^5 J_t^e, \tag{4.63}$$

where the processes J_t^1, J_t^2, J_t^3 are respectively defined by

$$\begin{aligned} J_t^1 &= \Gamma_t \int_0^t \Lambda_{\eta(s)}\partial V V\big(y^n_{\eta(s)}\big)\delta x_{\eta(s),s}\, dx_s, \\ J_t^2 &= \Gamma_t \int_0^t (\Lambda_s - \Lambda_{\eta(s)})\partial V V\big(y^n_{\eta(s)}\big)\delta x_{\eta(s),s}\, dx_s, \\ J_t^3 &= \Gamma_t \int_0^t \Lambda_s\, dI_s, \end{aligned} \tag{4.64}$$

and where J_t^4, J_t^5 are given by

$$J_t^4 = -H\cdot\Gamma_t \int_0^t (\Lambda_s - \Lambda_{\eta(s)})\partial V V\big(y^n_{\eta(s)}\big)\cdot\big(s-\eta(s)\big)^{2H-1}\, ds,$$

$$J_t^5 = -H\cdot\Gamma_t \int_0^t \Lambda_{\eta(s)}\partial V V\big(y^n_{\eta(s)}\big)\cdot\big(s-\eta(s)\big)^{2H-1}\, ds.$$

REMARK 4.15. For notational sake, we have stated our result and we will perform our computations in a one-dimensional setting. For completeness, let us now show how some of the terms look in a multidimensional setting. For instance the matrix multiplication in (4.62) should be interpreted as:

$$\partial V \partial V V(y^n_{\eta(s)})\, dx_s = \left(\sum_{i',i''=1}^{m} \sum_{j,j'=1}^{d} \partial_{i'} V^i_j \partial_{i''} V^{i'}_{j'} V^{i''}_{j'}(y^n_{\eta(s)})\, dx^j_s, i = 1, \dots, m \right),$$

$$\partial^2 V(y^n_v)\, dy^n_v\, dy^n_u\, dx_s = \left(\sum_{j=1}^{d} \sum_{i',i''=1}^{m} \partial^2_{i'i''} V^i_j (y^n_v)\, dy^{n,i'}_v\, dy^{n,i''}_u\, dx^j_s, i = 1, \dots, m \right).$$

Similarly, in (4.64) we have

$$\Gamma_t \int_0^t \Lambda_{\eta(s)} \partial V V(y^n_{\eta(s)}) \delta x_{\eta(s),s}\, dx_s$$
$$= \left(\sum_{i',i'',i'''=1}^{m} \sum_{j,j'=1}^{d} \Gamma^{ii'}_t \int_0^t \Lambda^{i'i''}_{\eta(s)} \partial_{i'''} V^{i''}_j V^{i'''}_{j'}(y^n_{\eta(s)}) \delta x^{j'}_{\eta(s),s}\, dx^j_s, i = 1, \dots, m \right).$$

We let the patient reader figure out how the remaining multidimensional quantities would look like.

PROOF OF LEMMA 4.14. As mentioned in Remark 4.15, for notational sake we will perform our computations in a one-dimensional setting. We first recall that the continuous time Euler scheme defined in (4.25) can be written, for $s \in [0, T]$, as

$$\delta y^n_{\eta(s),s} = V(y^n_{\eta(s)}) \delta x_{\eta(s),s} + \frac{1}{2} \partial V V(y^n_{\eta(s)}) \cdot (s - \eta(s))^{2H}, \tag{4.65}$$

where we recall that $\eta(s) = t_k$ whenever $s \in [t_k, t_{k+1})$. One can also write equation (4.65) in integral form, which yields an expression of the form

$$y^n_t = y_0 + \int_0^t V(y^n_{\eta(s)})\, dx_s + H \int_0^t \partial V V(y^n_{\eta(s)})(s - \eta(s))^{2H-1}\, ds. \tag{4.66}$$

Gathering (4.66) with equation (2.7) for which we omit the drift term, we get

$$\begin{aligned} y_t - y^n_t = {} & \int_0^t (V(y_s) - V(y^n_s))\, dx_s + \int_0^t (V(y^n_s) - V(y^n_{\eta(s)}))\, dx_s \\ & - H \cdot \int_0^t \partial V V(y^n_{\eta(s)}) \cdot (s - \eta(s))^{2H-1}\, ds. \end{aligned} \tag{4.67}$$

Next we will consider a decomposition of the quantity $V(y^n_s) - V(y^n_{\eta(s)})$ in (4.67). Namely we apply the chain rule twice to obtain

$$\begin{aligned} V(y^n_s) - V(y^n_{\eta(s)}) &= \int_{\eta(s)}^s \partial V(y^n_u)\, dy^n_u \\ &= \partial V(y^n_{\eta(s)}) \delta y^n_{\eta(s),s} + \int_{\eta(s)}^s \int_{\eta(s)}^u \partial^2 V(y^n_v)\, dy^n_v\, dy^n_u. \end{aligned} \tag{4.68}$$

Plugging (4.65) into (4.68) and then integrating in x we thus get

$$\begin{aligned} & \int_0^t (V(y^n_s) - V(y^n_{\eta(s)}))\, dx_s \\ & = \int_0^t \partial V(y^n_{\eta(s)}) \left(V(y^n_{\eta(s)}) \delta x_{\eta(s),s} + \frac{1}{2} \partial V V(y^n_{\eta(s)}) \cdot (s - \eta(s))^{2H} \right) dx_s \\ & \quad + \int_0^t \left(\int_{\eta(s)}^s \int_{\eta(s)}^u \partial^2 V(y^n_v)\, dy^n_v\, dy^n_u \right) dx_s. \end{aligned} \tag{4.69}$$

Recalling the definition of I_t^1, I_t^2 in (4.62), equation (4.69) can also be read as

$$\int_0^t \big(V(y_s^n) - V(y_{\eta(s)}^n)\big)\,dx_s = \int_0^t \partial VV(y_{\eta(s)}^n)\delta x_{\eta(s),s}\,dx_s + I_t^1 + I_t^2. \tag{4.70}$$

We now decompose the quantity $V(y_s) - V(y_s^n)$ in (4.67). Specifically we write

$$V(y_t) - V(y_t^n) = \int_0^1 \partial V\big(\theta y_t + (1-\theta)y_t^n\big)\,d\theta \cdot \big(y_t - y_t^n\big) = \tilde{V}(t)\cdot\big(y_t - y_t^n\big), \tag{4.71}$$

where we recall that the process $\tilde{V}$ has been introduced in Definition 4.11.

We are ready to plug (4.70) and (4.71) into (4.67) in order to get the following linear equation for $y - y^n$:

$$y_t - y_t^n = \int_0^t \tilde{V}(s)\cdot\big(y_s - y_s^n\big)\,dx_s + K_t, \tag{4.72}$$

where the process K_t is given by

$$K_t = \int_0^t \partial VV(y_{\eta(s)}^n)\delta x_{\eta(s)s}\,dx_s + I_t^1 + I_t^2 - H\int_0^t \partial VV(y_{\eta(s)}^n)\big(s-\eta(s)\big)^{2H-1}\,ds. \tag{4.73}$$

Eventually we recall that Γ solves the Jacobian type equation (4.48) and that $\Lambda_t = \Gamma_t^{-1}$. Hence applying Duhamel's principle in order solve (4.72), we get

$$y_t - y_t^n = \Gamma_t \int_0^t \Lambda_s\,dK_s.$$

Thanks to our expression (4.73), the above equation can be written more explicitly as

$$\begin{aligned} y_t - y_t^n = {} & \Gamma_t \int_0^t \Lambda_s \partial VV(y_{\eta(s)}^n)\delta x_{\eta(s)s}\,dx_s + \Gamma_t \int_0^t \Lambda_s\,d\big(I_s^1 + I_s^2\big) \\ & - H\cdot\Gamma_t \int_0^t \Lambda_s \partial VV(y_{\eta(s)}^n)\cdot\big(s-\eta(s)\big)^{2H-1}\,ds. \end{aligned} \tag{4.74}$$

With relation (4.74) in hand, we can now easily identify the terms in (4.63). Indeed, the second term on the right-hand side of equation (4.74) is exactly J_t^3. Also, in the same equation, by plugging the decomposition $\Lambda_s = \Lambda_{\eta(s)} + (\Lambda_s - \Lambda_{\eta(s)})$ into the first and third terms we identify the first and third terms as $J_t^1 + J_t^2$ and $J_t^4 + J_t^5$, respectively. We thus conclude the identity (4.63). The proof is complete. □

4.6. *The weak convergence of the Euler scheme.* We can now gather all the previous preliminary estimates in order to obtain our main result. This is summarized in the theorem below.

THEOREM 4.16. *Consider a vector field $V \in C_b^4$ and a driving fBm x with Hurst parameter $H > 1/3$. Let y be the solution of the rough differential equation (2.7). The corresponding interpolated Euler scheme is y^n, displayed in (4.25). Then for any $f \in C_b^4(\mathbb{R}^d)$ and $t \in [0,T]$ there is a constant $C > 0$ independent of n such that*

$$\big|\mathbb{E}f(y_t^n) - \mathbb{E}f(y_t)\big| \le \frac{C}{n^{4H-1-\varepsilon}}. \tag{4.75}$$

PROOF. For conciseness we will prove the theorem for the case $V_0 \equiv 0$ only. The general case can be considered in the similar way and is left to the patient reader.

Let f be a generic C_b^4 function. For $t \in [0,T]$ we define an interpolated process

$$f_1(t) = \int_0^1 \partial f\big(\lambda y_t + (1-\lambda)y_t^n\big)\,d\lambda. \tag{4.76}$$

Here note that in order to alleviate notation, we still drop indices and perform our computation as if our quantities were real-valued. Next a simple application of the fundamental theorem of calculus plus Lemma 4.14 reveal that

$$f(y_t) - f(y_t^n) = f_1(t) \cdot (y_t - y_t^n) = \sum_{e=1}^{5} f_1(t) J_t^e. \tag{4.77}$$

The remainder of the proof is dedicated to estimate the five terms in the right-hand side of (4.77). For sake of conciseness we prove (4.75) for $t \in [\![0, T]\!]$ only. The proof for $t \in [0, T]$ follows the same lines and is left to patient reader.

Step 1: Estimating J_t^1 and J_t^5. In this step, we consider the first and fifth term in (4.77). Note that the integrals in the expressions for J_t^1 and J_t^5 are in fact discrete sums. We can thus combine those two terms in order to get

$$f_1(t)(J_t^1 + J_t^5) = \sum_{t_k < t} f_1(t)\Gamma_t \Lambda_{t_k} \partial V V(y_{t_k}^n)\left(x_{t_k t_{k+1}}^2 - \frac{1}{2}\Delta^{2H}\right). \tag{4.78}$$

Let us say a few words about the term $\psi_k \equiv x_{t_k t_{k+1}}^2 - \frac{1}{2}\Delta^{2H}$ in the right-hand side of (4.78). First we highlight again the fact that we are performing 1-d type computations in order to simplify notation. In a d-dim setting we would consider random variables of the form

$$\psi_k^{ij} = x_{t_k t_{k+1}}^{2,ij} - \frac{1}{2}\Delta^{2H} \mathbf{1}_{\{i=j\}}, \quad \text{for } i, j \in \{1, \dots, d\}.$$

Here we will just focus on the terms $\psi_k \equiv \psi_k^{ii}$, which are the most demanding ones. We leave the off-diagonal terms ψ_k^{ij} to the patient reader for sake of conciseness. Next we should also have in mind the fact that ψ_k can be written as

$$\psi_k = \frac{1}{2}(\delta x_{t_k t_{k+1}})^2 - \frac{1}{2}\Delta^{2H} = \frac{1}{2}\Delta^{2H} H_2\left(\frac{(\delta x_{t_k t_{k+1}})^2}{\Delta^{2H}}\right),$$

where H_2 stands for the Hermite polynomial $H_2(x) = x^2 - 1$. Invoking [33], Page 23, we thus get

$$\psi_k = \delta^{\diamond,2}(\beta_{t_k t_{k+1}}),$$

where $\beta_{t_k t_{k+1}}$ is defined by (4.9) and $\delta^{\diamond,2}$ stands for a double Skorohod integral (see Section 2.3 for Malliavin calculus notation). Hence applying twice the integration by parts (2.17), we end up with

$$\mathbb{E}[f_1(t)(J_t^1 + J_t^5)] = \sum_{t_k < t} \mathbb{E}[\langle D^2[f_1(t)\Gamma_t \Lambda_{t_k} \partial V V(y_{t_k}^n)], \beta_{t_k t_{k+1}}\rangle_{\mathcal{H}^{\otimes 2}}],$$

where recall that β is defined in (4.9). Applying Lemma 4.3 with φ given by

$$\varphi = D^2[f_1(t)\Gamma_t \Lambda_{t_k} \partial V V(y_{t_k}^n)],$$

and recalling that f_1 is the process in (4.76), we obtain

$$|\mathbb{E}[f_1(t)(J_t^1 + J_t^5)]| \le \sum_{t_k < t} n^{-4H} \mathbb{E}[\| D^2[f_1(t)\Gamma_t \Lambda_{t_k} \partial V V(y_{t_k}^n)]\|_\infty]. \tag{4.79}$$

The integrability results Theorem 3.8 and Lemma 4.13 (b) guarantee the uniform integrability in n of the sup-norm in the inequality (4.79). Therefore, we have the estimate

$$|\mathbb{E}[f_1(t)(J_t^1 + J_t^5)]| \le C \sum_{t_k < t} n^{-4H} = Cn^{1-4H}. \tag{4.80}$$

Step 2: Estimating J_t^2. We turn to the estimate of J_t^2 in (4.77) and Lemma 4.14. Observe that according to the fact that $\Lambda = \Gamma^{-1}$ and Λ solves (4.48), we have $\Lambda_s - \Lambda_{\eta(s)} = -\int_{\eta(s)}^{s} \Lambda_u \tilde{V}(u)\, dx_u$. Substituting this into J_t^2 we obtain

$$J_t^2 = -\Gamma_t \int_0^t \left(\int_{\eta(s)}^{s} \Lambda_u \tilde{V}(u)\, dx_u\right) \partial V V\big(y^n_{\eta(s)}\big) \delta x_{\eta(s)s}\, dx_s. \tag{4.81}$$

Now let us write

$$\Lambda_u \tilde{V}(u) = \big(\Lambda_u \tilde{V}(u) - \Lambda_{\eta(s)} \tilde{V}\big(\eta(s)\big)\big) + \Lambda_{\eta(s)} \tilde{V}\big(\eta(s)\big).$$

Reporting this relation into our expression (4.81) for J_t^2 yields the decomposition:

$$\begin{aligned} J_t^2 = &-\Gamma_t \int_0^t \int_{\eta(s)}^{s} \big(\Lambda_u \tilde{V}(u) - \Lambda_{\eta(s)} \tilde{V}\big(\eta(s)\big)\big)\, dx_u \cdot \partial V V\big(y^n_{\eta(s)}\big) \delta x_{\eta(s)s}\, dx_s \\ &- \sum_{t_k < t} \Gamma_t \int_{t_k}^{t_{k+1}} \Lambda_{t_k} \tilde{V}(t_k) \delta x_{t_k s} \partial V V\big(y^n_{t_k}\big) \delta x_{t_k s}\, dx_s =: J_t^{21} + J_t^{22}. \end{aligned} \tag{4.82}$$

We now proceed to the analysis of J_t^{21} and J_t^{22} above.

In order to bound the term J_t^{22} in our decomposition (4.82), observe that this term is of the form $\sum_{t_k<t} f_{t_k} \delta g_{t_k t_{k+1}}$ as in Lemma 2.5. Precisely, we have

$$J_t^{22} = -\sum_{t_k < t} \underbrace{\Gamma_t \Lambda_{t_k} \tilde{V}(t_k) \partial V V\big(y^n_{t_k}\big)}_{= f_{t_k}} \cdot \underbrace{\int_{t_k}^{t_{k+1}} \delta x_{t_k s} \delta x_{t_k s}\, dx_s}_{= \delta g_{t_k t_{k+1}}}. \tag{4.83}$$

Moreover, according to (4.49) and the L^p-estimates for $\mathcal{M}_0$, $\mathcal{M}_1$, it is readily checked that for all $p \geq 1$ and $(u, v) \in \mathcal{S}_2(\llbracket 0, T \rrbracket)$ we have

$$\big(\mathbb{E}\big[|\delta f_{uv}|^{2p}\big]\big)^{\frac{1}{2p}} \lesssim |v - u|^{H-\varepsilon}. \tag{4.84}$$

In addition g has to be seen as a triple iterated integral of x. It has been shown in [29], Lemma 4.3, that for all $(u, v) \in \mathcal{S}_2(\llbracket 0, T \rrbracket)$ we have

$$\big(\mathbb{E}\big[|\delta g_{uv}|^{2p}\big]\big)^{\frac{1}{2p}} \lesssim \frac{|v - u|^{1/2}}{n^{3H-1/2}}. \tag{4.85}$$

Since we are considering points u, v on the grid $\llbracket 0, T \rrbracket$, it is readily checked that $v - u \geq T/n$. Hence one can play with the exponents in (4.85) and write

$$\big(\mathbb{E}\big[|\delta g_{uv}|^{2p}\big]\big)^{\frac{1}{2p}} \lesssim \frac{|v - u|^{1-H+2\varepsilon}}{n^{4H-1-2\varepsilon}}. \tag{4.86}$$

It follows that gathering (4.84) and (4.86) one can apply Lemma 2.5 to (4.83) and get

$$\big|\mathbb{E}\big[f_1(t) J_t^{22}\big]\big| \leq \frac{C}{n^{4H-1-\varepsilon}}. \tag{4.87}$$

In order to bound $\mathbb{E}[f_1(t) J_t^{21}]$, where J_t^{21} is defined in (4.82), we need to make a further decomposition. Using the product rule plus equation (4.48) for Λ, Definition 4.11 for $\tilde{V}$, as well as relation (2.7) and (4.25) for y and y^n, we can write

$$\Lambda_u \tilde{V}(u) - \Lambda_{\eta(s)} \tilde{V}\big(\eta(s)\big) = \int_{\eta(s)}^{u} f_2(v)\, dx_v + \int_{\eta(s)}^{u} f_3(v)\, d\big(v - \eta(v)\big)^{2H}, \tag{4.88}$$

where we have set

$$f_2(v) = -\Lambda_v \tilde{V}(v) \tilde{V}(v) + \Lambda_v \partial \tilde{V}(v), \tag{4.89}$$

$$f_3(v) = \frac{1}{4} \Lambda_v \int_0^1 \partial\partial V\big(\theta y_v + (1-\theta) y^n_v\big)(1-\theta) \partial V V(y_{\eta(s)}), \tag{4.90}$$

and where we denote

$$\partial \tilde{V}(v) = \int_0^1 \partial\partial V\big(\theta y_v + (1-\theta) y_v^n\big)\big(\theta V(y_v) + (1-\theta) V\big(y_{\eta(v)}^n\big)\big)\, d\theta.$$

Then we write

$$\int_{\eta(s)}^u f_2(v)\, dx_v = f_2\big(\eta(s)\big)\delta x_{\eta(s),u} + \int_{\eta(s)}^u \big(f_2(v) - f_2\big(\eta(s)\big)\big)\, dx_v.$$

Substituting the above into J_t^{21} we obtain a weighted sum of two fourth and one fifth order multiple integral in the form $\sum_{0\le t_k<t} h_k$. Precisely, we have $J_t^{21} = -(J_t^{211} + J_t^{212} + J_t^{213})$, where

$$\begin{aligned} J_t^{211} &= \sum_{0\le t_k<t} \Gamma_t \int_{t_k}^{t_{k+1}\wedge t} \int_{t_k}^s f_2(t_k)\delta x_{t_k,u}\, dx_u \cdot \partial V V\big(y_{t_k}^n\big)\delta x_{t_k s}\, dx_s \\ &\equiv \sum_{0\le t_k<t} \Gamma_t h_t^{211}, \\ J_t^{212} &= \sum_{0\le t_k<t} \Gamma_t \int_{t_k}^{t_{k+1}\wedge t} \int_{t_k}^s \int_{t_k}^u \big(f_2(v) - f_2(t_k)\big)\, dx_v\, dx_u \cdot \partial V V\big(y_{t_k}^n\big)\delta x_{t_k s}\, dx_s \\ &\equiv \sum_{0\le t_k<t} \Gamma_t h_k^{212}, \\ J_t^{213} &= -\Gamma_t \int_0^t \int_{\eta(s)}^s \int_{\eta(s)}^u f_3(v)\, d\big(v - \eta(v)\big)^{2H}\, dx_u \cdot \partial V V\big(y_{\eta(s)}^n\big)\delta x_{\eta(s)s}\, dx_s. \end{aligned} \tag{4.91}$$

We now proceed to estimate the terms J_t^{211}, J_t^{212} and J_t^{213}.

One can easily analyze the term J_t^{211} by writing

$$h_k^{211} = f_2(t_k)\partial V V\big(y_{t_k}^n\big) x_{t_k,t_{k+1}\wedge t}^4,$$

where x_{st}^4 denotes the fourth order iterated integral over the interval $[s,t]$. It follows that $\|h_k^{211}\|_{L^p} \lesssim \frac{1}{n^{4H}}$ for $p \ge 1$. This implies that

$$\mathbb{E}\big[f_1(t) J_t^{211}\big] \le \sum_{t_k<t} C \cdot n^{-4H} \le \frac{C}{n^{4H-1}}. \tag{4.92}$$

In the same way we can show that the bound (4.92) also holds for J_t^{213}.

As far as J_t^{212} is concerned, one can recast the term h_k^{212} as $h_k^{212} = \partial V V(y_{t_k}^n)\hat{h}_k^{212}$, with

$$\hat{h}_k^{212} = \int_{t_k}^{t_{k+1}\wedge t} \int_{t_k}^s \int_{t_k}^u \big(f_2(v) - f_2(t_k)\big)\, dx_v\, dx_u \delta x_{t_k s}\, dx_s. \tag{4.93}$$

The quantity $\hat{h}_k^{212}$ has to be seen as a fifth order iterated integral. One way to quantify $\hat{h}_k^{212}$ is to resort to Fubini's theorem for multiple rough integrals:

$$\int_{t_k}^{t_{k+1}} \int_{t_k}^s \int_{t_k}^u g(v)\delta x_{t_k s}\, dx_v\, dx_u\, dx_s = \int_{t_k}^{t_{k+1}} \int_v^{t_{k+1}} \int_u^{t_{k+1}} g(v)\delta x_{t_k s}\, dx_s\, dx_u\, dx_v, \tag{4.94}$$

where we have denoted $g(v) = f_2(v) - f_2(t_k)$. Note that relation (4.94) can be shown by taking limits along smooth approximations of x, plus using the fact that x is a geometric rough path. Applying (4.94) to (4.93) gives

$$\hat{h}_k^{212} = \int_{t_k}^{t_{k+1}} z_v^{t_k t_{k+1}}\, dx_v, \quad \text{with } z_v^{t_k t_{k+1}} = \big(f_2(v) - f_2(t_k)\big) \int_v^{t_{k+1}} dx_u \int_u^{t_{k+1}} \delta x_{t_k s}\, dx_s.$$

Using the rough path property of x recalled in Section 2.2 and the definition of f_2 in (4.88), it is readily checked that $z^{t_k t_{k+1}}$ is of order $(1/n)^{4H-\varepsilon}$ for any $\varepsilon > 0$. Reporting this information in (4.93), one gets the almost sure relation

$$|\hat{h}_k^{212}| \le \frac{G}{n^{5(H-\varepsilon)}},$$

where $G \in \bigcap_{p\ge 1} L^p(\Omega)$. With relation (4.91) in mind and taking into account the definition (4.76) of f_1, we discover that

$$|\mathbb{E}[f_1(t) J_t^{212}]| \le \frac{C}{n^{-1+5(H-\varepsilon)}} \le \frac{C}{n^{4H-1}}. \tag{4.95}$$

Summarizing our considerations for the term J_t^2, we gather our estimates (4.92) and (4.95). This yields the desired estimate

$$|\mathbb{E}[f_1(t) J_t^2]| \le \frac{C}{n^{4H-1-\varepsilon}}. \tag{4.96}$$

Step 3: Estimating J_t^3. In this step, we consider the term J_t^3 defined in Lemma 4.14. Also recall that I_t has been decomposed into $I_t^1 + I_t^2$ in (4.62). Accordingly we shall write

$$J_t^3 = J_t^{31} + J_t^{32} \equiv \Gamma_t \int_0^t \Lambda_s \, dI_s^1 + \Gamma_t \int_0^t \Lambda_s \, dI_s^2,$$

and estimate J_t^{31}, J_t^{32} separately. Resorting to expression (4.62) for I^2, let us write J_t^{32} as

$$J_t^{32} = \Gamma_t \int_0^t \int_{\eta(s)}^s \int_{\eta(s)}^u \Lambda_s \partial^2 V(y_v^n) \, dy_v^n \, dy_u^n \, dx_s.$$

In this way, it is readily checked that J_t^{32} exhibits the same type of regularity as J_t^2 defined by (4.81). The complete analysis of J_t^{32} thus follows the same steps as J_t^2. It relies on another discretization procedure, similar to (4.82). Namely one writes $J_t^{32} = J_t^{321} + J_t^{322}$, with

$$J_t^{321} = \Gamma_t \int_0^t \int_{\eta(s)}^s \int_{\eta(s)}^u \big(\Lambda_s \partial^2 V(y_v^n) - \Lambda_{\eta(s)} \partial^2 V(y_{\eta(v)})\big) \, dy_v^n \, dy_u^n \, dx_s,$$

$$J_t^{322} = \Gamma_t \sum_{t_k < t} \Lambda_{t_k} \partial^2 V(y_{t_k}) \int_{t_k}^{t_{k+1}} \int_{\eta(s)}^s \int_{\eta(s)}^u dy_v^n \, dy_u^n \, dx_s.$$

In addition, along the same lines as for (4.83) and resorting to the discrete dynamics (4.25) of y^n, one can express J_t^{321} as a weighted sum of triple integrals of x. We can thus proceed as in the estimation of J_t^2 and get the same inequalities as in (4.87), (4.92) and (4.95). Details are left to the reader for sake of conciseness. We obtain

$$|\mathbb{E}[f_1(t) J_t^{32}]| \le \frac{C}{n^{4H-1-\varepsilon}}. \tag{4.97}$$

In order to bound the term J_t^{31}, we first use another step of discretization. That is, we decompose J_t^{31} as $J_t^{311} + J_t^{312}$ with

$$J_t^{311} = \frac{1}{2} \Gamma_t \int_0^t \Lambda_{\eta(s)} \partial V \partial V V(y_{\eta(s)}^n) \cdot (s - \eta(s))^{2H} \, dx_s,$$

$$J_t^{312} = \frac{1}{2} \Gamma_t \int_0^t (\Lambda_s - \Lambda_{\eta(s)}) \partial V \partial V V(y_{\eta(s)}^n) \cdot (s - \eta(s))^{2H} \, dx_s.$$

Note that by applying Lemma 2.5 we can bound $\mathbb{E}[f_1(t) J_t^{311}]$ by $\frac{1}{n^{4H-1-\varepsilon}}$. Indeed, we can write

$$f_1(t) J_t^{311} = \frac{1}{2} f_1(t) \Gamma_t \sum_{0 \le t_k < t} \Lambda_{t_k} \partial V \partial V V(y_{t_k}^n) \cdot \nu_{t_k, t_{k+1} \wedge t}, \tag{4.98}$$

where the increment ν is defined by

$$\nu_{uv} = \int_u^v \big(s-\eta(s)\big)^{2H}\, dx_s. \tag{4.99}$$

Next recall the following result from Lemma 4.6 in [29]: For a fBm x with Hurst parameter H and f such that $\|f\|_\gamma \in L^p$ for all $\gamma < H$ and $p \geq 1$, we have

$$\left\{\mathbb{E}\left[\left|\sum_{0\le t_k<t} f_{t_k}\nu_{t_k,t_{k+1}\wedge t}\right|^p\right]\right\}^{1/p} \le \frac{CT}{n^{4H-1-\varepsilon}}. \tag{4.100}$$

One can apply directly this estimate to (4.98) in order to get

$$\mathbb{E}\big[f_1(t)J_t^{311}\big] \le \frac{C}{n^{4H-1-\varepsilon}}. \tag{4.101}$$

The term J_t^{312} has to be compared to J_t^{21} in (4.82). We can thus follow some computations which are very similar to (4.88)–(4.91). We end up with second and third integrals involving x and the increment ν in (4.99). Having the regularity (4.100) of ν into account we let the reader check that

$$\mathbb{E}\big[f_1(t)J_t^{312}\big] \le \frac{C}{n^{4H-1-\varepsilon}}, \tag{4.102}$$

similar to (4.92) and (4.95). We can thus conclude this step by gathering (4.101) and (4.102). This yields

$$\big|\mathbb{E}\big[f_1(t)J_t^{3}\big]\big| \le Cn^{1-4H+\varepsilon}. \tag{4.103}$$

Step 4: Estimating J_t^4. We now turn to an estimate of the term J_t^4 in Lemma 4.14. According to the expression therein and equation (4.48) for Λ, observe that

$$\mathbb{E}\big[f_1(t)J_t^{4}\big] = -H\cdot\int_0^t Q_s^t\cdot\big(s-\eta(s)\big)^{2H-1}\, ds, \tag{4.104}$$

where the quantity Q_s^t is given by

$$Q_s^t = \mathbb{E}\left[f_1(t)\left(\int_{\eta(s)}^s \Gamma_t\Lambda_u\tilde{V}(u)^T\, dx_u\right)\partial VV\big(y^n_{\eta(s)}\big)\right].$$

As for J_t^{31} we can show that

$$\left|\mathbb{E}\left[f_1(t)\Gamma_t\left(\int_{\eta(s)}^s \Lambda_u\tilde{V}(u)^T\, dx_u\right)\partial VV\big(y^n_{\eta(s)}\big)\right]\right| \le \frac{C}{n^{2H-\varepsilon}}. \tag{4.105}$$

Indeed, by writing and substituting

$$\Lambda_u\tilde{V}(u)^T = \Lambda_{\eta(s)}\tilde{V}\big(\eta(s)\big)^T + \big(\Lambda_u\tilde{V}(u)^T - \Lambda_{\eta(s)}\tilde{V}\big(\eta(s)\big)^T\big)$$

into (4.105), we decompose (4.105) into two components. The second component obtained is a double integral over the interval $[\eta(s), s]$, which is bounded by $\frac{1}{n^{2H-\varepsilon}}$. On the other hand, the first component is of the form $\mathbb{E}[F\delta x_{\eta(s)s}]$, where

$$F = f_1(t)\Gamma_t\big(\Lambda_{\eta(s)}\tilde{V}\big(\eta(s)\big)^T\big)\partial VV\big(y^n_{\eta(s)}\big).$$

Note that F is an integrable variable whose Malliavin derivative DF is also integrable. So applying integration by parts to $\mathbb{E}[F\delta x_{\eta(s)s}]$ and then Lemma 4.4 with $\varphi = DF$, together with the upper-bound estimates in Lemma 4.13 and Theorem 2.12, we obtain the bound $\frac{1}{n^{2H}}$. Gathering those consideration and (4.105) into (4.104), we end up with

$$\mathbb{E}\big[f_1(t)J_t^{4}\big] \le \frac{C}{n^{4H-1-\varepsilon}}. \tag{4.106}$$

Step 5: Conclusion. Taking expectations on both sides of (4.77) and reporting (4.80), (4.96), (4.103), and (4.106) we discover that (4.75) holds true. This finishes the proof. □

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