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Pathwise definition of second-order SDEs

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Abstract

In this article, a class of second-order differential equations on [0, 1], driven by a γ-Hölder continuous function for any value of $\gamma \in (0, 1)$ and with multiplicative noise, is considered. We first show how to solve this equation in a pathwise manner, thanks to Young integration techniques. We then study the differentiability of the solution with respect to the driving process and consider the case where the equation is driven by a fractional Brownian motion, with two aims in mind: show that the solution that we have produced coincides with the one which would be obtained with Malliavin calculus tools, and prove that the law of the solution is absolutely continuous with respect to the Lebesgue measure.

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1. Introduction

During the past few years, a growing activity has emerged, aiming at solving stochastic PDEs beyond the Brownian case. In some special situations, namely in linear (additive noise) or bilinear (noisy term of the form $u\,\dot{B}$) cases, stochastic analysis techniques can be applied [15, 29]. When the driving process of the equation exhibits a Hölder continuity exponent greater than $1/2$, Young integration or fractional calculus tools also allow us to solve those equations in a satisfactory way [11,17,22,7]. Eventually, when one wishes to tackle non-linear problems in

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which the driving noise is only Hölder continuous with Hölder regularity exponent $\leq 1/2$, rough paths analysis must come into the picture. This situation is addressed in [5,12,27].

It should be mentioned however that all the articles mentioned above only handle the case of parabolic or hyperbolic systems, leaving aside the case of elliptic equations. This is of course due to the special physical relevance of heat and wave equations, but also stems from a specific technical difficulty inherent to elliptic equations. Indeed, even in the usual Brownian case, the notions of filtration and the adapted process are useless for solving non-linear elliptic systems, so Itô's integration theory is not sufficient in this situation. A natural idea in this context is then to use the power of anticipative calculus, based on Malliavin type techniques (see e.g. [18]). This method has however a serious drawback in our context, mainly because the Picard type estimates involve Malliavin derivatives of any order, and cannot be closed. To the best of our knowledge, all the stochastic elliptic equations considered up to now involve thus a mere additive noise. Let us mention for instance the pioneering works [3,19] for the existence and uniqueness of solutions, the study of Markov's property [6,19], and the numerical approximations of [16,25,28], as well as the recent and deep contribution [4], which relates stochastic elliptic systems, anticipative Girsanov transforms and deterministic methods.

With these preliminary considerations in mind, the aim of the current paper is twofold:

(i) We wish to solve a non-linear elliptic equation of the form

$$\partial^2_{tt} z_t = \sigma(z_t)\dot{x}_t, \quad t \in [0,1], z_0 = z_1 = 0, \tag{1}$$

where σ is a smooth enough function from $\mathbb{R}$ to $\mathbb{R}$, and x is a Hölder continuous noisy input with any Hölder continuity exponent $\gamma \in (0,1)$. With this purpose, we shall write Eq. (1) in a variant of the so-called mild form, under which it becomes obvious that the system can be solved in the space $\mathcal{C}^\kappa$ of κ-Hölder continuous functions, for any $1-\gamma < \kappa < 1$ (see Section 2.1 for a precise definition of this space).

Let us observe however that, when dealing with a non-linear multiplicative noise, one is not allowed to use the monotonicity methods invoked in [3]. This forces us to use contraction type arguments, which can be applied only provided the Hölder norm of x is small enough. In order to overcome this restriction, we shall introduce a positive constant M, and replace the diffusion coefficient σ by a function $\sigma_M : \mathbb{R} \times \mathcal{C}^\gamma \to \mathbb{R}$ such that $y \mapsto \sigma_M(y,x)$ is regular enough and $\sigma_M(\cdot, x) \equiv 0$ whenever $\|x\|_\gamma \geq M+1$. We shall thus produce a *local* solution to Eq. (1), in the sense given for instance in [18] concerning the localization of the divergence operator on the Wiener space. Once this change is made, a proper definition of the solution plus a fixed point argument leads to the existence and uniqueness of the solution for Eq. (1).

(ii) Having produced a unique solution to our system in a reasonable class of functions, one may wonder whether this solution could have been obtained using Malliavin calculus techniques, in spite of the fact that a direct application of those techniques to our equation does not yield a satisfactory solution in terms of fixed point arguments. In order to answer this question, we shall prove that, when x is a fractional Brownian motion (fBm in the sequel), the solution is differentiable enough in the Malliavin calculus sense, so the stochastic integrals involved in the mild formulation of (1) can be interpreted as Skorohod integrals plus a trace term, or better, as Stratonovich integrals. This will be achieved by differentiating the deterministic equation (1) with respect to the driving noise x and identifying this derivative with the usual Malliavin derivative, as was done in [2,14,21]. As a by-product, we will also be able to study the density of the random variable z_t for a fixed time $t \in (0,1)$.

We shall thus obtain the following result, which is stated here in a rather loose form (the reader is sent to the corresponding sections for detailed statements):

Theorem 1.1. *Consider $x \in \mathcal{C}^\gamma$ for a given $\gamma > 0$, a constant $M > 0$ and a $\mathcal{C}^4(\mathbb{R})$ function σ, such that $\|\sigma^{(j)}\|_\infty \le \frac{c_j}{M+1}$ for any $j = 0, 1, 2$ with some small enough constants c_j. Let σ_M be the localized diffusion coefficient alluded to above (see Definition 2.6 for more details). Then:*

(1) *The equation*

$$\partial_{tt}^2 z_t = \sigma_M(x, z_t)\dot{x}_t, \quad t \in [0, 1], z_0 = z_1 = 0 \tag{2}$$

admits a unique solution, lying in a space of the form $\mathcal{C}^\kappa$ for any $1 - \gamma < \kappa < 1$.

(2) *Assume x to be the realization of a fractional Brownian motion with Hurst parameter $H > 1/2$. Then for any $t \in [0, 1]$, z_t is an element of the Malliavin–Sobolev space $\mathbb{D}^{1,2}$ and the integral form of (2) can be interpreted by means of Skorohod integrals plus trace terms (see Section 4 for further definitions).*

(3) *Still in the fBm context, with a slight modification of our cutoff coefficient σ_M and under the non-degeneracy condition $|\sigma(y)| \ge \sigma_0 > 0$ for all $y \in \mathbb{R}$, one gets the following result: for any $t \in (0, 1)$ and $a > 0$, the restriction of $\mathcal{L}(z_t)$ to $\mathbb{R} \setminus (-a, a)$ admits a density with respect to Lebesgue's measure.*

The reader might wonder why we have made the assumption of a *small* coefficient σ here, through the assumption $\|\sigma^{(j)}\|_\infty \le \frac{c_j}{M+1}$. This is due to the fact that monotonicity methods, which are essential in the deterministic literature (see e.g. [8]) as well as in the stochastic references quoted above, are ruled out here by the presence of the diffusion coefficient in front of the noise $\dot{x}$. We have thus focused on contraction type properties, which are also mentioned in [19]. Let us also say a word about possible generalizations to elliptic equations in dimension $d = 2, 3$: the main additional difficulty lies in the fact that the fundamental solution to the elliptic equation exhibits some singularities on the diagonal, which should be dealt with. In particular, if one wishes to handle the case of a general Hölder continuous signal x, rough paths arguments in higher dimensions should be used. This possibility goes far beyond the current article.

At a technical level, let us mention that the first part of Theorem 1.1 above relies on an appropriate formulation of the equation, which enables one to quantify the increments of the candidate solution in a reasonable way, plus some classical contraction arguments. As far as the Malliavin differentiability of the solution is concerned, it hinges on rather standard methods (see [14,21]). However, our density result for $\mathcal{L}(z_t)$ is rather delicate, for two main reasons:

- The lack of a real time direction or filtration in Eq. (2) makes many usual lower bounds on the Malliavin derivatives rather clumsy.
- One has to take care of the derivatives of our cutoff function σ_M with respect to the driving process, for which upper bounds are to be provided and compared to some leading terms in the Malliavin derivatives.

Solutions to these additional problems are given at Section 4, which can be seen as the most demanding part of our paper. It should also be pointed out that we are able to solve Eq. (2) for any Hölder regularity of the driving noise x, while our stochastic analysis part is devoted to fBm with Hurst parameter $H > 1/2$. This is only due to the fact that Malliavin calculus is much easier to handle in the latter situation, and we firmly believe that our results could be generalized to $H < 1/2$.

Here is how our article is structured. Our equation is defined and solved in Section 2. Differentiation properties of its solution with respect to the driving process are investigated in Section 3. Finally, the Malliavin calculus aspects for fractional Brownian motion, including the existence of a density, are handled in Section 4.

Unless otherwise stated, any constant c or C appearing in our computations below is understood as a generic constant which might change from line to line without further mention.

2. Existence and uniqueness of the solution

Recall that we wish to solve the one-dimensional second-order differential equation (1). Towards this end, we shall change its formulation a little using some heuristic considerations, and introduce our localization coefficient σ_M. We will then be able to solve the equation thanks to a fixed point argument.

2.1. Heuristic considerations

Assume for the moment that x is a smooth function defined on $[0, 1]$. Hence, if σ is small and regular enough, it is easily shown (see [19] for similar arguments) that Eq. (1) can be solved thanks to contraction arguments.

It is also well-known in this case that Eq. (1) can be understood in the mild sense. Specifically, let the kernel $K : [0, 1]^2 \to [0, 1]$ be the fundamental solution of the linear elliptic equation with Dirichlet boundary conditions, and notice that this kernel is explicitly given by

$$K(t, \xi) = t \wedge \xi - t\xi, \quad t, \xi \in [0, 1]. \tag{3}$$

Then $\{z_t, t \in [0, 1]\}$ solves (1) if it satisfies the integral equation

$$z_t = \int_0^1 K(t, \xi)\sigma(z_\xi)\mathrm{d}x_\xi, \quad t \in [0, 1], \tag{4}$$

where the integrals above are understood in the Riemann sense as soon as x is continuously differentiable.

Still assuming that x is continuously differentiable, let us retrieve some more information about the increments of the solution z to our elliptic equation. In order to do so, set first

$$\delta f_{st} = f_t - f_s, \quad 0 \le s \le t \le 1,$$

for any continuous function f. Let us also give an expression for the increments of K, by noticing that this kernel can be differentiated with respect to its first variable. Indeed, one has

$$\partial_u K(u, \xi) = \mathbf{1}_{\{u \le \xi\}} - \xi \Longrightarrow K(t, \xi) - K(s, \xi) = \int_s^t \left(\mathbf{1}_{\{u \le \xi\}} - \xi\right) \mathrm{d}u.$$

Then, thanks to an obvious application of Fubini's theorem, the increments of z can be written as

$$\begin{aligned}
(\delta z)_{st} &= \int_0^1 \left(\int_s^t (\mathbf{1}_{\{u \le \xi\}} - \xi)\mathrm{d}u\right) \sigma(z_\xi)\mathrm{d}x_\xi \\
&= \int_s^t \mathrm{d}u \int_0^1 (\mathbf{1}_{\{u \le \xi\}} - \xi)\sigma(z_\xi)\mathrm{d}x_\xi \\
&= \int_s^t \mathrm{d}u \left(\int_u^1 \sigma(z_\xi)\mathrm{d}x_\xi\right) - (t - s) \int_0^1 \xi\sigma(z_\xi)\mathrm{d}x_\xi.
\end{aligned}$$

The latter equation is the one which is amenable to generalization to a non-smooth setting, and we will thus interpret our elliptic system in this way: we say that a continuous function $z : [0, 1] \to \mathbb{R}$ is a solution to (1) if, for any $0 \le s \le t \le 1$,

$$\delta z_{st} = \int_s^t \mathrm{d}u \left(\int_u^1 \sigma(z_\xi)\mathrm{d}x_\xi \right) - (t-s) \int_0^1 \xi\sigma(z_\xi)\mathrm{d}x_\xi, \tag{5}$$

where the integrals with respect to the driving noise x are interpreted in the Young sense.

2.2. *Hölder spaces and the cutoff*

Though it could be intuited from the original equation, our formulation (5) of the elliptic system indicates clearly that the candidate solution should be κ-Hölder continuous for any $\kappa < 1$, independently of the smoothness of x.

More precisely, let $\mathcal{C}^\gamma$ be the space of continuous functions $f \in \mathcal{C}([0, 1])$ such that $\|f\|_\gamma < +\infty$, where

$$\|f\|_\gamma = \|f\|_\infty + \sup_{0 \le s < t \le 1} \frac{|\delta f_{st}|}{|t-s|^\gamma},$$

and where we recall that $\delta f_{st} = f_t - f_s$. We shall define the integrals in (5) thanks to the following classical proposition (see [30]):

Proposition 2.1. *Let $f \in \mathcal{C}^\gamma$, $g \in \mathcal{C}^\kappa$ with $\gamma + \kappa > 1$, and $0 \le s \le t \le 1$. Then the integral $\int_s^t g_\xi \mathrm{d}f_\xi$ is well-defined as a limit of Riemann sums along partitions of $[s, t]$. Moreover, the following estimate is fulfilled:*

$$\left| \int_s^t g_\xi \mathrm{d}f_\xi \right| \le c_{\gamma,\kappa} \|f\|_\gamma \|g\|_\kappa |t-s|^\gamma, \tag{6}$$

where the constant $c_{\gamma,\kappa}$ only depends on γ and κ. A sharper estimate is also available:

$$\left| \int_s^t g_\xi \mathrm{d}f_\xi \right| \le |g_s| \, \|f\|_\gamma |t-s|^\gamma + c_{\gamma,\kappa} \|f\|_\gamma \|g\|_\kappa |t-s|^{\gamma+\kappa}. \tag{7}$$

The following straightforward property will also be used in the sequel: if $f, g \in \mathcal{C}^\gamma$, then the product fg defines an element in $\mathcal{C}^\gamma$ such that $\|fg\|_\gamma \le \|f\|_\gamma \|g\|_\gamma$.

Remark 2.2. It might be clear to the reader that the solution to our elliptic system will live in fact in a space of Lipschitz functions. We have chosen here to work in the Young setting because this does not induce any additional difficulty, and is more likely to be generalizable to higher dimensions of the parameter t.

The following Fubini type theorem for Young integrals is a slight modification of [14, Proposition 2.6] and will be needed in the sequel:

Proposition 2.3. *Consider $\gamma_i, \lambda_i \in (0, 1)$, $i = 1, 2$, such that $\gamma_i + \lambda_j > 1$ for all $i, j = 1, 2$. Let $g \in \mathcal{C}^{\gamma_1}$ and $f \in \mathcal{C}^{\gamma_2}$, and $h : \{(t, s) \in [0, 1]^2; 0 \le s \le t \le 1\} \to \mathbb{R}$ be a function such that $h(\cdot, t)$ (resp. $h(t, \cdot)$) belongs to $\mathcal{C}^{\lambda_1}([t, 1])$ (resp. $\mathcal{C}^{\lambda_2}([0, t])$) uniformly in $t \in [0, 1]$, and*

$$\|h(r_1, \cdot) - h(r_2, \cdot)\|_{\lambda_1} \le C|r_1 - r_2|^{\lambda_1} \qquad \|h(\cdot, u_1) - h(\cdot, u_2)\|_{\lambda_2} \le C|u_1 - u_2|^{\lambda_2}. \tag{8}$$

Then

$$\int_s^t \int_s^r h(r,u)\, \mathrm{d}g_u \mathrm{d}f_r = \int_s^t \int_u^t h(r,u)\, \mathrm{d}f_r \mathrm{d}g_u, \quad 0 \le s \le t \le T, \tag{9}$$

and

$$\int_s^t \int_u^1 h(r,u)\, \mathrm{d}f_r \mathrm{d}g_u = \int_s^1 \int_s^{t\wedge r} h(r,u)\, \mathrm{d}g_u \mathrm{d}f_r, \quad 0 \le s \le t \le T.$$

We also label the following three-dimensional Fubini type relation for further use:

Proposition 2.4. *Let k be a κ-Hölder function with $\kappa > 1/2$ and $\psi : [0,1]^2 \to \mathbb{R}$ be a measurable and bounded function. Then we have*

$$\int_0^1 \mathrm{d}\zeta \int_0^\zeta \mathrm{d}\eta\, \psi_{\zeta\eta} \int_\eta^\zeta \mathrm{d}k_r = \int_0^1 \mathrm{d}k_r \int_0^r \mathrm{d}\zeta \int_r^1 \mathrm{d}\eta\, \psi_{\zeta\eta}. \tag{10}$$

Proof. Let us approximate ψ and k by two sequences $\{\psi^n, k^n;\ n \ge 1\}$ of smooth functions, such that $\lim_{n\to\infty} \psi^n = \psi$ in $L^\infty([0,1]^2)$ and $\lim_{n\to\infty} k^n = k$ in $\mathcal{C}^\kappa$. For a fixed n, relation

$$\int_0^1 \mathrm{d}\zeta \int_0^\zeta \mathrm{d}\eta\, \psi^n_{\zeta\eta} \int_\eta^\zeta \mathrm{d}k^n_r = \int_0^1 \mathrm{d}k^n_r \int_0^r \mathrm{d}\zeta \int_r^1 \mathrm{d}\eta\, \psi^n_{\zeta\eta} \tag{11}$$

is an immediate consequence of Fubini's theorem. In addition, one is allowed to take limits in the left-hand side of (11) by a simple application of the Dominated Convergence Theorem.

In order to take limits in the right-hand side of (11), set

$$a : [0,1] \to \mathbb{R}, \qquad r \mapsto \int_0^r \mathrm{d}\zeta \int_r^1 \mathrm{d}\eta\, \psi_{\zeta\eta},$$

which means that the right-hand side of (11) can be written as $\int_0^1 \mathrm{d}k_r\, a_r$. It is then readily checked that a is a Lipschitz function, and thus $\int_0^1 \mathrm{d}k_r\, a_r$ can be interpreted as a Young integral. Furthermore, defining the function a^n by $a^n_r = \int_0^r \mathrm{d}\zeta \int_r^1 \mathrm{d}\eta\, \psi^n_{\zeta\eta}$, we have $\lim_n a^n = a$ in the space of Lipschitz functions. Therefore, owing to the fact that $\lim_{n\to\infty} k^n = k$ in $\mathcal{C}^\kappa$, we have

$$\lim_{n\to\infty} \int_0^1 \mathrm{d}k^n_r\, a^n_r = \int_0^1 \mathrm{d}k_r\, a_r.$$

In other words, we are allowed to take limits in the right-hand side of (11).

In conclusion, one can pass to the limit in relation (11), which yields our claim (10). □

Let us describe now our cutoff procedure for the coefficient σ. Recall that we wish to produce a smooth function $\sigma_M : \mathbb{R} \times \mathcal{C}^\gamma \to \mathbb{R}$ such that $\sigma_M(\cdot, x) \equiv 0$ whenever $\|x\|_\gamma \ge M+1$. This also means that the Hölder norm of x should enter into the picture in a smooth manner. With this purpose, let us consider the Sobolev type norm

$$\|f\|_{\gamma,p} := \left(\int_0^1 \int_0^1 \frac{(f(\zeta) - f(\eta))^{2p}}{|\zeta - \eta|^{2p\gamma+2}}\, \mathrm{d}\zeta \mathrm{d}\eta \right)^{\frac{1}{2p}}, \quad \text{for } p \ge 1.$$

It will be seen below that $\|f\|^{2p}_{\gamma,p}$ can be differentiated with respect to f in a suitable sense. Furthermore, if one assumes that $f_0 = 0$, Garsia's lemma (see e.g. [10, Lemma 1]) lets us infer that, whenever $2p\gamma > 1$, we have $\|f\|_\gamma \le C \|f\|_{\gamma,p}$. Otherwise stated, we have the following:

Remark 2.5. Let $\gamma \in (0, 1)$. Assume that $\varepsilon > 0$ and $p \geq 1$ satisfy $\varepsilon > \frac{1}{2p}$, and $f_0 = 0$. Then,

$$f \in \mathcal{C}^{\gamma+\varepsilon} \Longrightarrow \|f\|_{\gamma,p} < \infty.$$

This property will play an essential role in the sequel. For this reason, without loss of generality, we assume that the driving noise x in Eq. (5) satisfies $x_0 = 0$ (this will be the case for the fractional Brownian motion). In fact, we will consider such a hypothesis also in Section 3 for all functions acting as controls in the equations appearing therein.

This being said, our local coefficient is built in the following manner: let $M > 0$ be an arbitrary strictly positive number. We introduce a smooth cutoff function φ_M satisfying:

Definition 2.6. We consider a function $\varphi_M \in \mathcal{C}_b^\infty((0,\infty))$ such that $\varphi_M(r) = 0$ for all $r > M + 1$, and $\varphi_M(r) = 1$ for $r < M$. For any $x : [0, 1] \to \mathbb{R}$ for which $\|x\|_{\gamma,p} < \infty$, for some $\gamma \in (0, 1)$ and $p \geq 1$, set

$$G_M(x) := \varphi_M(\|x\|_{\gamma,p}^{2p}). \tag{12}$$

Eventually, for such x and any $y \in \mathbb{R}$, we define

$$\sigma_M(x, y) := G_M(x)\sigma(y). \tag{13}$$

Hence, in particular, if we choose M large enough, $\sigma_M(x, y) = 0$ whenever $\|x\|_{\gamma,p}^{2p} \geq M + 1$.

We shall consider now the modified elliptic integral equation:

$$\delta z_{st} = \int_s^t \mathrm{d}u \left(\int_u^1 \sigma_M(x, z_\xi)\mathrm{d}x_\xi \right) - (t - s) \int_0^1 \xi\sigma_M(x, z_\xi)\mathrm{d}x_\xi, \quad 0 \leq s \leq t \leq 1. \tag{14}$$

That is, we will solve Eq. (5) for any control $x \in \mathcal{C}^\gamma$ such that $\|x\|_{\gamma,p}^{2p} < M$. Notice in particular that the solution z to (14) depends on all the parameters involved in the cutoff procedure, though we have avoided most of the explicit references to this fact for the sake of the notation.

2.3. The fixed point argument

Following the preliminary considerations of Sections 2.1 and 2.2, we now consider a driving signal x in a Hölder space $\mathcal{C}^\gamma$ satisfying $x_0 = 0$, and we will seek for a unique solution to Eq. (14) in $\mathcal{C}^\kappa$ with $1 - \gamma < \kappa < 1$.

As will be illustrated in the proof of Theorem 2.8, we will need some regularity properties of σ when considered as a map defined on $\mathcal{C}^\kappa$ with values in itself. More precisely, we will make use of the following result:

Lemma 2.7. *Suppose that $\sigma : \mathbb{R} \to \mathbb{R}$ is a bounded function that belongs to $\mathcal{C}^2(\mathbb{R})$ and has bounded derivatives. Then, for any $\kappa \in (0, 1)$, $\sigma : \mathcal{C}^\kappa \to \mathcal{C}^\kappa$ satisfies the following properties: for all $y, z \in \mathcal{C}^\kappa$,*

$$\|\sigma(y)\|_\kappa \leq \|\sigma'\|_\infty \|y\|_\kappa + \|\sigma\|_\infty,$$
$$\|\sigma(y) - \sigma(z)\|_\kappa \leq C\|y - z\|_\kappa \left\{\|\sigma'\|_\infty + \|\sigma''\|_\infty(\|y\|_\kappa + \|y - z\|_\kappa)\right\}.$$

Proof. The first part in the statement is an immediate consequence of the fact that σ and σ' are bounded functions.

For the second part, let us fix $s, t \in [0,1]$ and $y, z \in \mathcal{C}^\kappa$, so we need to analyze the increment

$$\delta(\sigma(y)-\sigma(z))_{st} = \sigma(y_t)-\sigma(z_t)-\sigma(y_s)+\sigma(z_s).$$

To this end, let us consider the following path: for any $\lambda, \mu \in [0,1]$, set

$$a(\lambda,\mu) = y_s + \lambda(z_s - y_s) + \mu(y_t - y_s) + \lambda\mu(y_s - y_t - z_s + z_t).$$

Notice that, in particular, $a(0,0) = y_s$, $a(0,1) = y_t$, $a(1,0) = z_s$ and $a(1,1) = z_t$. Then, we can write

$$\begin{aligned}
&\delta(\sigma(z)-\sigma(y))_{st} \\
&\quad = \int_0^1 \mathrm{d}\lambda \int_0^1 \mathrm{d}\mu\, \partial_\lambda \partial_\mu \sigma(a(\lambda,\mu)) \\
&\quad = \int_0^1 \mathrm{d}\lambda \int_0^1 \mathrm{d}\mu \left[\sigma'(a(\lambda,\mu))\partial_\lambda\partial_\mu a(\lambda,\mu) + \sigma''(a(\lambda,\mu))\partial_\lambda a(\lambda,\mu)\partial_\mu a(\lambda,\mu)\right]. \qquad (15)
\end{aligned}$$

On the other hand, we have the following estimates:

$$\begin{aligned}
&|\partial_\lambda\partial_\mu a(\lambda,\mu)| \le \|y-z\|_\kappa |t-s|^\kappa, \\
&|\partial_\lambda a(\lambda,\mu)\partial_\mu a(\lambda,\mu)| \le C\|y-z\|_\kappa(\|y\|_\kappa + \|y-z\|_\kappa)|t-s|^\kappa.
\end{aligned}$$

Using these bounds and expression (15), we end up with

$$|\delta(\sigma(y)-\sigma(z))_{st}| \le C\|y-z\|_\kappa \left\{\|\sigma'\|_\infty + \|\sigma''\|_\infty(\|y\|_\kappa + \|y-z\|_\kappa)\right\}|t-s|^\kappa.$$

Therefore, we conclude the proof. □

We are now in a position to state the following existence and uniqueness result for Eq. (14):

Theorem 2.8. *Let $\gamma, \kappa \in (0,1)$ be such that $\gamma+\kappa > 1$. Assume that $\varepsilon > 0$ and $p \ge 1$ satisfy $\varepsilon > \frac{1}{2p}$ and let $x \in \mathcal{C}^{\gamma+\varepsilon}$ with $x_0 = 0$. Suppose that $\sigma : \mathbb{R} \to \mathbb{R}$ is bounded, belongs to $\mathcal{C}^2(\mathbb{R})$ and has bounded derivatives. Suppose also that the derivatives of σ satisfy the following condition:*

$$\|\sigma^{(j)}\|_\infty \le \frac{c_1}{M+1}, \qquad j = 0,1,2, \tag{16}$$

for a small enough constant $c_1 < 1$. Then, there exists a unique solution of Eq. (14) in $\mathcal{C}^\kappa$. Moreover, it holds that

$$\|z\|_\kappa \le C, \tag{17}$$

where C is a positive constant depending on M and σ.

Remark 2.9. It would certainly have been possible to handle Eq. (14) by means of fractional calculus techniques, as was done in for instance [20]. It seemed however easier to invoke Hölder spaces and Young integration tools (look e.g. at the simple form of our basic estimates (6) and (7)), and this is why we have chosen to work under this framework.

Proof of Theorem 2.8. As mentioned above, we will apply a fixed point argument. Let us thus consider the following map on $\mathcal{C}^\kappa$: for any $z \in \mathcal{C}^\kappa$, $\Gamma(z)$ is the element of $\mathcal{C}([0,1])$ given by

$$\Gamma(z)_t = \int_0^t \mathrm{d}u \left(\int_u^1 \sigma_M(x, z_\xi)\mathrm{d}x_\xi\right) - t\int_0^1 \xi\,\sigma_M(x, z_\xi)\mathrm{d}x_\xi, \qquad t \in [0,1].$$

Owing to Lemma 2.7 and the definition of σ_M, one easily proves that, for all $z \in \mathcal{C}^\kappa$, $\Gamma(z)$ is well-defined and belongs to $\mathcal{C}^\kappa$. We aim to prove that $\Gamma : \mathcal{C}^\kappa \to \mathcal{C}^\kappa$ has a unique fixed point. For this, we will find an invariant ball in $\mathcal{C}^\kappa$ under Γ and check that Γ, restricted to that ball, defines a contraction.

To begin with, let us fix a real number $K > 1$ and consider the following closed ball in the Hölder space $\mathcal{C}^\kappa$:

$$\mathcal{B}_K := \{z \in \mathcal{C}^\kappa, \|z\|_\kappa \le K\}.$$

Next, for $z \in \mathcal{B}_K$, we are going to analyze the norm $\|\Gamma(z)\|_\kappa$. Indeed, for any $s, t \in [0, 1]$, $s < t$, we have that

$$\begin{aligned} |\delta(\Gamma(z))_{st}| &\le \int_s^t \mathrm{d}u \left| \int_u^1 \sigma_M(x, z_\xi)\mathrm{d}x_\xi \right| + |t-s| \left| \int_0^1 \xi\, \sigma_M(x, z_\xi)\mathrm{d}x_\xi \right| \\ &\le C_1 G_M(x)\|x\|_\gamma \|\sigma(z)\|_\kappa |t-s| \\ &\le C_1 G_M(x)\|x\|_\gamma (\|\sigma'\|_\infty \|z\|_\kappa + \|\sigma\|_\infty)|t-s|, \end{aligned}$$

where in the last inequality we have applied Lemma 2.7 and C_1 denotes a positive constant. Furthermore, the above estimate lets us also infer that

$$\|\Gamma(z)\|_\infty \le C_1 G_M(x)\|x\|_\gamma (\|\sigma'\|_\infty \|z\|_\kappa + \|\sigma\|_\infty).$$

Hence,

$$\|\Gamma(z)\|_\kappa \le C_1 G_M(x)\|x\|_\gamma (\|\sigma'\|_\infty \|z\|_\kappa + \|\sigma\|_\infty). \tag{18}$$

Since $z \in \mathcal{B}_K$ and $G_M(x)\|x\|_\gamma < M$ (fixing M large enough if necessary), we get

$$\|\Gamma(z)\|_\kappa \le C_1 M(K\|\sigma'\|_\infty + \|\sigma\|_\infty) \le C_1\, c_1(K+1),$$

thanks to (16). Moreover, recall that we have chosen a constant $K > 1$. Therefore, by the hypothesis on σ, if we take for instance $c_1 < (2C_1)^{-1}$, we obtain $\|\Gamma(z)\|_\kappa \le (K+1)/2$, and thus

$$\|\Gamma(z)\|_\kappa \le K \quad \text{whenever } \|z\|_\kappa \le K.$$

This implies that $\mathcal{B}_K$ is invariant under Γ.

Let us now prove that $\Gamma_{|\mathcal{B}_K} : \mathcal{B}_K \to \mathcal{B}_K$ is a contraction. For this, it suffices to show that $\Gamma_{|\mathcal{B}_K}$ is Lipschitz with a Lipschitz constant smaller than 1. Namely, we shall prove the existence of a constant $L < 1$ such that, for all $y, z \in \mathcal{B}_K$,

$$\|\Gamma(y) - \Gamma(z)\|_\kappa \le L\|y - z\|_\kappa.$$

Let $s, t \in [0, 1]$, $s < t$, and $y, z \in \mathcal{B}_K$. Then,

$$\begin{aligned} \delta(\Gamma(y) - \Gamma(z))_{st} = \int_s^t \mathrm{d}u \left(\int_u^1 [\sigma_M(x, y_\xi) - \sigma_M(x, z_\xi)]\mathrm{d}x_\xi \right) \\ - (t-s) \int_0^1 \xi\, [\sigma_M(x, y_\xi) - \sigma_M(x, z_\xi)]\mathrm{d}x_\xi. \end{aligned} \tag{19}$$

By Lemma 2.7 and the properties of the Young integral, it turns out that the absolute value of both terms on the right-hand side of (19) can be bounded, up to some positive constant, by

$$G_M(x)\|x\|_\gamma \|y - z\|_\kappa \left\{ \|\sigma'\|_\infty + \|\sigma''\|_\infty (\|y\|_\kappa + \|y - z\|_\kappa) \right\} |t - s|.$$

We have a similar bound for $\|\Gamma(y) - \Gamma(z)\|_\infty$ as well. Thus, because $y, z \in \mathcal{B}_K$, we eventually end up with

$$\|\Gamma(y) - \Gamma(z)\|_\kappa \leq C_2 M K (\|\sigma'\|_\infty + \|\sigma''\|_\infty)\|y - z\|_\kappa.$$

It suffices now to consider that $\|\sigma'\|_\infty$ and $\|\sigma''\|_\infty$ are sufficiently small (that is we can take $c_1 < (C_2K + 1)^{-1} \wedge (2C_1)^{-1}$, where C_1 is the constant of the first part of the proof) so that the right-hand side above is bounded by $L\|y - z\|_\kappa$, with $L < 1$. Therefore, Γ has a unique fixed point in $\mathcal{B}_K$, which means that Eq. (14) has a unique solution in $\mathcal{C}^\kappa$.

Eventually, using (18) one proves that

$$\|z\|_\kappa \leq C_1 M (\|\sigma'\|_\infty \|z\|_\kappa + \|\sigma\|_\infty).$$

In addition, invoking (16) and the fact that $c_1 \leq (2C_1)^{-1}$, we obtain

$$\|z\|_\kappa \leq \frac{C_1 M \|\sigma\|_\infty}{1 - C_1 M \|\sigma'\|_\infty} \leq C(M),$$

which concludes the proof. □

Remark 2.10. Having been able to solve Eq. (14) in $\mathcal{C}^\kappa$ for any $\kappa \in (1-\gamma, 1)$, one can now apply the Fubini type Proposition 2.3 in order to establish that z is the unique solution to the integral equation

$$z_t = \int_0^1 K(t,\xi)\sigma_M(x, z_\xi)\,\mathrm{d}x_\xi, \quad t \in [0,1],$$

where we recall that the kernel $K(t,\xi)$ is defined by $K(t,\xi) = t \wedge \xi - t\xi$.

3. Differentiability of the solution with respect to the control

This section is devoted to showing that the solution of Eq. (14) is differentiable, in the sense of Fréchet, when considered as a function of the control x driving the equation. For this, we need two auxiliary results.

Let us recall that the diffusion coefficient under consideration (see Eq. (14)) is introduced in our Definition 2.6. Moreover, as mentioned in the previous section, in all results of the present one we assume that any function acting as a control of any equation considered here vanishes at the zero. We make this hypothesis without further mentioning.

The following differentiation rule holds true:

Proposition 3.1. *Let $\gamma, \kappa \in (0,1)$ be such that $\gamma + \kappa > 1$, $p \geq 1$ and $\varepsilon > \frac{1}{2p}$. Assume that $\sigma \in \mathcal{C}^4(\mathbb{R})$ is bounded together with all its derivatives and let σ_M be given by Definition 2.6. Consider x an element of $\mathcal{C}^{\gamma+\varepsilon}$ and define the following map:*

$$F : \mathcal{C}^{\gamma+\varepsilon} \times \mathcal{C}^\kappa \longrightarrow \mathcal{C}^\kappa,$$

where, for all $h \in \mathcal{C}^{\gamma+\varepsilon}$ and $z \in \mathcal{C}^\kappa$,

$$F(h,z)_t := z_t - \int_0^t \mathrm{d}u \left(\int_u^1 \sigma_M(x+h, z_\xi)\mathrm{d}(x+h)_\xi \right) + t \int_0^1 \xi\, \sigma_M(x+h, z_\xi)\mathrm{d}(x+h)_\xi.$$

Then, the map F is Fréchet differentiable with respect to the first and second variable and the Fréchet derivatives are given by, respectively: for all $t \in [0,1]$, $k \in \mathcal{C}^{\gamma+\varepsilon}$ and $g \in \mathcal{C}^{\kappa}$,

$$\begin{aligned}(D_1 F(h,z)\cdot k)_t &= -\int_0^t \mathrm{d}u \left[\int_u^1 \sigma_M(x+h, z_\xi)\,\mathrm{d}k_\xi + \int_u^1 (DG_M(x+h)\cdot k)\,\sigma(z_\xi)\,\mathrm{d}(x+h)_\xi\right] \\ &\quad + t\left[\int_0^1 \xi\,\sigma_M(x+h,z_\xi)\,\mathrm{d}k_\xi + \int_0^1 \xi\,(DG_M(x+h)\cdot k)\,\sigma(z_\xi)\,\mathrm{d}(x+h)_\xi\right], \end{aligned} \tag{20}$$

and

$$\begin{aligned}(D_2 F(h,z)\cdot g)_t &= g_t - \int_0^t \mathrm{d}u\left[\int_u^1 G_M(x+h)\,\sigma'(z_\xi)\,g_\xi\,\mathrm{d}(x+h)_\xi\right] \\ &\quad + t\int_0^1 \xi\,G_M(x+h)\,\sigma'(z_\xi)\,g_\xi\,\mathrm{d}(x+h)_\xi. \end{aligned} \tag{21}$$

Remark 3.2. In the above formulae (20) and (21), the Fréchet derivative of $G_M(\cdot)$ is well-defined and can be computed explicitly. Indeed, G_M is defined on the Hölder space $\mathcal{C}^{\gamma+\varepsilon}$, takes values in $\mathbb{R}$ and is defined by $G_M(x) = \varphi_M(\|x\|_{\gamma,p}^{2p})$, with some $p \ge 1$. Moreover, φ_M is a smooth function which fulfills the hypotheses of Definition 2.6. Hence, the Fréchet derivative $DG_M(x)$ at any point $x \in \mathcal{C}^{\gamma+\varepsilon}$ defines a linear map on $\mathcal{C}^{\gamma+\varepsilon}$ with values in $\mathbb{R}$, and it is straightforward to check that it is given by

$$DG_M(x)\cdot k = 2p\,\varphi_M'(\|x\|_{\gamma,p}^{2p})\int_0^1\int_0^1 \frac{(x_\zeta - x_\eta)^{2p-1}(k_\zeta - k_\eta)}{|\zeta-\eta|^{2\gamma p+2}}\mathrm{d}\zeta\,\mathrm{d}\eta, \quad k \in \mathcal{C}^{\gamma+\varepsilon}.$$

Moreover, we have that

$$\|DG_M(x)\| := \|DG_M(x)\|_{\mathcal{L}(\mathcal{C}^{\gamma+\varepsilon};\,\mathbb{R})} \le C\,\|x\|_{\gamma+\varepsilon}^{2p-1}, \tag{22}$$

where the norm on the left-hand side denotes the corresponding operator norm.

Remark 3.3. As in Remark 2.10, one can apply Fubini's theorem for Young integrals in order to obtain some more compact expressions for the derivatives of F. Indeed, it is readily checked that

$$\begin{aligned}(D_1 F(h,z)\cdot k)_t &= -G_M(x+h)\int_0^1 K(t,\xi)\sigma(z_\xi)\,\mathrm{d}k_\xi \\ &\quad - (DG_M(x+h)\cdot k)\int_0^1 K(t,\xi)\sigma(z_\xi)\,\mathrm{d}(x+h)_\xi \end{aligned} \tag{23}$$

and

$$(D_2 F(h,z)\cdot g)_t = g_t - G_M(x+h)\int_0^1 K(t,\xi)\sigma'(z_\xi)g_\xi\,\mathrm{d}(x+h)_\xi,$$

where K is the kernel defined by (3).

Remark 3.4. As will be explained later on in the paper, we will apply the results of this section to the case where x is a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, defined on a complete probability space $(\Omega, \mathcal{F}, P)$. In particular, the paths of x are almost surely γ-Hölder continuous for all $\gamma < H$, with γ-Hölder norm in $L^p(\Omega)$ for any $p \geq 1$. Thus, if we fix $\gamma < H$, we will be able to find $\varepsilon > 1/(2p)$ satisfying $\gamma + \varepsilon < H$. This opens the possibility of applying the results of the current section to this particular case.

Proof of Proposition 3.1. Though the following considerations might be mostly standard (see [14,21] for similar calculations), we include most of the details here for the sake of clarity. We will develop the proof in several steps.

Step 1. First of all, let us prove that F is continuous. For this, let $h, \tilde{h} \in \mathcal{C}^{\gamma+\varepsilon}$ and $z, \tilde{z} \in \mathcal{C}^{\kappa}$, so we need to study the increment $\delta(F(h,z) - F(\tilde{h},\tilde{z}))_{st}$, for $0 \leq s < t \leq 1$. Indeed, we have that

$$|\delta(F(h,z) - F(\tilde{h},\tilde{z}))_{st}| \leq A_1 + A_2 + A_3, \tag{24}$$

where

$$\begin{aligned} A_1 &= |\delta(z-\tilde{z})_{st}|, \\ A_2 &= \int_s^t \mathrm{d}u \left| \int_u^1 \sigma_M(x+h, z_\xi)\,\mathrm{d}(x+h)_\xi - \int_u^1 \sigma_M(x+\tilde{h}, \tilde{z}_\xi)\,\mathrm{d}(x+\tilde{h})_\xi \right|, \\ A_3 &= (t-s) \left| \int_0^1 \xi\, \sigma_M(x+h, z_\xi)\,\mathrm{d}(x+h)_\xi - \int_0^1 \xi\, \sigma_M(x+\tilde{h}, \tilde{z}_\xi)\,\mathrm{d}(x+\tilde{h})_\xi \right|. \end{aligned}$$

It is clear that

$$A_1 \leq \|z-\tilde{z}\|_\kappa (t-s)^\kappa.$$

On the other hand, the term A_2 can be decomposed as $A_2 \leq A_{11} + A_{12}$, with

$$\begin{aligned} A_{11} &= \int_s^t \mathrm{d}u \left| \int_u^1 \Big(\sigma_M(x+h, z_\xi) - \sigma_M(x+\tilde{h}, \tilde{z}_\xi)\Big)\,\mathrm{d}(x+h)_\xi \right| \\ A_{12} &= \int_s^t \mathrm{d}u \left| \int_u^1 \sigma_M(x+\tilde{h}, \tilde{z}_\xi)\,\mathrm{d}(h-\tilde{h})_\xi \right|. \end{aligned} \tag{25}$$

In addition, our bound (6) on Young type integrals easily yields

$$A_{12} \leq G_M(x+\tilde{h})(\|\sigma'\|_\infty \|\tilde{z}\|_\kappa + \|\sigma\|_\infty)\|h-\tilde{h}\|_{\gamma+\varepsilon}(t-s). \tag{26}$$

We still need to bound the term A_{11} by a sum $B_1 + B_2$, where the latter terms are defined by

$$\begin{aligned} B_1 &= \int_s^t \mathrm{d}u \left| \int_u^1 \big(\sigma_M(x+h, z_\xi) - \sigma_M(x+h, \tilde{z}_\xi)\big)\,\mathrm{d}(x+h)_\xi \right|, \\ B_2 &= \int_s^t \mathrm{d}u \left| \int_u^1 \Big(\sigma_M(x+h, \tilde{z}_\xi) - \sigma_M(x+\tilde{h}, \tilde{z}_\xi)\Big)\,\mathrm{d}(x+h)_\xi \right|. \end{aligned}$$

Now, invoking Lemma 2.7, we get

$$\begin{aligned} B_1 &\leq C G_M(x+h)\|x+h\|_{\gamma+\varepsilon}\|\sigma(z)-\sigma(\tilde{z})\|_\kappa (t-s) \\ &\leq C G_M(x+h)\|x+h\|_{\gamma+\varepsilon}(\|\sigma'\|_\infty + \|\sigma''\|_\infty(\|z\|_\kappa \\ &\quad + \|z-\tilde{z}\|_\kappa))\|z-\tilde{z}\|_\kappa (t-s). \end{aligned} \tag{27}$$

Concerning the term B_2, notice that we clearly have

$$B_2 \le \left| G_M(x+h) - G_M(x+\tilde{h}) \right| \int_s^t \left| \int_u^1 \sigma(\tilde{z}_\xi)\, \mathrm{d}(x+h)_\xi \right| \mathrm{d}u$$
$$\le \left| G_M(x+h) - G_M(x+\tilde{h}) \right| \|x+h\|_{\gamma+\varepsilon} (\|\sigma'\|_\infty \|\tilde{z}\|_\kappa + \|\sigma\|_\infty)(t-s).$$

Let us finally analyze the difference $|G_M(x+h) - G_M(x+\tilde{h})|$ on the right-hand side above: by definition of G_M and the properties of φ_M summarized in Definition 2.6, we can argue as follows:

$$\begin{aligned} |G_M(x+h) - G_M(x+\tilde{h})| &= |\varphi_M(\|x+h\|_{\gamma,p}^{2p}) - \varphi_M(\|x+\tilde{h}\|_{\gamma,p}^{2p})| \\ &\le C_{M,p} |\|x+h\|_{\gamma,p} - \|x+\tilde{h}\|_{\gamma,p}| \\ &\le C_{M,p} \|h-\tilde{h}\|_{\gamma,p} \end{aligned} \tag{28}$$

and this last term may be bounded, up to some constant, by $\|h-\tilde{h}\|_{\gamma+\varepsilon}$, because we have chosen ε to be small but verifying $\varepsilon > \frac{1}{2p}$ (see Remark 2.5). This implies that

$$B_2 \le C\|h-\tilde{h}\|_{\gamma+\varepsilon} \|x+h\|_{\gamma+\varepsilon} (\|\sigma'\|_\infty \|\tilde{z}\|_\kappa + \|\sigma\|_\infty)(t-s). \tag{29}$$

Plugging the bounds (26), (27) and (29) in (25), we obtain that

$$\begin{aligned} A_2 \le\ & C_1(\|\sigma'\|_\infty \|\tilde{z}\|_\kappa + \|\sigma\|_\infty)(G_M(x+\tilde{h}) + \|x+h\|_{\gamma+\varepsilon})\|h-\tilde{h}\|_{\gamma+\varepsilon}(t-s) \\ & + C_2 G_M(x+h)\|x+h\|_{\gamma+\varepsilon}(\|\sigma'\|_\infty \\ & + \|\sigma''\|_\infty(\|z\|_\kappa + \|z-\tilde{z}\|_\kappa))\|z-\tilde{z}\|_\kappa(t-s), \end{aligned} \tag{30}$$

where C_1, C_2 denote some positive constants.

The analysis for the term A_3 is very similar to that of A_2 and, indeed, for the former we end up with a bound similar to that in (30). Therefore, going back to expression (24), we have proved that

$$\|F(h,z) - F(\tilde{h},\tilde{z})\|_\kappa \le C(M,\sigma,x,z,\tilde{z},h,\tilde{h})(\|z-\tilde{z}\|_\kappa + \|h-\tilde{h}\|_{\gamma+\varepsilon}),$$

which implies that F is continuous.

Step 2. Let us prove now that the Fréchet derivative of F with respect to h is given by (20). First of all, let us check that $D_1F(h,z) : \mathcal{C}^{\gamma+\varepsilon} \to \mathcal{C}^\kappa$, as defined by expression (20), is a continuous map. Indeed, owing to inequality (6) and Remark 3.2, one can easily check from expressions (20) and (22) that

$$\|D_1F(h,z)\cdot k\|_\kappa \le C(\|\sigma'\|_\infty \|z\|_\kappa + \|\sigma\|_\infty)(G_M(x+h) + \|x+h\|_{\gamma+\varepsilon}^2)\|k\|_{\gamma+\varepsilon},$$

which implies that $D_1F(h,z)$ is continuous.

In order to prove that (20) also represents the Fréchet derivative of F with respect to the first variable, we fix $h \in \mathcal{C}^{\gamma+\varepsilon}$ and $z \in \mathcal{C}^\kappa$, so we need to prove that

$$\lim_{\|k\|_{\gamma+\varepsilon}\to 0} \frac{\|F(h+k,z) - F(h,z) - D_1F(h,z)\cdot k\|_\kappa}{\|k\|_{\gamma+\varepsilon}} = 0. \tag{31}$$

For this, let $0 \le s < t \le 1$, $h, k \in \mathcal{C}^{\gamma+\varepsilon}$ and $z \in \mathcal{C}^\kappa$, and we proceed to analyze the increment

$$|\delta(F(h+k,z) - F(h,z) - D_1F(h,z)\cdot k)_{st}|. \tag{32}$$

According to (20), the above increment can be split into a sum of four terms, which we denote by E_i, $i = 1, \dots, 4$, and are defined as follows:

$$E_1 = \int_s^t \mathrm{d}u \int_u^1 [G_M(x+h) - G_M(x+h+k) + DG_M(x+h)\cdot k]\,\sigma(z_\xi)\,\mathrm{d}(x+h)_\xi,$$

$$E_2 = \int_s^t \mathrm{d}u \int_u^1 [G_M(x+h) - G_M(x+h+k)]\,\sigma(z_\xi)\,\mathrm{d}k_\xi,$$

$$E_3 = (t-s)\int_0^1 [G_M(x+h+k) - G_M(x+h) - DG_M(x+h)\cdot k]\,\xi\,\sigma(z_\xi)\,\mathrm{d}(x+h)_\xi,$$

$$E_4 = (t-s)\int_0^1 [G_M(x+h+k) - G_M(x+h)]\,\xi\,\sigma(z_\xi)\,\mathrm{d}k_\xi.$$

We will only deal with the study of the terms E_1 and E_2, since the remaining ones involve analogous arguments. First, note that we have the following estimates:

$$\begin{aligned} |E_1| &\le |G_M(x+h+k) - G_M(x+h) - DG_M(x+h)\cdot k| \left| \int_s^t \mathrm{d}u \int_u^1 \sigma(z_\xi)\,\mathrm{d}(x+h)_\xi \right| \\ &\le C\,|G_M(x+h+k) - G_M(x+h) - DG_M(x+h)\cdot k| \\ &\quad \times \|x+h\|_{\gamma+\varepsilon}(\|\sigma'\|_\infty \|z\|_\kappa + \|\sigma\|_\infty)(t-s). \end{aligned} \tag{33}$$

By Remark 3.2, the map $G_M : \mathcal{C}^{\gamma+\varepsilon} \to \mathbb{R}$ is Fréchet differentiable and its derivative can be computed explicitly. Hence,

$$\lim_{\|k\|_{\gamma+\varepsilon}\to 0} \frac{|G_M(x+h+k) - G_M(x+h) - DG_M(x+h)\cdot k|}{\|k\|_{\gamma+\varepsilon}} = 0,$$

and this implies that the contribution of $|E_1|$ is of order $o(\|k\|_{\gamma+\varepsilon})$.

On the other hand, using the same arguments as in (28), we have

$$\begin{aligned} |E_2| &\le C\,|G_M(x+h) - G_M(x+h+k)|\,\|k\|_{\gamma+\varepsilon}(\|\sigma'\|_\infty \|z\|_\kappa + \|\sigma\|_\infty)(t-s) \\ &\le C\|k\|_{\gamma+\varepsilon}^2(\|\sigma'\|_\infty \|z\|_\kappa + \|\sigma\|_\infty)(t-s), \end{aligned} \tag{34}$$

which is obviously also of order $o(\|k\|_{\gamma+\varepsilon})$.

For the terms $|E_3|$ and $|E_4|$ we obtain, respectively, the same bounds as in (33) and (34). Eventually, plugging all these estimates in (32), we end up with the limit (31).

Step 3. In this part, we prove that the Fréchet derivative of F with respect to the second variable is given by (21). The continuity of $D_2F(h,z)$ in (21) can be proved as we have done in Step 2 for $D_1F(h,z)$. Hence, we will check that, for all $h \in \mathcal{C}^{\gamma+\varepsilon}$ and $z \in \mathcal{C}^\kappa$, it holds that

$$\lim_{\|g\|_\kappa\to 0} \frac{\|F(h,z+g) - F(h,z) - D_2F(h,z)\cdot g\|_\kappa}{\|g\|_\kappa} = 0. \tag{35}$$

Throughout this step we will use the fact that σ, considered as a map defined on and taking values in $\mathcal{C}^\kappa$, is Fréchet differentiable and its derivative is given by (see Lemma 3.5 below)

$$(D\sigma(z)\cdot g)_t = \sigma'(z_t)g_t, \quad z, g \in \mathcal{C}^\kappa.$$

This means that, for all $z \in \mathcal{C}^\kappa$,

$$\lim_{\|g\|_\kappa\to 0} \frac{\|\sigma(z+g) - \sigma(z) - D\sigma(z)\cdot g\|_\kappa}{\|g\|_\kappa} = 0.$$

In order to prove (35), let us fix $0 \le s < t \le 1$ and observe that

$$\left|\delta\left(F(h,z+g) - F(h,z) - D_2F(h,z)\cdot g\right)_{st}\right| \le F_1 + F_2,$$

where

$$F_1 := \left|\int_s^t \int_u^1 G_M(x+h)[\sigma(z_\xi+g_\xi) - \sigma(z_\xi) - \sigma'(z_\xi)g_\xi]\mathrm{d}(x+h)_\xi \,\mathrm{d}u\right|$$
$$\le\ CG_M(x+h)\|x+h\|_{\gamma+\varepsilon}\|\sigma(z+g) - \sigma(z) - \sigma'(z)g\|_\kappa (t-s)$$

and

$$F_2 := (t-s)\left|\int_0^1 G_M(x+h)\xi\,[\sigma(z_\xi+g_\xi) - \sigma(z_\xi) - \sigma'(z_\xi)g_\xi]\mathrm{d}(x+h)_\xi\right|,$$

for which the same inequality as for F_1 is available. Therefore, we obtain that

$$\|F(h,z+g) - F(h,z) - D_2F(h,z)\cdot g\|_\kappa$$
$$\le CG_M(x+h)\|x+h\|_{\gamma+\varepsilon}\|\sigma(z+g) - \sigma(z) - \sigma'(z)g\|_\kappa,$$

and the latter κ-norm, as we have mentioned above, is of order $o(\|g\|_\kappa)$ whenever $\|g\|_\kappa$ tends to zero. This implies that (35) holds, and ends the proof. □

Let us quote now the relation needed in the previous proof in order to compute the Fréchet derivative of the process $\sigma(z)$:

Lemma 3.5. *Let $\sigma \in \mathcal{C}^4(\mathbb{R})$ be a bounded function with bounded derivatives. Then σ, understood as a map $\sigma : \mathcal{C}^\kappa \to \mathcal{C}^\kappa$, is Fréchet differentiable and its derivative is given by*

$$(D\sigma(z)\cdot g)_t = \sigma'(z_t)g_t, \quad z, g \in \mathcal{C}^\kappa.$$

Proof. We refer the reader to [14, Proposition 3.5] for the proof of this fact, and in particular for the identification of $(D\sigma(z)\cdot g)_t$ with the quantity $\sigma'(z_t)\cdot g_t$. □

As in [21], a crucial step for differentiating z with respect to the driving noise x is solving the following class of linear elliptic PDEs. The proof of this result is very similar to that of Theorem 2.8 and, therefore, it will be omitted.

Proposition 3.6. *Let $\gamma, \kappa \in (0,1)$ be such that $\gamma + \kappa > 1$. Assume that we are given $x \in \mathcal{C}^{\gamma+\varepsilon}$ with $x_0 = 0$ and $w, R \in \mathcal{C}^\kappa$ such that the κ-norm of R verifies*

$$\|R\|_\kappa < \frac{c_2}{M+1}, \tag{36}$$

for some small enough constant $c_2 < 1$. Then, there exists a unique solution $\{y_t, t \in [0,1]\}$ in $\mathcal{C}^\kappa$ of the following linear integral equation:

$$y_t = w_t - G_M(x)\int_0^1 K(t,\xi)R_\xi\, y_\xi\,\mathrm{d}x_\xi, \quad t \in [0,1].$$

Moreover, there exists a positive constant $c(M)$ that only depends on M such that

$$\|y\|_\kappa \le c(M)\|w\|_\kappa. \tag{37}$$

At this point, we can proceed to state and prove the main result of the section.

Theorem 3.7. *Let $\gamma, \kappa \in (0, 1)$ be such that $\gamma + \kappa > 1$. Let $\varepsilon > 0$ and a sufficiently large $p \geq 1$ be such that $\varepsilon > \frac{1}{2p}$. Assume that $\sigma \in \mathcal{C}^4(\mathbb{R})$ is a bounded function with bounded derivatives such that*

$$\|\sigma^{(j)}\|_\infty \leq \frac{c_3}{M+1}, \quad j = 0, 1, 2, \tag{38}$$

for some constant $c_3 < \frac{c_2}{1+C(M)} \wedge c_1$, where c_1 and $C(M)$ are the constants in the statement of Theorem 2.8, and c_2 the one of Proposition 3.6.

Let $z(x) = \{z_t, t \in [0, 1]\}$ be the solution of Eq. (14) with control $x \in \mathcal{C}^{\gamma+\varepsilon}$ and diffusion coefficient σ_M (see (12) and (13)). Then, the map $x \mapsto z(x)$, defined in $\mathcal{C}^{\gamma+\varepsilon}$ with values in $\mathcal{C}^\kappa$, is Fréchet differentiable. Moreover, for all $h \in \mathcal{C}^{\gamma+\varepsilon}$, the Fréchet derivative of $z(x)$ is given by

$$(Dz(x) \cdot h)_t = \int_0^1 \Phi_s(t) \mathrm{d}h_s, \tag{39}$$

where the kernels $\Phi_s(t)$ satisfy the following equation:

$$\Phi_s(t) = \Psi_s(t) + G_M(x) \int_0^1 K(t, \xi)\sigma'(z_\xi)\Phi_s(\xi)\, \mathrm{d}x_\xi, \tag{40}$$

with

$$\Psi_s(t) = G_M(x)\,\sigma(z_s)\,K(t, s) + 2\varphi'_M(\|x\|^{2p}_{\gamma,p})\,\mu_s\, z_t, \tag{41}$$

and

$$\mu_s := \int_0^s \int_s^1 \rho_{\zeta\eta}\, \mathrm{d}\zeta\, \mathrm{d}\eta, \quad \textit{where } \rho_{\zeta\eta} = 2p \frac{(x_\zeta - x_\eta)^{2p-1}}{|\zeta - \eta|^{2\gamma p+2}}. \tag{42}$$

Proof. We will adapt the arguments used in the proof of Proposition 4 in [21]. That is, we will apply the Implicit Function Theorem to the functional F defined in the statement of Proposition 3.1. For this, notice first that we have proved there that, for any $h \in \mathcal{C}^{\gamma+\varepsilon}$ and $z \in \mathcal{C}^\kappa$, $F(h, z)$ belongs to $\mathcal{C}^\kappa$ and F is Fréchet differentiable with partial derivatives with respect to h and z given by (20) and (21), respectively. Moreover, since z is the solution of (14), we have that $F(0, z) = 0$.

We need to check now that $D_2F(0, z)$ defines a linear homeomorphism from $\mathcal{C}^\kappa$ into itself for which, by the Open Map Theorem, it suffices to prove that it is bijective (we already know that it is continuous). For this, we apply Proposition 3.6 to the case where $R_\xi = \sigma'(z_\xi)$, so

$$(D_2F(0, z) \cdot g)_t = g_t - G_M(x) \int_0^1 K(t, \xi)\sigma'(z_\xi) g_\xi\, \mathrm{d}x_\xi \tag{43}$$

defines a one-to-one mapping. Indeed, observe that condition (38) guarantees that (36) in Proposition 3.6 is satisfied. On the other hand, if we fix $w \in \mathcal{C}^\kappa$, applying again Proposition 3.6 we deduce that there exists $g \in \mathcal{C}^\kappa$ such that $w = D_2F(0, z) \cdot g$, which implies that $D_2F(0, z)$ is onto and therefore a bijection.

Hence, by the Implicit Function Theorem, the map $x \mapsto z(x)$ is continuously Fréchet differentiable and

$$Dz(x) = -D_2F(0, z)^{-1} \circ D_1F(0, z). \tag{44}$$

Moreover, by (43), for any $h \in \mathcal{C}^{\gamma+\varepsilon}$, $Dz(x) \cdot h$ is the unique solution to the differential equation

$$(Dz(x) \cdot h)_t = w_t + G_M(x) \int_0^1 K(t, \xi)\sigma'(z_\xi)(Dz(x) \cdot h)_\xi \, \mathrm{d}x_\xi,$$

with $w_t = -(D_1 F(0, z) \cdot h)_t$.

Let us proceed to prove (39). Consider Eq. (40) and integrate both sides with respect to some $h \in \mathcal{C}^{\gamma+\varepsilon}$:

$$\int_0^1 \Phi_s(t)\mathrm{d}h_s = \int_0^1 \Psi_s(t)\mathrm{d}h_s + G_M(x) \int_0^1 \left[\int_0^1 K(t, \xi)\sigma'(z_\xi)\Phi_s(\xi)\mathrm{d}x_\xi \right] \mathrm{d}h_s. \tag{45}$$

At this point, we can use the same arguments as in the proof of Proposition 4 in [21]: we apply our Fubini type Proposition 2.3 to the last term in the right-hand side of (45), which yields

$$\int_0^1 \Phi_s(t)\mathrm{d}h_s = \int_0^1 \Psi_s(t)\mathrm{d}h_s + G_M(x) \int_0^1 K(t, \xi)\sigma'(z_\xi) \left[\int_0^1 \Phi_s(\xi)\mathrm{d}h_s \right] \mathrm{d}x_\xi. \tag{46}$$

In order to conclude the proof, thanks to the uniqueness part of Proposition 3.6, it is now sufficient to show that $w = -D_1 F(0, z) \cdot h$ can be represented in the form

$$w_t = \int_0^1 \Psi_s(t)\mathrm{d}h_s. \tag{47}$$

For this, let us observe that, by (23), it holds that

$$(D_1 F(0, z) \cdot h)_t = -G_M(x) \int_0^1 K(t, \xi)\sigma(z_\xi)\mathrm{d}h_\xi - (DG_M(x) \cdot h)\, z_t.$$

Hence, owing to Lemma 3.8 below, we obtain the representation (47) with $\Phi_s(t)$ given by (41), which concludes the proof. □

We close this section by giving an expression for $DG_M(x)$, which has already been used in the proof above.

Lemma 3.8. *For all $h \in \mathcal{C}^{\gamma+\varepsilon}$, it holds that*

$$DG_M(x) \cdot h = 2\varphi_M'(\|x\|_{\gamma,p}^{2p}) \int_0^1 \mu_s \, \mathrm{d}h_s, \tag{48}$$

where the function μ is defined at Eq. (42).

Proof. As we have mentioned in Remark 3.2, the map $G_M : \mathcal{C}^{\gamma+\varepsilon} \to \mathbb{R}$ is Fréchet differentiable at any point $x \in \mathcal{C}^{\gamma+\varepsilon}$, and its Fréchet derivative is given by

$$DG_M(x) \cdot k = 2p\, \varphi_M'(\|x\|_{\gamma,p}^{p}) \int_0^1 \int_0^1 \frac{(x_\zeta - x_\eta)^{2p-1}(k_\zeta - k_\eta)}{|\zeta - \eta|^{2\gamma p+2}} \mathrm{d}\zeta\mathrm{d}\eta, \quad k \in \mathcal{C}^{\gamma+\varepsilon}.$$

According to the definition of $\rho_{\zeta\eta}$, this derivative can be written in the form

$$DG_M(x) \cdot k = \varphi_M'(\|x\|_{\gamma,p}^{p}) \int_0^1 \int_0^1 \rho_{\zeta\eta}\, (k_\zeta - k_\eta)\mathrm{d}\zeta\mathrm{d}\eta. \tag{49}$$

Then, invoking relation (10), one can argue as follows:

$$\begin{aligned}
&\int_0^1 \int_0^1 \rho_{\zeta\eta}\,(k_\zeta - k_\eta)\mathrm{d}\zeta\,\mathrm{d}\eta \\
&= \int_0^1 \int_0^1 \rho_{\zeta\eta}\left(\int_\eta^\zeta \mathrm{d}k_r\right)\mathbf{1}_{\{\eta\le\zeta\}}\mathrm{d}\zeta\,\mathrm{d}\eta + \int_0^1 \int_0^1 \rho_{\zeta\eta}\left(\int_\zeta^\eta \mathrm{d}k_r\right)\mathbf{1}_{\{\zeta\le\eta\}}\mathrm{d}\zeta\,\mathrm{d}\eta \\
&= \int_0^1 \left[\int_0^r \mathrm{d}\zeta \int_r^1 \mathrm{d}\eta\,\rho_{\zeta\eta}\right]\mathrm{d}k_r + \int_0^1 \left[\int_0^r \mathrm{d}\eta \int_r^1 \mathrm{d}\zeta\,\rho_{\zeta\eta}\right]\mathrm{d}k_r \\
&= 2\int_0^1 \left[\int_0^r \mathrm{d}\zeta \int_r^1 \mathrm{d}\eta\,\rho_{\zeta\eta}\right]\mathrm{d}k_r.
\end{aligned}$$

Plugging this expression in (49) we obtain (48) and we conclude the proof. □

4. Stochastic elliptic equations driven by a fractional Brownian motion

Let us first describe the probabilistic setting in which we will apply the results obtained in the previous section. For some fixed $H \in (0, 1)$, we consider $(\Omega, \mathcal{F}, P)$, the canonical probability space associated with the fractional Brownian motion with Hurst parameter H. That is, $\Omega = \mathcal{C}_0([0, 1])$ is the Banach space of continuous functions vanishing at 0 equipped with the supremum norm, $\mathcal{F}$ is the Borel sigma-algebra and P is the unique probability measure on Ω such that the canonical process $B = \{B_t,\ t \in [0, 1]\}$ is a fractional Brownian motion with Hurst parameter H. Recall that this means that B is a centered Gaussian process with covariance

$$R_H(t, s) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}).$$

In particular, the paths of B are γ-Hölder continuous for all $\gamma \in (0, H)$. Then, we consider Eq. (14) where the driving trajectory is a path of B. Namely,

$$\delta z_{st} = \int_s^t \mathrm{d}u \left(\int_u^1 \sigma_M(B, z_\xi)\mathrm{d}B_\xi\right) - (t - s)\int_0^1 \xi\sigma_M(B, z_\xi)\mathrm{d}B_\xi, \quad 0 \le s < t \le 1,$$

which can be written in the reduced form

$$z_t = G_M(B)\int_0^1 K(t, \xi)\sigma(z_\xi)\,\mathrm{d}B_\xi, \quad t \in [0, 1]. \tag{50}$$

Assuming that $\sigma \in \mathcal{C}^2(\mathbb{R})$ is bounded, has bounded derivatives and satisfies (16), Theorem 2.8 implies that Eq. (50) has a unique solution $z = \{z_t,\ t \in [0, 1]\}$ such that $z \in \mathcal{C}^\kappa$ for any $\kappa \in (1 - \gamma, 1)$, and almost surely in $\omega \in \Omega$.

4.1. Malliavin differentiability of the solution

This subsection is devoted to presenting the Malliavin calculus setting which we shall work in, so that we will be able to obtain that the solution of (50) belongs to the domain of the Malliavin derivative. Notice that, in spite of the fact that we can solve Eq. (50) driven by a fBm with arbitrary Hurst parameter, our Malliavin calculus section will be restricted to the range $H \in (1/2, 1)$. This is due to the fact that stochastic analysis of fractional Brownian motion becomes onerous for $H < 1/2$, and we have thus imposed this restriction for the sake of conciseness.

Consider then a fixed parameter $H > 1/2$, and let us start by briefly describing the abstract Wiener space introduced for Malliavin calculus purposes (for a more general and complete description, we refer the reader to [21, Section 3]).

Let $\mathcal{E}$ be the set of $\mathbb{R}$-valued step functions on $[0, 1]$ and $\mathcal{H}$ the completion of $\mathcal{E}$ with respect to the semi-inner product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} := R_H(s, t), \quad s, t \in [0, 1].$$

The space $\mathcal{H}$ may contain distributions. Then, one constructs an isometry $K_H^* : \mathcal{H} \to L^2([0, 1])$ such that $K_H^*(\mathbf{1}_{[0,t]}) = \mathbf{1}_{[0,t]} K_H(t, \cdot)$, where the kernel K_H is given by

$$K_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u - s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \, \mathrm{d}u$$

and verifies that $R_H(t, s) = \int_0^{s\wedge t} K_H(t, r) K_H(s, r) \, \mathrm{d}r$, for some constant c_H. In fact, let us observe that, for any element $\varphi \in \mathcal{H}$ which defines a function on $[0, 1]$, $K_H^* \varphi$ can be represented in the following form:

$$[K_H^* \varphi]_t = \int_t^1 \varphi_r \partial_r K_H(r, t) \, \mathrm{d}r.$$

The fractional Cameron–Martin space can be introduced in the following way: let $\mathcal{K}_H : L^2([0, 1]) \to \mathcal{H}_H := \mathcal{K}_H(L^2([0, 1]))$ be the operator defined by

$$[\mathcal{K}_H h](t) := \int_0^t K_H(t, s) h(s) \, \mathrm{d}s, \quad h \in L^2([0, 1]).$$

Then, $\mathcal{H}_H$ is the reproducing kernel Hilbert space associated with the fractional Brownian motion B. Observe that, in the case of the classical Brownian motion, one has that $K_H(t, s) = \mathbf{1}_{[0,t]}(s)$, K_H^* is the identity operator in $L^2([0, 1])$ and $\mathcal{H}_H$ is the usual Cameron–Martin space.

In order to deduce that $(\Omega, \mathcal{H}, P)$ defines an abstract Wiener space, we remark that $\mathcal{H}$ is continuously and densely embedded in Ω. In fact, one proves that the operator $\mathcal{R}_H : \mathcal{H} \to \mathcal{H}_H$ given by

$$\mathcal{R}_H \psi := \int_0^{\cdot} K_H(\cdot, s) [K_H^* \psi](s) \, \mathrm{d}s$$

defines a dense and continuous embedding from $\mathcal{H}$ into Ω; this is due to the fact that $\mathcal{R}_H \psi$ is H-Hölder continuous (for details, see [21, p. 9]).

At this point, we can introduce the Malliavin derivative operator on the Wiener space $(\Omega, \mathcal{H}, P)$. Namely, we first let $\mathcal{S}$ be the family of smooth functionals F of the form

$$F = f(B(h_1), \ldots, B(h_n)),$$

where $h_1, \ldots, h_n \in \mathcal{H}$, $n \geq 1$, and f is a smooth function having polynomial growth together with all its partial derivatives. Then, the Malliavin derivative of such a functional F is the $\mathcal{H}$-valued random variable defined by

$$\mathcal{D}F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B(h_1), \ldots, B(h_n)) h_i.$$

For all $p > 1$, it is known that the operator $\mathcal{D}$ is closable from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$ (see e.g. [18, Section 1]). We will still denote by $\mathcal{D}$ the closure of this operator, whose domain is usually denoted by $\mathbb{D}^{1,p}$ and is defined as the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} := \left(E(|F|^p) + E(\|\mathcal{D}F\|_{\mathcal{H}}^p)\right)^{\frac{1}{p}}.$$

The local property of the operator $\mathcal{D}$ allows us to define the localized version of $\mathbb{D}^{1,p}$, as follows. By definition, $F \in \mathbb{D}^{1,p}_{loc}$ if there is a sequence $\{(\Omega_n, F_n), n \geq\}$ in $\mathcal{F} \times \mathbb{D}^{1,p}$ such that Ω_n increases to Ω with probability 1 and $F = F_n$ on Ω_n. In this case, one sets $\mathcal{D}F := \mathcal{D}F_n$ on Ω_n.

We will first prove now that the solution of (50) at any $t \in [0, 1]$ belongs to $\mathbb{D}^{1,p}_{loc}$. For this, we need to introduce the notion of differentiability of a random variable F in the directions of $\mathcal{H}$, and we shall apply a classical result of Kusuoka (see [13] or [18, Proposition 4.1.3]). Indeed, a random variable F is $\mathcal{H}$-differentiable if, by definition, for almost all $\omega \in \Omega$ and for any $h \in \mathcal{H}$, the map $\nu \mapsto F(\omega + \nu \mathcal{R}_H h)$ is differentiable. Then, the above-mentioned result of Kusuoka states that any $\mathcal{H}$-differentiable random variable F belongs to the space $\mathbb{D}^{1,p}_{loc}$, for any $p > 1$. We have the following result:

Proposition 4.1. *Let $\gamma, \kappa \in (0, 1)$ be such that $\gamma + \kappa > 1$. Let $\varepsilon > 0$ and a sufficiently large $p \geq 1$ be such that $\varepsilon > \frac{1}{2p}$ and $\gamma + \varepsilon < H$ (this latter condition guarantees that $B \in \mathcal{C}^{\gamma+\varepsilon}$). Assume that σ satisfies the hypotheses of Theorem 3.7.*

Let $z = \{z_t,\ t \in [0, 1]\} \in \mathcal{C}^{\kappa}$ be the unique solution of Eq. (50). Then, for any $t \in [0, 1]$, $z_t \in \mathbb{D}^{1,2}_{loc}$ and we have

$$\langle \mathcal{D}z_t, h\rangle_{\mathcal{H}} = [Dz(B)(\mathcal{R}_H h)]_t, \quad h \in \mathcal{H}. \tag{51}$$

Proof. Recall that the process B is γ-Hölder continuous for any $\gamma \in (0, H)$. Hence, in the statement of Theorem 3.7, we will be able to find ε (choosing p therein sufficiently large) such that $\gamma + \varepsilon < H$ and $\|B\|_{\gamma,p}$ is finite almost surely.

On the other hand, note that for all $h \in \mathcal{H}$, we have

$$|(\mathcal{R}_H h)(t) - (\mathcal{R}_H h)(s)| = \left(E(|B_t - B_s|^2)\right)^{\frac{1}{2}} \|h\|_{\mathcal{H}} \leq |t-s|^H \|h\|_{\mathcal{H}}.$$

Consequently, by Theorem 3.7 and Lemma 4.2 below, we can infer that z_t is $\mathcal{H}$-differentiable. Therefore, Kusuoka's result implies that $z_t \in \mathbb{D}^{1,2}_{loc}$ and we have

$$\langle \mathcal{D}z_t, h\rangle_{\mathcal{H}} = \left.\frac{d}{d\nu} z_t(\omega + \nu \mathcal{R}_H h)\right|_{\nu=0} \quad \text{a.s.}, \tag{52}$$

which, together with Lemma 4.2, allows us to conclude that

$$\langle \mathcal{D}z_t, h\rangle_{\mathcal{H}} = Dz_t(B)(\mathcal{R}_H h) = [Dz(B)(\mathcal{R}_H h)](t). \quad \square$$

Lemma 4.2. *Let $\gamma < H$ and $\varepsilon > 0$ be such that $\gamma + \varepsilon < H$, as in the statement of Theorem 3.7. Let z be the solution of (50) and $t \in [0, 1]$. Then $x \mapsto z_t(x)$ is Fréchet differentiable from $\mathcal{C}^{\gamma+\varepsilon}$ into $\mathbb{R}$. Furthermore, for $x \in \mathcal{C}^{\gamma+\varepsilon}$, it holds that*

$$Dz_t(x)(k) = [Dz(x)(k)]_t, \quad k \in \mathcal{C}^{\gamma+\varepsilon}.$$

Proof. It is very similar to that of [14, Lemma 4.2]. Indeed, the following estimates are readily checked:

$$\begin{aligned}|z(x+k)_t - z(x)_t - [Dz(x)k]_t| &\leq \|z(x+k) - z(x) - [Dz(x)k]\|_{\infty} \\ &\leq \|z(x+k) - z(x) - [Dz(x)k]\|_{\gamma+\varepsilon}.\end{aligned}$$

In addition, Theorem 3.7 ensures that the latter term is of order $o(\|k\|_{\gamma+\varepsilon})$, from which our claim is easily deduced. □

At this point, let us go a step further and prove that the solution z_t of Eq. (50) does indeed belong to $\mathbb{D}^{1,2}$.

Proposition 4.3. *Let $\gamma, \kappa \in (0,1)$ be such that $\gamma + \kappa > 1$. Let $\varepsilon > 0$ and a sufficiently large $p \geq 1$ be such that $\varepsilon > \frac{2}{p}$ and $\gamma+\varepsilon < H$. Assume that σ satisfies the hypotheses of Theorem 3.7.*

Let $z = \{z_t, t \in [0,1]\}$ be the unique solution of Eq. (50). Then, for any $t \in [0,1]$, z_t belongs to $\mathbb{D}^{1,2}$.

Proof. By (51), formula (39) and the definition and properties of $\mathcal{R}_H$, we have the following equalities: for any $h \in \mathcal{H}$,

$$\begin{aligned}\langle \mathcal{D}z_t, h\rangle_{\mathcal{H}} &= [Dz(B)(\mathcal{R}_H h)]_t = \int_0^1 \Phi_s(t)\,\mathrm{d}(\mathcal{R}_H h)_s \\ &= \int_0^1 \Phi_s(t)\left(\int_0^s \frac{\partial K_H}{\partial s}(s,r)(K_H^* h)(r)\,\mathrm{d}r\right)\mathrm{d}s \\ &= \int_0^1 (K_H^* \Phi_\cdot(t))(s)(K_H^* h)(s)\,\mathrm{d}s = \langle \Phi_\cdot(t), h\rangle_{\mathcal{H}}.\end{aligned}$$

This implies that, as elements of $\mathcal{H}$, $\mathcal{D}z_t = \Phi_\cdot(t)$.

On the other hand, let us observe that $L^{\frac{1}{H}}([0,1]) \subset \mathcal{H}$ continuously (see e.g. [18, Lemma 5.1.1]), and clearly any Hölder space $\mathcal{C}^\kappa$ is continuously embedded in $L^{\frac{1}{H}}([0,1])$. Therefore, if we aim to prove that $E(\|\mathcal{D}z_t\|^2_{\mathcal{H}}) < +\infty$ (see [18, Lemma 4.1.2]), it suffices to verify that $E(\|\Phi_\cdot(t)\|^2_\kappa) < \infty$, for any $\kappa \in (0,1)$.

Taking into account that $\Phi_s(t)$ satisfies the linear equation (40), we are in position to apply Proposition 3.6, so we end up with

$$\|\Phi_\cdot(t)\|_\kappa \leq C(M)\|\Psi_\cdot(t)\|_\kappa, \tag{53}$$

where we recall that $\Psi_s(t)$ has been defined in (41). By the boundedness of G_M and φ'_M, the fact that $K(t,\cdot)$ is Lipschitz with Lipschitz constant bounded by $1-t$, Lemma 2.7 and estimate (17), we can infer that

$$\|\Psi_\cdot(t)\|_\kappa \leq C(1+\|\mu\|_\kappa), \tag{54}$$

for some constant C depending on M and σ. Hence, it remains to study the κ-Hölder regularity of μ (recall that this process is defined by (42)). Namely, for any $0 \leq s_1 < s_2 \leq 1$, one easily verifies that

$$\mu_{s_2} - \mu_{s_1} = \int_{s_1}^{s_2}\int_{s_2}^1 \rho_{\zeta\eta}\,\mathrm{d}\zeta\,\mathrm{d}\eta - \int_0^{s_1}\int_{s_1}^{s_2} \rho_{\zeta\eta}\,\mathrm{d}\zeta\,\mathrm{d}\eta.$$

At this point, let us observe that, in the statement, the condition relating p and ε is slightly stronger than the one considered in Proposition 4.1. In fact, the former allows us to infer that $\rho_{\zeta\eta} \leq C\|B\|^{2p-1}_{\gamma+\varepsilon}$, almost surely, which guarantees that $\mu \in \mathcal{C}^\kappa$ and

$$\|\mu\|_\kappa \leq C\|B\|^{2p-1}_{\gamma+\varepsilon}. \tag{55}$$

Plugging this bound in (54) and using (53), we end up with

$$E(\|\mathcal{D}z_t\|_{\mathcal{H}}^2) \le E(\|\Phi_\cdot(t)\|_\kappa^2) \le C\left(1 + E(\|B\|_{\gamma+\varepsilon}^{4p-2})\right),$$

and the latter is a finite quantity since $\gamma+\varepsilon < H$ and $\|B\|_{\gamma+\varepsilon}$ has moments of any order by Fernique's lemma [9, Theorem 1.2.3]. This concludes the proof. □

4.2. The Stratonovich interpretation of the fractional elliptic equation

Up to now, we have succeeded in solving Eq. (50) by interpreting any integral with respect to B in the Young (pathwise) sense. In this particular situation, it is a well-known fact [23] that our approach is equivalent to Russo–Vallois kinds of techniques. Namely, if for a process V the integral $\int_0^T V_s\,\mathrm{d}B_s$ can be defined in the Young sense, then one also has almost surely

$$\int_0^T V_s\,\mathrm{d}B_s = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^T V_s\ (B_{s+\varepsilon} - B_{s-\varepsilon})\,\mathrm{d}s.$$

The latter limit is usually called the Stratonovich integral with respect to B (see [18, Definition 5.2.2]), and is denoted by $\int_0^T V_s \circ \mathrm{d}B_s$.

Our point of view in this section is slightly different: we wish to show that the integrals with respect to B in Eq. (50) can also be interpreted as the sum of a Skorohod integral and a trace term. As we shall see below (see Proposition 4.4), this gives another definition of the Russo–Vallois symmetric integral in the particular case of smooth integrands in the Malliavin calculus sense. In particular we shall see that, at least a posteriori, Malliavin calculus might have been applied in order to solve our original elliptic equation, though a direct application of these techniques leads to non-closed estimations.

Let us thus introduce the space $|\mathcal{H}|$, which is composed of measurable functions $\varphi : [0,1] \to \mathbb{R}$ such that

$$\|\varphi\|_{|\mathcal{H}|}^2 := \alpha_H \int_0^1\int_0^1 |\varphi_r||\varphi_u||r-u|^{2H-2}\mathrm{d}r\mathrm{d}u < +\infty,$$

where $\alpha_H = H(2H-1)$, and we denote by $\langle\cdot,\cdot\rangle_{|\mathcal{H}|}$ the associated inner product. We define Stratonovich integrals thanks to the following result, borrowed from [1, Proposition 3]:

Proposition 4.4. *Let $\{u_t, t\in[0,1]\}$ be a stochastic process in $\mathbb{D}^{1,2}(|\mathcal{H}|)$ such that*

$$\int_0^1\int_0^1 |\mathcal{D}_s u_t||t-s|^{2H-2}\mathrm{d}s\mathrm{d}t < +\infty \quad a.s. \tag{56}$$

Then, the Stratonovich integral $\int_0^1 u_t \circ \mathrm{d}B_t$ exists and can be written as

$$\int_0^1 u_t\circ\mathrm{d}B_t = \delta(u) + \int_0^1\int_0^1 \mathcal{D}_s u_t|t-s|^{2H-2}\mathrm{d}s\mathrm{d}t, \tag{57}$$

where $\delta(u)$ stands for the Skorohod integral of u.

We are now in a position to apply this result to our elliptic equation:

Proposition 4.5. *Let $z=\{z_t, t\in[0,1]\}$ be the solution to Eq. (50). Under the same hypothesis as in Proposition 4.3, the process z belongs to $\mathbb{D}^{1,2}(|\mathcal{H}|)$ and also satisfies the equation*

$$z_t = G_M(B) \int_0^1 K(t,\xi)\sigma(z_\xi) \circ \mathrm{d}B_\xi, \quad t \in [0,1], \tag{58}$$

where the Stratonovich stochastic integral with respect to B is interpreted as in (57).

Proof. Note first that the norm of z in $\mathbb{D}^{1,2}(|\mathcal{H}|)$ is given by

$$\|z\|^2_{\mathbb{D}^{1,2}(|\mathcal{H}|)} = E(\|z\|^2_{|\mathcal{H}|}) + E(\|\mathcal{D}z\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|}).$$

By Theorem 2.8 (see (17) therein), we have

$$\begin{aligned} E(\|z\|^2_{|\mathcal{H}|}) &= \int_0^1 \int_0^1 E(|z_r z_u|)|r-u|^{2H-2}\mathrm{d}r\mathrm{d}u \\ &\le C E(\|z\|^2_\infty) \int_0^1 \int_0^1 |r-u|^{2H-2}\mathrm{d}r\mathrm{d}u \le C. \end{aligned} \tag{59}$$

On the other hand, owing to (53)–(55) we can infer that, for any $r, u \in [0,1]$,

$$E(|\mathcal{D}_u z_r|) \le C,$$

for some positive constant C. Thus

$$\begin{aligned} &E(\|\mathcal{D}z\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|}) \qquad (60) \\ &= \int_0^1 \int_0^1 \mathrm{d}r_1\mathrm{d}r_2\, |r_1-r_2|^{2H-2} \int_0^1 \int_0^1 \mathrm{d}u_1\mathrm{d}u_2\, E\left(|\mathcal{D}_{r_1} z_{u_1}|\,|\mathcal{D}_{r_2} z_{u_2}|\right) |u_1-u_2|^{2H-2} \\ &\le C \int_0^1 \int_0^1 \mathrm{d}r_1\mathrm{d}r_2\, |r_1-r_2|^{2H-2} \int_0^1 \int_0^1 \mathrm{d}u_1\mathrm{d}u_2\, |u_1-u_2|^{2H-2} < +\infty. \qquad (61) \end{aligned}$$

Putting together (59) and (61), we have seen that $z \in \mathbb{D}^{1,2}(|\mathcal{H}|)$, and one also deduces that (56) holds. By Proposition 4.4, this implies that z belongs to the domain of the Stratonovich integral. Therefore, thanks to the regularity properties of σ and the fact that $K(t,\cdot)$ is a deterministic function, we obtain that the Stratonovich integral $\int_0^1 K(t,\xi)\sigma(z_\xi) \circ \mathrm{d}B_\xi$ is well-defined. By Russo and Vallois [24, Section 2.2, Proposition 3], this Stratonovich integral coincides with the pathwise Young integral on the right-hand side of (50), for which we can conclude that z solves (58). □

4.3. A modified elliptic equation

One of the major obstacles on our way to get the absolute continuity of $\mathcal{L}(z_t)$ is the following: associated with Eq. (50) is the process μ defined by (42), appearing in the expression for $\mathcal{D}z_t$. This process happens to have some fluctuations around $s=0$ which are too high for guaranteeing the strict positivity of $\mathcal{D}z_t$ at least in a small interval. This is why we consider in this section a slight modification of our elliptic equation (50) and we will prove that its solution, at any instant t, has a law which is absolutely continuous with respect to the Lebesgue measure. Specifically, the cutoff term $G_M(B)$ in Eq. (50) will be replaced by a new $\tilde{G}_M(B)$, whose motivation relies on a variation of Garsia's lemma given below:

Proposition 4.6. *Let f be a continuous function defined on* $[0,1]$. *Set, for $p \ge 1$,*

$$U_{\gamma,p}(f) := \left(\int_0^1 \mathrm{d}v \int_v^{4v\wedge 1} \frac{|\delta f_{uv}|^{2p}}{|v-u|^{2\gamma p+2}}\, \mathrm{d}u \right)^{1/2p}, \tag{62}$$

and assume $U_{\gamma,p}(f) < \infty$. *Then* $f \in C^{\gamma}([0,1])$*; more precisely,*

$$\|f\|_{\gamma} \le C\, U_{\gamma,p}(f), \tag{63}$$

for some positive constant C.

Proof. Let $0 \le s < t \le 1$. We wish to show that

$$|\delta f_{st}| \le C\, U_{\gamma,p}(f)\, |t-s|^{\gamma}. \tag{64}$$

To this end, let us construct a sequence of points $(s_k)_{k\ge 0}$, $s_k \in [0,1]$, converging to t in the following way: set $s_0 = s$, suppose by induction that $s_0, \ldots, s_k \le t$ have been constructed, and let

$$V_k := [a_k, b_k], \quad \text{with } a_k = 2s_k \wedge \left(\frac{s_k + t}{2}\right),\ b_k = 3s_k \wedge t. \tag{65}$$

Notice that the main differences between our proof an the original one by Garsia (or better, the one given by Stroock in [26]) stems from this definition of a_k, b_k. Indeed, in the classical proof, $a_k = \frac{s_k+t}{2}$ and $b_k = t$. Define then

$$A_k := \left\{ v \in V_k \mid I(v) > \frac{6\, U^{2p}_{\gamma,p}(f)}{|v - s_k|} \right\} \tag{66}$$

and

$$B_k := \left\{ v \in V_k \mid \frac{|\delta f_{s_k v}|^{2p}}{|v - s_k|^{2\gamma p+2}} > \frac{6\, I(s_k)}{|v - s_k|} \right\} \tag{67}$$

where we have set

$$I(v) := \int_v^{t\wedge 4v} \frac{|\delta f_{uv}|^{2p}}{|v-u|^{2\gamma p+2}}\, \mathrm{d}u.$$

Let us prove now that $V_k \setminus (A_k \cup B_k)$ is not empty: observe that, for $t \in [0,1]$,

$$\int_v^{4v\wedge 1} \frac{|\delta f_{uv}|^{2p}}{|v-u|^{2\gamma p+2}}\, \mathrm{d}u \ge \int_v^{4v\wedge t} \frac{|\delta f_{uv}|^{2p}}{|v-u|^{2\gamma p+2}}\, \mathrm{d}u = I(v),$$

and thus

$$U^{2p}_{\gamma,p}(f) \ge \int_{A_k} I(v)\, \mathrm{d}v > \frac{6\, U^{2p}_{\gamma,p}(f)}{|b_k - s_k|} \mu(A_k).$$

Moreover,

$$\begin{aligned} I(s_k) &= \int_{s_k}^{4s_k\wedge t} \frac{|\delta f_{s_k u}|^{2p}}{|u-s_k|^{2\gamma p+2}}\, \mathrm{d}u \ge \int_{B_k} \frac{|\delta f_{s_k u}|^{2p}}{|u-s_k|^{2\gamma p+2}}\, \mathrm{d}u \\ &> \int_{B_k} \frac{6\, I(s_k)}{|u - s_k|}\, \mathrm{d}u \ge \frac{6\, I(s_k)}{|b_k - s_k|}\, \mu(B_k). \end{aligned}$$

Altogether one has obtained $\mu(A_k), \mu(B_k) < \frac{|b_k - s_k|}{6}$, so $\mu(A_k) + \mu(B_k) < \frac{|b_k - s_k|}{3}$.

Next we show that $|b_k - s_k| = 2\mu(V_k) = 2|b_k - a_k|$. This study can be separated into two cases:

(i) If $s_k \le t/3$, then $a_k = 2s_k$ and $b_k = 3s_k$. Thus $b_k - a_k = s_k$ and $b_k - s_k = 2s_k$. This obviously yields $|b_k - s_k| = 2\mu(V_k)$.

(ii) If $s_k > t/3$, then $a_k = \frac{s_k+t}{2}$ and $b_k = t$. Thus $b_k - a_k = \frac{t-s_k}{2}$ and $b_k - s_k = t - s_k$. Here again, we get $|b_k - s_k| = 2\mu(V_k)$.

We have thus proved that $\mu(A_k) + \mu(B_k) < \frac{2\mu(V_k)}{3}$, which means that $V_k \setminus (A_k \cup B_k)$ is not empty. Let us thus choose s_{k+1} arbitrarily in this set. Note that, by construction, $s_k \to t$ while staying inside $[s, t]$.

Now, for an arbitrary $n \ge 1$, decompose δf_{st} into

$$\delta f_{st} = \delta f_{s_{n+1}t} + \sum_{k=0}^{n} \delta f_{s_k s_{k+1}}. \tag{68}$$

Applying $(67)_k$ and $(66)_{k-1}$, one gets

$$\frac{|\delta f_{s_k s_{k+1}}|^{2p}}{|s_{k+1} - s_k|^{2\gamma p+2}} \le \frac{c\, I(s_k)}{|s_{k+1} - s_k|} \le \frac{c\, U_{\gamma,p}^{2p}(f)}{|s_{k+1} - s_k||s_k - s_{k-1}|},$$

and hence

$$|\delta f_{s_k s_{k+1}}|^{2p} \le c\, Q_k\, U_{\gamma,p}^{2p}(f)\, |s_{k+1} - s_k|^{2\gamma p}, \quad \text{where } Q_k := \frac{|s_{k+1} - s_k|}{|s_k - s_{k-1}|}. \tag{69}$$

Notice that in our definition (65), we have $a_k = 2s_k$ instead of $(s_k + t)/2$ iff $s_k < t/3$. Therefore, we can distinguish three cases in order to bound the quantity Q_k above:

(i) If $s_{k-1} > t/3$, then $s_{k+1} - s_k \le t - s_k$ and $s_k - s_{k-1} \ge (t - s_k)/4$. Thus $Q_k \le 4$.

(ii) If $s_k \le t/3$, then $s_{k+1} - s_k \le 3s_k - s_k = 2s_k$ and $s_k - s_{k-1} \ge s_k - s_k/2 = s_k/2$. Thus $Q_k \le 4$ again.

(iii) If $s_{k-1} \le t/3$ and $s_k > t/3$, then $s_{k+1} - s_k \le t - s_k \le 3s_k - s_k = 2s_k$ and $s_k - s_{k-1} \ge s_k/2$. Thus $Q_k \le 4$.

Putting those estimates together, we end up with $Q_k \le 4$ in all cases, and plugging this inequality into (69), we obtain

$$|\delta f_{s_k s_{k+1}}|^{2p} \le c\, U_{\gamma,p}^{2p}(f)\, |s_{k+1} - s_k|^{2\gamma p}.$$

Now (68) reads

$$\left|\delta f_{st}\right| \le \left|\delta f_{s_{n+1}t}\right| + \sum_{k=0}^{n} \left|\delta f_{s_k s_{k+1}}\right| \le \left|\delta f_{s_{n+1}t}\right| + U_{\gamma,p}(f) \sum_{k=0}^{n} |s_{k+1} - s_k|^{\gamma}. \tag{70}$$

It remains to bound $\sum_{k=0}^{n} |s_{k+1} - s_k|^{\gamma}$ for an arbitrary n. This is achieved by separating cases again:

(i) If $s > t/3$, then it is easily shown that $a_k = \frac{s_k+t}{2}$ and $b_k = t$, for all k, for which we have $s_{k+1} \in [(s_k + t)/2, t]$. This implies that $t - s_{k+1} \le (t - s_k)/2$ and hence $t - s_k \le 2^{-k}(t - s)$, for any $k \ge 0$. Therefore

$$s_{k+1} - s_k \le t - \frac{t + s_{k-1}}{2} = \frac{t - s_{k-1}}{2} \le \frac{t - s}{2^k}.$$

Plugging this into (70),

$$|\delta f_{st}| \le |\delta f_{s_{n+1}t}| + U_{\gamma,p}(f) \sum_{k=0}^{n} \frac{1}{2^{\gamma k}} |t - s|^{\gamma}.$$

Let $n \to \infty$ and use the continuity of f and the fact that $s_{n+1} \to t$. Then

$$|\delta f_{st}| \le C U_{\gamma,p}(f)|t-s|^{\gamma},$$

where C denotes a positive constant which may depend on γ. This concludes the proof in the case $s > t/3$.

(ii) If $s \le t/3$, then by definition of s_{k+1} we will have, for small enough k, that $s_{k+1} = \beta_{k+1} s_k$ for some $\beta_{k+1} \in [2,3]$. Thus

$$s_k = \left(\prod_{j=1}^{k} \beta_j\right) s.$$

Set $M := \inf\{k \in \mathbb{N};\ \prod_{j=1}^{k} \beta_j \ge t/(3s)\}$, so we wish to evaluate $\sum_{k=0}^{M-1} |s_{k+1} - s_k|^{\gamma}$:

$$\sum_{k=0}^{M-1} |s_{k+1} - s_k|^{\gamma} = \sum_{k=0}^{M-1} \left(\prod_{j=1}^{k} \beta_j s\right)^{\gamma} (\beta_{k+1} - 1)^{\gamma} \le 2^{\gamma} s^{\gamma} \sum_{k=0}^{M-1} \left(\prod_{j=1}^{k} \beta_j\right)^{\gamma}.$$

Notice that $b_M := \prod_{j=0}^{M-1} \beta_j \le t/(3s)$, by definition of M, and

$$\prod_{j=0}^{M-l} \beta_j \le \frac{b_M}{2^{l-1}}, \quad \text{for } l = 1, \ldots, M,$$

since $\beta_j \ge 2$. Hence

$$\sum_{k=0}^{M-1} \left(\prod_{j=0}^{k} \beta_j\right)^{\gamma} \le \sum_{l=1}^{M} \frac{b_M^{\gamma}}{2^{(l-1)\gamma}} < C b_M^{\gamma} \le C(t/s)^{\gamma}$$

and

$$\sum_{k=0}^{M-1} |s_{k+1} - s_k|^{\gamma} \le 2^{\gamma} s^{\gamma} C(t/s)^{\gamma} \le C t^{\gamma}.$$

Observe that $t - s > 2t/3$ whenever $s < t/3$. Therefore $t < \frac{3}{2}(t-s)$ and

$$\sum_{k=0}^{M-1} |s_{k+1} - s_k|^{\gamma} \le C(t-s)^{\gamma}.$$

Let us go back now to (70) and write

$$\begin{aligned} \left|\delta f_{st}\right| \le\ & \left|\delta f_{s_{n+1}t}\right| + U_{\gamma,p}(f) \left(\sum_{k=0}^{M-1} |s_{k+1} - s_k|^{\gamma} + \sum_{k=M}^{n} |s_{k+1} - s_k|^{\gamma} \right) \\ :=\ & \left|\delta f_{s_{n+1}t}\right| + U_{\gamma,p}(f)\,(A_M + B_M). \end{aligned} \tag{71}$$

We have just seen that $A_M \le C(t-s)^{\gamma}$, and one can also prove that $B_M \le C(t-s)^{\gamma}$ uniformly in n by means of the same kind of argument as for step (i). This ends the proof on taking limits in (71). □

We will now take advantage of the previous proposition in order to build a slight modification of our elliptic equation which is amenable to density results. Namely, as before, let $M > 0$ be any real number and $\varphi_M \in \mathcal{C}_b^\infty((0,\infty))$ such that $\varphi_M(r) = 0$ for all $r > M+1$, and $\varphi_M(r) = 1$ for $r < M$. For any $x : [0,1] \to \mathbb{R}$ for which $U_{\gamma,p}(x) < \infty$, for some $\gamma \in (0,1)$ and $p \geq 1$, set

$$\tilde{G}_M(x) := \varphi_M(U_{\gamma,p}(x)^{2p}),$$

and, for such x and any $z \in \mathbb{R}$, we define

$$\tilde{\sigma}_M(x,z) := \tilde{G}_M(x)\sigma(z). \tag{72}$$

We shall thus consider another kind of modified elliptic integral equation driven by the fractional Brownian motion B:

$$\delta z_{st} = \int_s^t \mathrm{d}u \left(\int_u^1 \tilde{\sigma}_M(B, z_\xi)\mathrm{d}B_\xi \right) - (t-s) \int_0^1 \xi \tilde{\sigma}_M(B, z_\xi)\mathrm{d}B_\xi, \quad 0 \leq s \leq t \leq 1. \tag{73}$$

This equation can be equivalently formulated in its compact form:

$$z_t = \tilde{G}_M(B) \int_0^t K(t,\xi)\sigma(z_\xi)\mathrm{d}B_\xi, \quad t \in [0,1]. \tag{74}$$

We will prove that the probability law of the solution to (74) taken at $t \in (0,1)$ is absolutely continuous with respect to the Lebesgue measure.

First of all, let us point out that the results of Sections 2.3 and 3 remain valid for the solution of Eq. (74). Moreover, using exactly the same arguments as in the proof of Proposition 4.3, one obtains that, for all $t \in [0,1]$, the solution z_t belongs to the domain of the Malliavin derivative. Combining all of this, we can state the following result:

Theorem 4.7. *Let $\gamma, \kappa \in (0,1)$ be such that $\gamma + \kappa > 1$. Let $\varepsilon > 0$ and a sufficiently large $p \geq 1$ be such that $\varepsilon > \frac{2}{p}$ and $\gamma + \varepsilon < H$. Assume that σ satisfies the hypotheses of Theorem 3.7.*

Then, there exists a unique solution $z = \{z_t,\ t \in [0,1]\}$ of (74), which is an element of $\mathcal{C}^\kappa$. For any $t \in [0,1]$, z_t belongs to $\mathbb{D}^{1,2}$ and the Malliavin derivative $\mathcal{D}z_t$ satisfies the following linear integral equation:

$$\mathcal{D}_s z_t = \Psi_s(t) + \tilde{G}_M(B) \int_0^1 K(t,\xi)\sigma'(z_\xi)\mathcal{D}_s z_\xi \,\mathrm{d}B_\xi, \quad s \in [0,1], \tag{75}$$

with

$$\Psi_s(t) = \tilde{G}_M(B)\,\sigma(z_s)\,K(t,s) + 2\varphi_M'(U_{\gamma,p}(B)^{2p})\,\tilde{\mu}_s\,z_t, \tag{76}$$

and

$$\tilde{\mu}_s := \int_{\frac{s}{4}}^s \int_s^{4\eta\wedge 1} \rho_{\zeta\eta}\,\mathrm{d}\zeta\,\mathrm{d}\eta, \quad \textit{where } \rho_{\zeta\eta} = (2p-1)\frac{(B_\zeta - B_\eta)^{2p-1}}{|\zeta-\eta|^{2\gamma p+2}}. \tag{77}$$

Remark 4.8. The term $\tilde{\mu}_s$ in (76) comes from the fact that, as one can easily verify, the Fréchet derivative of $\tilde{G}_M$ at $x \in \mathcal{C}^{\gamma+\varepsilon}$ is given by

$$D\tilde{G}_M(x)\cdot h = 2\varphi_M'(U_{\gamma,p}(x)^{2p}) \int_0^1 \tilde{\mu}_s \mathrm{d}h_s, \quad h \in \mathcal{C}^{\gamma+\varepsilon}.$$

We can now give the technical justification for our change in the elliptic equation that we consider: the lemma below (whose proof can be immediately deduced from (77)) shows that $\tilde{\mu}$

can be made of order s^q for an arbitrary large q and s in a neighborhood of 0. This simple fact will enable us to upper bound $|\mathcal{D}_s z_t|$ for $s \to 0$ in a satisfactory way. The following result will thus be important in the sequel:

Lemma 4.9. *Assume that the hypotheses of Theorem 4.7 are satisfied. Then, for all $s \in (0, \frac{1}{4})$ and $p \geq 1$,*

$$\tilde{\mu}_s \leq \|B\|_{\gamma+\varepsilon}^{2p-1} s^\beta \quad a.s., \tag{78}$$

where $\beta = (2p-1)\varepsilon - \gamma$.

Fix $t \in (0, 1)$, and consider z_t, the solution to (74). Observe that the random variable z_t cannot have a density $p_t(y)$ at $y = 0$, since $P(z_t = 0) > 0$ due to our cutoff procedure. Hence we will prove the existence of density for the law of the random variable z_t on the subset of Ω defined by $\Omega_a := \{|z_t| \geq a\}$, for all $a > 0$. The fact that we are restricting our analysis to Ω_a implies the following simple but useful properties:

Lemma 4.10. *On Ω_a, we have*

$$\|B\|_\gamma \leq C_1 \quad \text{and} \quad \tilde{G}_M(B) \geq C_2, a.s.$$

where C_1, C_2 denote some positive constants depending on a and M.

Proof. Note that on Ω_a we must clearly have that

$$\tilde{G}_M(B) = \varphi_M(U_{\gamma,p}(B)^{2p}) > 0 \quad \text{a.s.}$$

Thus, by definition of φ_M, we get $U_{\gamma,p}(B)^{2p} < M+1$ a.s. in Ω_a, and the first part of the statement follows after applying Proposition 4.6.

Let us also estimate the integral appearing in Eq. (74): by (6), Lemma 2.7, Theorem 2.8, and the first part of the lemma, on Ω_a we have

$$\left| \int_0^1 K(t, \xi)\sigma(z_\xi)\, \mathrm{d}B_\xi \right| \leq C\|K(t, \cdot)\|_\kappa (\|\sigma'\|_\infty \|z\|_\kappa + \|\sigma\|_\infty)\|B\|_\gamma \leq C, \quad \text{a.s.} \tag{79}$$

where the constant C is positive, depends on $M, \sigma, \kappa, \gamma$ and indeed can be small enough whenever $\|\sigma\|_\infty$ and $\|\sigma'\|_\infty$ are small. Note that here we have used the fact that $\|K(t, \cdot)\|_\kappa \leq Ct^{1-\kappa}$, which can be easily deduced from the explicit expression for the kernel K.

On the other hand, still playing with Eq. (74),

$$\tilde{G}_M(B) \left| \int_0^1 K(t, \xi)\sigma(z_\xi)\, \mathrm{d}B_\xi \right| \geq a \quad \text{a.s. on } \Omega_a.$$

Hence, (79) yields $\tilde{G}_M(B) \geq \frac{a}{C}$ almost surely on Ω_a, which concludes the proof. □

4.4. *The absolute continuity of the law*

With the previous changes in the equation that we are considering, we are now ready to state and prove our result concerning the density of the law for z_t:

Theorem 4.11. *Assume that σ satisfies the hypothesis of Theorem 3.7 and that $|\sigma(y)| \geq \sigma_0 > 0$ for all $y \in \mathbb{R}$, for some constant σ_0. For any $t \in (0, 1)$, we consider the random variable $z_t \in \mathbb{D}^{1,2}$ and $a > 0$. Then, we have that $\|\mathcal{D}z_t\|_{\mathcal{H}} > 0$ a.s. on Ω_a.*

As a consequence, the law of z_t restricted to $\mathbb{R} \setminus (-a, a)$ is absolutely continuous with respect to the Lebesgue measure.

Let us say a few words about the methodology that we have followed in order to prove the result above: as in many instances, our density result will be obtained by bounding the Malliavin derivatives from below. Let us go back thus to Eq. (75), which is the one satisfied by the Malliavin derivative $\mathcal{D}z_t$. We wish to prove that a density exists for the random variable z_t under a non-degeneracy condition of the form $\sigma(y) > \sigma_0$ for any $y \in \mathbb{R}$; we can assume, without losing generality, that σ is positive. Our strategy will be based on the fact that $\mathcal{D}z_t$ is a continuous function, and we will prove that, almost surely on Ω_a, the Malliavin derivative is negative on some non-trivial interval. This necessarily implies that the norm $\|\mathcal{D}z_t\|_{\mathcal{H}}$ cannot vanish. Let us however make the following observations:

(i) We will take advantage of the leading term $\Psi_s(t)$ in Eq. (75) and we will analyze its increments. According to expression (76), these can only be assumed to be strictly negative when s is small enough: we have not imposed any condition on $\tilde{\mu}_s$, and thus we can only rely on the upper bound (78), which is valid for s close enough to 0. Let us insist again here on the fact that our change of cutoff in the elliptic equation that we consider is meant to have $\tilde{\mu}_s$ very small in a neighborhood of 0.

(ii) The estimation of the integral part in Eq. (75) involves some Hölder norms of the function $\xi \mapsto \mathcal{D}_s z_\xi$. It is thus natural to think that the same should occur on the left-hand side of this equation. Therefore, we are induced to consider increments of the form $\mathcal{D}_s z_{t_2} - \mathcal{D}_s z_{t_1}$ and perform our estimations on these quantities.

(iii) We shall tackle those increment estimates in a slightly more abstract setting, similar to that of Proposition 3.6: consider a function $(t, \eta) \mapsto w_t^\eta$, depending on two parameters $t, \eta \in [0, 1]$. For $\eta \in [0, 1]$, let z^η be the solution to

$$z_t^\eta = w_t^\eta - \tilde{G}_M(B) \int_0^1 K(t,\xi)\, R_\xi\, z_\xi^\eta \, \mathrm{d}B_\xi. \tag{80}$$

In the equation above, w^η and R satisfy some suitable Hölder continuity assumption, and we assume the increments of w^η to be also bounded from below. Notice that, for $\eta \le t$, the function $t \mapsto \mathcal{D}_\eta z_t$ satisfies an equation of the form (80). Our aim is then to get an appropriate lower bound on the increments of z^η. This will be a consequence of the following lemma:

Lemma 4.12. *Let $\gamma < H$ and $\kappa \in (0,1)$ be such that $\gamma + \kappa > 1$. For any $\eta \in [0, 1]$, let w^η be a function in $\mathcal{C}^\kappa$ satisfying the relation $|\delta w_{t_1 t_2}^\eta| \le c_1 |t_2 - t_1|\, \eta$ for any $\eta \le t_1 \le t_2 \le 1$ and $c_1 < 1$ small enough. Moreover, let $R \in \mathcal{C}^\kappa$ be such that*

$$\|R\|_\kappa \le \frac{c_2}{M+1}, \tag{81}$$

for a small enough constant $c_2 < 1$ (see Proposition 3.6). Then the solution z^η to Eq. (80) is such that for all $\eta \le t_1 \le t_2 \le 1$,

$$|\delta z_{t_1,t_2}^\eta| \le |t_2 - t_1|\, \eta. \tag{82}$$

If we further suppose that $\delta w_{t_1 t_2}^\eta \le -c_1 |t_2 - t_1|\, \eta$ for any $\eta \le t_1 \le t_2 \le 1$ and c_1 large enough, then we also get the bound

$$\delta z_{t_1,t_2}^\eta \le -c|t_2 - t_1|\, \eta, \tag{83}$$

for all $\eta \le t_1 \le t_2 \le 1$ and a small positive constant c.

Proof. Let us start by proving (82): the solution z^η to Eq. (80) is obtained as the fixed point of an application Θ constructed as in the proof of Proposition 3.6. Namely, let us define the map $\Theta : \mathcal{C}^\kappa \to \mathcal{C}^\kappa$ by

$$\Theta(y)_t := w_t^\eta - \tilde{G}_M(B) \int_0^t \mathrm{d}u \left(\int_u^1 R_\xi \, y_\xi \mathrm{d}B_\xi \right) + t\, \tilde{G}_M(B) \int_0^1 \xi \, R_\xi \, y_\xi \mathrm{d}B_\xi .$$

Then under our standing assumptions, z^η can be seen as the fixed point of the map Θ. It is thus enough to check that, if y verifies $|\delta y_{t_1 t_2}| \le |t_2 - t_1|\, \eta$ for all $\eta \le t_1 \le t_2 \le 1$, then $\hat{y} := \Theta(y)$ fulfills the same condition.

Let us write then

$$\delta \hat{y}_{t_1 t_2} = A_{t_1 t_2} - C_{t_1 t_2} + D_{t_1 t_2},$$

with $A_{t_1 t_2} = \delta w^\eta_{t_1 t_2}$ and

$$C_{t_1 t_2} = \tilde{G}_M(B) \int_{t_1}^{t_2} \mathrm{d}u \left(\int_u^1 R_\xi \, y_\xi \mathrm{d}B_\xi \right), \qquad D_{t_1 t_2} = (t_2 - t_1) \tilde{G}_M(B) \int_0^1 \xi \, R_\xi \, y_\xi \mathrm{d}B_\xi .$$

We shall bound those three terms separately for $\eta \le t_1 \le t_2 \le 1$.

$|A_{t_1 t_2}|$ is bounded by assumption by $c_1 |t_2 - t_1|\, \eta$. Furthermore, $|C_{t_1 t_2}|$ is easily estimated as follows:

$$|C_{t_1 t_2}| \le \|R\|_\kappa \, \|y\|_{\mathcal{C}^\kappa([t_1,1])} M \, |t_2 - t_1| \le \|R\|_\kappa \, M \, |t_2 - t_1|\, \eta,$$

thanks to our induction hypothesis. Hence, by (81), we have

$$|C_{t_1,t_2}| \le c_2 |t_2 - t_1| \eta.$$

Some similar considerations also yield $|D_{t_1,t_2}| \le c_3 |t_2 - t_1| \eta$ for a small enough constant c_3. In order to complete the proof of (82), it suffices thus to consider c_1, c_2 small enough that $c_1 + c_2 + c_3 < 1$.

Let us turn now to the proof of (83): it is sufficient to go through the same computations as for (82) and take into account the lower bound on $\delta w^\eta_{t_1,t_2}$. The details are left to the reader. We only note that the constant c_1 has to be taken such that $c_1 > c_2 + c_3$, where the c_2, c_3 are the same constants as in the proof of (82). □

At this point, we already have the main tools for proving the main result of the section.

Proof of Theorem 4.11. Taking into account that the Malliavin derivative $\mathcal{D} z_t$ satisfies Eq. (75), we will apply (83) to the following situation: $z_t^s = \mathcal{D}_s z_t$, $R_\xi = \sigma(z_\xi)$ and $w_t^s = \Psi_s(t)$, where we recall that

$$\Psi_s(t) = \tilde{G}_M(B) \, \sigma(z_s) \, K(t,s) + 2\varphi'_M(U_{\gamma,p}(B)^{2p}) \, \tilde{\mu}_s \, z_t$$

and $\tilde{\mu}_s$ is defined by (77). We also recall that, throughout the proof, we have implicitly fixed ω belonging to Ω_a.

First, note that the hypotheses on σ guarantee that (81) is satisfied. Next, let us prove that $\delta w^s_{t_1,t_2} \le -c_1 |t_2 - t_1| s$, for all $s \le t_1 \le t_2 \le T$ with $s < 1/4$. To be precise, we clearly have that

$$\begin{aligned} \delta w^s_{t_1,t_2} &= \Psi_s(t_2) - \Psi_s(t_1) \\ &= -\tilde{G}_M(B) \sigma(z_s)(t_2 - t_1) s + 2\varphi'_M(U_{\gamma,p}(B)^{2p}) \tilde{\mu}_s (\delta z_{t_1,t_2}). \end{aligned} \tag{84}$$

By Lemma 4.10 and the non-degeneracy condition on σ, the first term on the right-hand side of (84) can be bounded by $-c_4 (t_2 - t_1) s$, where c_4 is some large enough constant (see the proof of Lemma 4.10). We will check now that, for all $s \le t_1 \le t_2 \le 1$ with $s < 1/4$,

$$2\varphi'_M(U_{\gamma,p}(B)^{2p})\tilde{\mu}_s(\delta z_{t_1,t_2}) \le c_5(t_2 - t_1)s, \tag{85}$$

for some (small) constant c_5 (which may depend on ω). For this, we use the boundedness of φ'_M, apply Lemma 4.9 and take into account the fact that, as can be deduced from the existence result Theorem 2.8, the solution z is indeed Lipschitz continuous (with Lipschitz constant depending on M). Altogether this yields

$$2\varphi'_M(U_{\gamma,p}(B)^{2p})\tilde{\mu}_s(\delta z_{t_1,t_2}) \le C\|B\|_{\gamma+\varepsilon}^{2p-1} s^{\beta}(t_2 - t_1) \le c_5(t_2 - t_1)s,$$

with $c_5 = C\|B\|_{\gamma+\varepsilon}^{2p-1} (1/4)^{\beta-1}$, where we recall that $\beta = (2p-1)\varepsilon - \gamma$.

Therefore, taking p large enough that $c_5 < c_4$ and plugging (85) into (84), we obtain

$$\delta w^s_{t_1,t_2} = \Psi_s(t_2) - \Psi_s(t_1) \le -c_1(t_2 - t_1)s, \quad \text{for all } s \le t_1 \le t_2 \le 1 \text{ with } s < \frac{1}{4},$$

where c_1 can be large enough (since c_4 can be as well).

Then, we are in position to apply (83) and we obtain that

$$\mathcal{D}_s z_{t_1} - \mathcal{D}_s z_{t_2} \le -(t_2 - t_1)s, \quad \text{for all } s \le t_1 \le t_2 \le 1 \text{ with } s < \frac{1}{4}.$$

Thus, taking $t_1 = t$ and $t_2 = 1$, we obtain that $D_s z_t < 0$ for all $s \in (0, \frac{1}{4} \wedge t)$. Therefore, we have that $\|\mathcal{D} z_t\|_{\mathcal{H}} > 0$ a.s. on Ω_a, by the continuity of $\mathcal{D} z_t$. This concludes the proof. □

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