Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2009, Vol. 45, No. 1, 157–174
DOI: 10.1214/07-AIHP159

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Trees and asymptotic expansions for fractional stochastic differential equations

A. Neuenkirch[a], I. Nourdin[b], A. Rößler[c] and S. Tindel[d]

[a]*Johann Wolfgang Goethe-Universität Frankfurt, FB 12 Institut für Mathematik, Robert-Mayer-Strasse 10, 60325 Frankfurt am Main, Germany. E-mail: neuenkir@math.uni-frankfurt.de*

[b]*Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, Boîte courrier 188, 4 Place Jussieu, 75252 Paris Cedex 5, France. E-mail: inourdin@gmail.com*

[c]*Technische Universität Darmstadt, Fachbereich Mathematik, Schlossgartenstrasse 7, 64289 Darmstadt, Germany. E-mail: roessler@mathematik.tu-darmstadt.de*

[d]*Institut Élie Cartan Nancy (IECN), Boîte postale 239, 54506 Vandœuvre-lès-Nancy Cedex, France. E-mail: tindel@iecn.u-nancy.fr*

Received 28 April 2007; revised 20 November 2007; accepted 29 November 2007

Abstract. In this article, we consider an n-dimensional stochastic differential equation driven by a fractional Brownian motion with Hurst parameter $H > 1/3$. We derive an expansion for $E[f(X_t)]$ in terms of t, where X denotes the solution to the SDE and $f:\mathbb{R}^n \to \mathbb{R}$ is a regular function. Comparing to F. Baudoin and L. Coutin, *Stochastic Process. Appl.* **117** (2007) 550–574, where the same problem is studied, we provide an improvement in three different directions: we are able to consider equations with drift, we parametrize our expansion with trees, which makes it easier to use, and we obtain a sharp estimate of the remainder for the case $H > 1/2$.

Résumé. Dans cet article, nous considérons une équation différentielle stochastique multidimensionnelle dirigée par un mouvement brownien fractionnaire d'indice de Hurst $H > 1/3$. Nous développons $E[f(X_t)]$ par rapport à t, où on note X la solution de l'EDS et où $f:\mathbb{R}^n \to \mathbb{R}$ est une fonction régulière. Par rapport à F. Baudoin et L. Coutin, *Stochastic Process. Appl.* **117** (2007) 550–574, où le même problème est étudié, nous améliorons leur résultat dans trois directions différentes: nous traîtons le cas d'une équation avec dérive, nous paramétrons notre développement à l'aide d'arbres, ce qui le rend plus facile à utiliser, et nous proposons un contrôle plus fin du reste quand $H > 1/2$.

MSC: 60H05; 60H07; 60G15

Keywords: Fractional Brownian motion; Stochastic differential equations; Trees expansions

1. Introduction

This article is concerned with a stochastic differential equation (SDE in short) of the following type:

$$X_t^a = a + \int_0^t \sigma\left(X_s^a\right) \mathrm{d}B_s + \int_0^t b\left(X_s^a\right) \mathrm{d}s, \quad t \in [0, T], \tag{1}$$

where B is a d-dimensional fractional Brownian motion (fBm in short) with Hurst index $H > 1/3$, $a \in \mathbb{R}^n$ is a non-random initial value and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ and $b : \mathbb{R}^n \to \mathbb{R}^n$ are smooth functions. In the last years, significant advances concerning the rigorous definition and the solution of such SDEs have been made: for instance, in the case $H > 1/2$ it is now well known that one can use the Young integral for integration with respect to fBm and, with this choice, the existence and uniqueness of the solution for Eq. (1) in the class of processes, which have α-Hölder continuous paths with $1-H<\alpha<H$, has been obtained e.g. in [23]. See also [10,17,24] for an approach based on fractional derivatives. When $1/4<H<1/2$, it is still possible to give a sense to equation (1), using the rough path theory, which was initiated by Lyons [7,8] and applied to the fBm case by Coutin and Qian [5]. In this setting, we also have existence and uniqueness in an appropriate class of processes. Moreover, by using a generalization of the symmetric Russo–Vallois integral (namely the Newton–Cotes integral corrected by a Lévy area) one can obtain existence and uniqueness for (1), but only in dimension $n=d=1$, see [14]. All the techniques mentioned above, which have been applied to fractional SDEs, are of pathwise (or semi-pathwise) type. However, in case of an equation in which the noise enters linearly, Eq. (1) can be solved in the Skorohod sense, by means of some purely probabilistic methods, see e.g. [15].

In this context, and with a rigorous definition of the solution to Eq. (1) at hand, it seems worthwhile to study the basic properties of the solution of this SDE. Some steps in this direction, such as moments estimates [10], existence of a density for the random variable X_t^a [13,18] or numerical approximations [11], have already been accomplished. The current article can be seen as a part of this ongoing general project, and we will focus here on the following problem: since the fBm B is not a Markov process when $H\neq 1/2$, one cannot expect the law of X_t^a to satisfy a partial differential equation. In order to cope with this problem, and to start an analysis of the law of the process X^a, Baudoin and Coutin [2] studied the asymptotic expansion with respect to t of the quantity $P_t f(a)$ defined by

$$P_t f(a)=\mathrm{E}\big(f\big(X_t^a\big)\big), \quad t\in[0,T], a\in\mathbb{R}^n, f\in \mathrm{C}_c^\infty\big(\mathbb{R}^n;\mathbb{R}\big), \tag{2}$$

where X^a is the solution of (1). Note that in the Brownian case $H=1/2$, this problem has already been addressed, and the Taylor expansion of the P_t, which defines in this case a semigroup, is well studied, see, e.g. [20,21]. There are several reasons to study the family of operators $(P_t, t\geq 0)$. As mentioned above, the knowledge of $P_t f(a)$ for a sufficiently large class of functions f characterizes the law of the random variable X_t^a. Moreover, the knowledge of $P_t f(a)$ helps, e.g., also in finding good sample designs for the reconstruction of fractional diffusions, see [11]. We will therefore take up here the program initiated in [2], and improve their result in several directions:

1. In [2], the authors considered the case $b\equiv 0$. Consequently, their formula contains only powers of t of the form t^{nH} with $n\in\mathbb{N}$. We are able to extend their result to the case of a general drift b, and we will obtain an expression containing powers of the type t^{nH+m} with $n,m\in\mathbb{N}$.
2. In the current article, we use rooted trees in order to obtain a nice representation of our formula. In contrast to the hierarchical set approach, where a cumulative sum of products of derivatives of the functions f, σ and b corresponds to each multi-index, by using trees we can identify each single summand of the expansion with exactly one corresponding rooted tree. As a result of this, rooted trees allow us to obtain a compact representation of the expansion of P_t.
3. In the case where $H>1/2$, we obtain a series expansion (20) of the operator P_t, which is not only valid for small times as in [2], but for any fixed time $t\geq 0$. This improvement will rely on a careful analysis of the behavior of multiple integrals with respect to the fractional Brownian motion.

The crucial point in the problem considered above is to control the remainders of the derived expansions. This depends of course heavily on the definition we use for the stochastic integrals with respect to the fractional Brownian motion. Here we have chosen to solve Eq. (1) by means of the rough path theory introduced by Gubinelli in [9]. This variant is based on an algebraic structure, which turns out to be useful for computational purposes, but has also its own interest, and is in fact a nice alternative to the meanwhile classical theory of rough paths initiated by Lyons [7,8]. More important for us, it leads to a simplification of some steps in the rough path analysis, and avoids some cumbersome discretization procedures of the fractional Brownian path. These simplifications will be essential for the analysis of the remainders.

The paper is organized as follows. In Section 2, we first present the basic setup of [9] and its application to fractional Brownian motion, and moreover we give a short introduction to rooted trees. Finally, we state our main result in Section 3 and prove it in Section 4.

2. Preliminaries

2.1. *Some elements of algebraic integration*

This section contains a summary of the algebraic integration introduced in [9]. We recall its main features here, since all our results are obtained in this setting.

In the sequel, we will denote by $\mathcal{L}^{d,n}$ the space of linear operators from $\mathbb{R}^d$ to $\mathbb{R}^n$, i.e., the space of matrices of $\mathbb{R}^{n\times d}$ and we denote by A^* the transpose of a vector or matrix A. Let x be a Hölder continuous $\mathbb{R}^d$-valued function of order γ, with $1/3 < \gamma \le 1/2$. The case where $\gamma > 1/2$ is classically covered by means of Young's integral, see e.g. [23], and is therefore omitted in the present section. Moreover, let $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$, $b : \mathbb{R}^n \to \mathbb{R}^n$ be two bounded and smooth functions. In the sequel, we shall consider the n-dimensional equation

$$\mathrm{d}y_t = \sigma(y_t)\,\mathrm{d}x_t + b(y_t)\,\mathrm{d}t, \qquad y_0 = a \in \mathbb{R}^n, \quad t \in [0, T]. \tag{3}$$

In order to rigorously define and solve this equation, we will need some algebraic and analytic notions. Therefore, we first present the basic algebraic structures which allow us to define a pathwise integral with respect to irregular functions. For an arbitrary real number $T > 0$, a vector space V and an integer $k \ge 1$ we denote by $\mathcal{C}_k(V)$ the set of functions $g : [0, T]^k \to V$ such that $g_{t_1,\dots,t_k} = 0$, whenever $t_i = t_{i+1}$ for some $i \le k-1$. Such a function will be called a $(k-1)$-*increment*, and we will set $\mathcal{C}_*(V) = \bigcup_{k\ge 1} \mathcal{C}_k(V)$. An important elementary operator is defined by

$$\delta : \mathcal{C}_k(V) \to \mathcal{C}_{k+1}(V), \qquad (\delta g)_{t_1,\dots,t_{k+1}} = \sum_{i=1}^{k+1} (-1)^{k-i} g_{t_1,\dots,\hat{t}_i,\dots,t_{k+1}}, \tag{4}$$

where $\hat{t}_i$ means that this particular argument is omitted. A fundamental property of δ, which is easily verified, is that $\delta\delta = 0$, where $\delta\delta$ is considered as an operator from $\mathcal{C}_k(V)$ to $\mathcal{C}_{k+2}(V)$. We will denote $\mathcal{ZC}_k(V) = \mathcal{C}_k(V) \cap \operatorname{Ker}\delta$ and $\mathcal{BC}_k(V) = \mathcal{C}_k(V) \cap \operatorname{Im}\delta$.

Note that our further discussion will mainly rely on k-increments with $k \le 2$. For the simplicity of the exposition, we will assume from now on that $V = \mathbb{R}^d$. We measure the size of these increments by Hölder norms, which are defined in the following way: for $f \in \mathcal{C}_2(V)$ let $\|f\|_\mu = \sup_{s,t\in[0,T]} |f_{s,t}|/|t-s|^\mu$ and $\mathcal{C}_2^\mu(V) = \{f \in \mathcal{C}_2(V);\ \|f\|_\mu < \infty\}$. Obviously, the usual Hölder spaces $\mathcal{C}_1^\mu(V)$ are determined in the following way: for a continuous function $g \in \mathcal{C}_1(V)$, we simply set $\|g\|_\mu = \|\delta g\|_\mu$, and we say that $g \in \mathcal{C}_1^\mu(V)$ iff $\|g\|_\mu$ is finite. Note that $\|\cdot\|_\mu$ is only a semi-norm on $\mathcal{C}_1(V)$, but we will work in general on spaces of the type $\mathcal{C}_{1,a}^\mu(V) = \{g : [0, T] \to V;\ g_0 = a, \|g\|_\mu < \infty\}$, for a given $a \in V$, on which $\|g\|_\mu$ is a norm. For $h \in \mathcal{C}_3(V)$ we set in the same way

$$\|h\|_{\gamma,\rho} = \sup_{s,u,t\in[0,T]} \frac{|h_{s,u,t}|}{|u-s|^\gamma |t-u|^\rho} \quad \text{and} \quad \|h\|_\mu = \inf \sum_i \|h_i\|_{\rho_i,\mu-\rho_i}, \tag{5}$$

where the infimum is taken over all sequences $\{h_i \in \mathcal{C}_3(V)\}$ such that $h = \sum_i h_i$ and for all choices of the numbers $\rho_i \in (0, \mu)$. Then $\|\cdot\|_\mu$ is easily seen to be a norm on $\mathcal{C}_3(V)$, and we set $\mathcal{C}_3^\mu(V) := \{h \in \mathcal{C}_3(V);\ \|h\|_\mu < \infty\}$. Eventually, let $\mathcal{C}_3^{1+}(V) = \bigcup_{\mu>1} \mathcal{C}_3^\mu(V)$, and note that the same kind of norms can be considered on the spaces $\mathcal{ZC}_3(V)$, leading to the definition of the spaces $\mathcal{ZC}_3^\mu(V)$ and $\mathcal{ZC}_3^{1+}(V)$.

With these notations in mind, the crucial point in the presented approach to pathwise integration of irregular paths is that the operator δ can be inverted under mild smoothness assumptions. This inverse is called Λ. The proof of the following proposition can be found in [9]:

Proposition 2.1. *There exists a unique linear map* $\Lambda : \mathcal{ZC}_3^{1+}(V) \to \mathcal{C}_2^{1+}(V)$ *such that* $\delta\Lambda = \mathrm{Id}_{\mathcal{ZC}_3^{1+}(V)}$ *and* $\Lambda\delta = \mathrm{Id}_{\mathcal{C}_2^{1+}(V)}$. *In other words, for any* $h \in \mathcal{C}_3^{1+}(V)$ *such that* $\delta h = 0$ *there exists a unique* $g = \Lambda(h) \in \mathcal{C}_2^{1+}(V)$ *such that* $\delta g = h$. *Furthermore, for any* $\mu > 1$, *the map* Λ *is continuous from* $\mathcal{ZC}_3^\mu(V)$ *to* $\mathcal{C}_2^\mu(V)$ *and we have* $\|\Lambda h\|_\mu \le \frac{1}{2^\mu - 2}\|h\|_\mu$ *for any* $h \in \mathcal{ZC}_3^\mu(V)$.

Moreover, Λ has a nice interpretation in terms of generalized Young integrals:

Corollary 2.2. *For any* 1*-increment* $g \in \mathcal{C}_2(V)$ *such that* $\delta g \in \mathcal{C}_3^{1+}$, *set* $\delta f = (\mathrm{Id} - \Lambda\delta) g$. *Then* $(\delta f)_{s,t} = \lim_{|\Pi_{t,s}| \to 0} \sum_{i=0}^{n} g_{t_i, t_{i+1}}$, *where the limit is over any partition* $\Pi_{s,t} = \{t_0 = s, \ldots, t_n = t\}$ *of* $[s,t]$ *whose mesh tends to zero. Thus, the* 1*-increment* δf *is the indefinite integral of the* 1*-increment* g.

Now, we will give the definition of generalized integrals with respect to a rough path of order 2, and then explain how to solve Eq. (3). Notice that, in the sequel, we will use both the notations $\int_s^t f\,\mathrm{d}g$ or $\mathcal{J}_{s,t}(f\,\mathrm{d}g)$ for the integral of a function f with respect to a given increment $\mathrm{d}g$ on the interval $[s,t]$. The structure we will demand for a possible solution of (3) is as follows:

Definition 2.3. *Let* z *be a path in* $\mathcal{C}_1^\kappa(\mathbb{R}^k)$ *with* $\kappa \le \gamma$ *and* $2\kappa + \gamma > 1$. *We say that* z *is a controlled path based on* x *if* $z_0 = a$ *(which is a given initial condition in* $\mathbb{R}^k$*) and if* $\delta z \in \mathcal{C}_2^\kappa(\mathbb{R}^k)$ *can be decomposed into*

$$\delta z = \zeta\,\delta x + r, \qquad \textit{i.e.}\ (\delta z)_{s,t} = \zeta_s (\delta x)_{s,t} + \rho_{s,t}, \quad s,t \in [0,T], \tag{6}$$

with $\zeta \in \mathcal{C}_1^\kappa(\mathbb{R}^{k\times d})$ *and* ρ *is a regular part belonging to* $\mathcal{C}_2^{2\kappa}(\mathbb{R}^k)$. *The space of classical controlled paths will be denoted by* $\mathcal{Q}_{\kappa,a}(\mathbb{R}^k)$, *and a path* $z \in \mathcal{Q}_{\kappa,a}(\mathbb{R}^k)$ *should be considered in fact as a couple* (z,ζ). *The semi-norm on* $\mathcal{Q}_{\kappa,a}(\mathbb{R}^k)$ *is given by* $\mathcal{N}[z; \mathcal{Q}_{\kappa,a}(\mathbb{R}^k)] = \|z\|_\kappa + \|\zeta\|_\infty + \|\zeta\|_\kappa + \|\rho\|_{2\kappa}$ *with* $\|\zeta\|_\infty = \sup_{0\le s\le T} |\zeta_s|_{\mathbb{R}^{k\times d}}$.

Having defined our algebraic and analytic framework, the strategy in order to solve Eq. (3) is now as follows:

(i) Verify the stability of $\mathcal{Q}_{\kappa,a}(\mathbb{R}^k)$ under a smooth map $\varphi : \mathbb{R}^k \to \mathbb{R}^n$;
(ii) Define rigorously the integral $\int z_u\,\mathrm{d}x_u = \mathcal{J}(z\,\mathrm{d}x)$ for a controlled path z and compute its decomposition (6);
(iii) Solve Eq. (3) in the space $\mathcal{Q}_{\kappa,a}(\mathbb{R}^k)$ by a fixed point argument.

Actually, for the second point we have to assume the following hypothesis on the driving rough path, which is standard in rough path type considerations:

Hypothesis 2.4. *The* $\mathbb{R}^d$*-valued* γ*-Hölder path* x *admits a Lévy area, that is a process* $\mathbf{x}^2 \in \mathcal{C}_2^{2\gamma}(\mathbb{R}^{d\times d})$ *satisfying*

$$\delta\mathbf{x}^2 = \delta x \otimes \delta x, \qquad \textit{i.e.}\ \big[\big(\delta\mathbf{x}^2\big)_{s,u,t}\big](i,j) = \big[\delta x^i\big]_{s,u}\big[\delta x^j\big]_{u,t}, \quad s,u,t \in [0,T],\ i,j \in \{1,\ldots,d\}.$$

Then, we have

Proposition 2.5. *Let* $z \in \mathcal{Q}_{\kappa,a}(\mathbb{R}^k)$ *with decomposition* (6), $\varphi \in C^2(\mathbb{R}^k;\mathbb{R}^n)$ *be bounded with bounded derivatives and set* $\hat{z} = \varphi(z)$, $\hat{a} = \varphi(a)$. *Then* $\hat{z} \in \mathcal{Q}_{\kappa,\hat{a}}(\mathbb{R}^n)$. *Moreover, it can be decomposed into* $\delta\hat{z} = \hat{\zeta}\,\delta x + \hat{r}$, *with* $\hat{\zeta} = \nabla\varphi(z)\zeta$ *and* $\hat{r} = \nabla\varphi(z) r + [\delta(\varphi(z)) - \nabla\varphi(z)\delta z]$. *Furthermore, it holds*

$$\mathcal{N}\big[\hat{z}; \mathcal{Q}_{\kappa,\hat{a}}\big(\mathbb{R}^n\big)\big] \le c_{\varphi,T}\big(1 + \mathcal{N}^2\big[z; \mathcal{Q}_{\kappa,a}\big(\mathbb{R}^n\big)\big]\big). \tag{7}$$

Concerning the integration of controlled paths, we have

Proposition 2.6. *For a given* $\gamma > 1/3$ *and* $\kappa < \gamma$, *let* x *be a process satisfying Hypothesis* 2.4. *Furthermore, let* $m \in \mathcal{Q}_{\kappa,b}(\mathcal{L}^{d,1})$ *with decomposition* $m_0 = b \in \mathcal{L}^{d,1}$ *and*

$$(\delta m)_{s,t} = \big[\mu_s(\delta x)_{s,t}\big]^* + r_{s,t}, \quad \textit{where}\ \mu \in \mathcal{C}_1^\kappa\big(\mathcal{L}^{d,d}\big), r \in \mathcal{C}_2^{2\kappa}\big(\mathcal{L}^{d,1}\big). \tag{8}$$

Define z *by* $z_0 = a \in \mathbb{R}$ *and* $\delta z = m\delta x + \mu\cdot\mathbf{x}^2 + \Lambda(r\delta x + \delta\mu\cdot\mathbf{x}^2)$. *Finally, set* $\mathcal{J}(m\,\mathrm{d}x) = \delta z$. *Then* z *is well-defined as an element of* $\mathcal{Q}_{\kappa,a}(\mathbb{R})$. *Moreover, the semi-norm of* z *in* $\mathcal{Q}_{\kappa,a}(\mathbb{R})$ *can be estimated as*

$$\mathcal{N}\big[z; \mathcal{Q}_{\kappa,a}(\mathbb{R})\big] \le c\big[\|x\|_\gamma + \big\|\mathbf{x}^2\big\|_{2\gamma}\big]\big(\|m\|_\infty + T^{\gamma-\kappa}\mathcal{N}\big[m; \mathcal{Q}_{\kappa,b}\big(\mathcal{L}^{d,1}\big)\big]\big), \tag{9}$$

for a universal constant c. *We also have*

$$\|\delta z\|_\kappa \le c\big[\|x\|_\gamma + \big\|\mathbf{x}^2\big\|_{2\gamma}\big]T^{\gamma-\kappa}\mathcal{N}\big[m; \mathcal{Q}_{\kappa,b}\big(\mathcal{L}^{d,1}\big)\big]. \tag{10}$$

Finally, it holds $\mathcal{J}_{s,t}(m\,\mathrm{d}x) = \lim_{|\Pi_{s,t}|\to 0}\sum_{i=0}^{n}[m_{t_i}(\delta x)_{t_i,t_{i+1}} + \mu_{t_i}\cdot \mathbf{x}^2_{t_i,t_{i+1}}]$ *for any* $0 \le s < t \le T$, *where the limit is taken over all partitions* $\Pi_{s,t} = \{s = t_0, \dots, t_n = t\}$ *of* $[s,t]$ *whose mesh tends to zero.*

Finally, we can prove the following result, using the strategy and the tools sketched above (see [9], Proposition 8):

Theorem 2.7. *Let* x *be a process satisfying Hypothesis* 2.4. *Let* $b:\mathbb{R}^n \to \mathbb{R}^n$ *and* $\sigma:\mathbb{R}^n \to \mathbb{R}^{n\times d}$ *be twice continuously differentiable and assume moreover that* σ *and* b *are bounded together with their derivatives. Then*:

1. *Equation* (3) *admits a unique solution* y *in* $\mathcal{Q}_{\kappa,a}(\mathbb{R}^n)$ *for any* $\kappa < \gamma$ *such that* $2\kappa + \gamma > 1$.
2. *The mapping* $(a, x, \mathbf{x}^2) \mapsto y$ *is continuous from* $\mathbb{R}^n \times \mathcal{C}_1^{\gamma}(\mathbb{R}^d) \times \mathcal{C}_2^{2\gamma}(\mathbb{R}^{d\times d})$ *to* $\mathcal{Q}_{\kappa,a}(\mathbb{R}^n)$.

In the sequel, it will also be crucial to have a change of variable formula. This will be achieved under the following additional assumption on $\mathbf{x}^2$:

Hypothesis 2.8. *Let* $\mathbf{x}^2$ *be the area process defined in Hypothesis* 2.4 *and denote by* $\mathbf{x}^{2,s}$ *the symmetric part of* $\mathbf{x}^2$, *i.e.* $\mathbf{x}^{2,s} = 1/2(\mathbf{x}^2 + (\mathbf{x}^2)^*)$. *Then, we assume that we have* $\mathbf{x}^{\mathbf{2,s}}_{s,t} = 1/2[\delta x]_{s,t} \otimes [\delta x]_{s,t}$ *for* $0 \le s < t \le T$.

Our change of variable formula, whose proof is left to the reader for the sake of conciseness, reads as follows:

Proposition 2.9. *Assume that* x *satisfies both Hypothesis* 2.4 *and* 2.8. *Let* $b:\mathbb{R}^n \to \mathbb{R}^n$ *and* $\sigma:\mathbb{R}^n \to \mathcal{L}^{d,n}$ *be twice continuously differentiable and assume moreover that* σ *and* b *are bounded together with their derivatives. Let* y *be the unique solution to* (3) *given by Theorem* 2.7. *If* $f \in C^2(\mathbb{R}^n;\mathbb{R})$ *is bounded together with its derivatives, then* $f(y_t)$ *can be decomposed, for any* $t \in [0,T]$, *as*

$$f(y_t) = f(a) + \int_0^t \nabla f(y_s) b(y_s)\,\mathrm{d}s + \int_0^t \nabla f(y_s)\sigma(y_s)\,\mathrm{d}x_s. \tag{11}$$

2.2. *Application to fractional Brownian motion*

A d-dimensional fractional Brownian motion (fBm in short) with Hurst parameter H is a centered Gaussian process which can be written as $B = \{B_t = (B_t^1, \dots, B_t^d); t \ge 0\}$, where $B^1, \dots, B^d$ are d independent one-dimensional fBm. The fBm verifies the following two important properties:

$$\text{(scaling)} \qquad \text{For any } c > 0, \quad B^{(c)} = c^H B_{\cdot/c} \text{ is a fBm,} \tag{12}$$

$$\text{(stationary increments)} \qquad \text{For any } h > 0, \quad B_{\cdot+h} - B_h \text{ is a fBm.} \tag{13}$$

All the results of the previous Section 2.1 rely on the specific assumptions we have made on the process x. For fBm, it can be checked (see, for instance, [12]) that

Proposition 2.10. *Let* B *be a* d*-dimensional fractional Brownian motion and suppose* $H > 1/3$. *Then almost all sample paths of* B *satisfy Hypothesis* 2.4 *and* 2.8.

2.3. *Rooted trees*

In order to obtain a compact representation of the expansion of P_t we will use rooted trees following the approach in [21,22].

Definition 2.11. *A monotonically labelled S-tree (stochastic tree)* $\mathbf{t}$ *with* $l = l(\mathbf{t}) \in \mathbb{N}$ *nodes is a pair of maps* $\mathbf{t} = (\mathbf{t}', \mathbf{t}'')$ *of the form* $\mathbf{t}' : \{2, \dots, l\} \longrightarrow \{1, \dots, l-1\}$ *and* $\mathbf{t}'' : \{1, \dots, l\} \longrightarrow \mathcal{A}$, *with* $\mathcal{A} = \{\gamma, \tau_0, \tau_{j_k}, k \in \mathbb{N}\}$ *where* j_k *is a variable index with* $j_k \in \{1, \dots, d\}$, *such that* $\mathbf{t}'(i) < i$, $\mathbf{t}''(1) = \gamma$ *and* $\mathbf{t}''(i) \in \mathcal{A}\setminus\{\gamma\}$ *for* $i = 2, \dots, l$. *Let LTS denote the set of all monotonically labelled S-trees.*

We will use the following notation: $d(\mathbf{t}) = |\{i : \mathbf{t}''(i) = \tau_0\}|$, $s(\mathbf{t}) = |\{i : \mathbf{t}''(i) = \tau_{j_k}, j_k \neq 0\}| = l(\mathbf{t}) - d(\mathbf{t}) - 1$ and $\rho(\mathbf{t}) = Hs(\mathbf{t}) + d(\mathbf{t})$, with $\rho(\gamma) = 0$. In the following we also denote by $LTS(S) \subset LTS$, where (S) stands for Stratonovich, the subset $LTS(S) = \{\mathbf{t} \in LTS : s(\mathbf{t}) = 2k, k \in \mathbb{N}_0\}$ with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ containing all trees having an even number of stochastic nodes.

Every monotonically labelled S-tree $\mathbf{t}$ can be represented as a graph, whose nodes are elements of $\{1, \ldots, l(\mathbf{t})\}$ and whose arcs are the pairs $(\mathbf{t}'(i), i)$ for $i = 2, \ldots, l(\mathbf{t})$. Here, $\mathbf{t}'$ defines a father-son relation between the nodes, i.e., $\mathbf{t}'(i)$ is the father of the son i. Further, $\gamma = \otimes$ denotes the root, $\tau_0 = \bullet$ is a deterministic node and $\tau_{j_k} = \bigcirc_{j_k}$ a stochastic node. Here, we have to point out that each tree $\mathbf{t} \in LTS$ depends on the variable indices $j_1, \ldots, j_{s(\mathbf{t})} \in \{1, \ldots, d\}^{s(\mathbf{t})}$, although this is not mentioned explicitly if we shortly write $\mathbf{t}$ for the tree.

Definition 2.12. *If $\mathbf{t}_1, \ldots, \mathbf{t}_k$ are coloured trees, then we denote by $(\mathbf{t}_1, \ldots, \mathbf{t}_k)$, $[\mathbf{t}_1, \ldots, \mathbf{t}_k]$ and $\{\mathbf{t}_1, \ldots, \mathbf{t}_k\}_j$ the tree in which $\mathbf{t}_1, \ldots, \mathbf{t}_k$ are each joined by a single branch to $\otimes$, $\bullet$ and $\bigcirc_j$, respectively (see also Fig. 2).*

Therefore proceeding recursively, for the two examples $\mathbf{t}_I$ and $\mathbf{t}_{II}$ in Fig. 1 we obtain $\mathbf{t}_I = ([\bigcirc_{j_2}^4]^2, \bigcirc_{j_1}^3)^1 = ([\tau_{j_2}^4]^2, \tau_{j_1}^3)^1$ and $\mathbf{t}_{II} = (\{\bullet^4, \bigcirc_{j_2}^3\}_{j_1}^2)^1 = (\{\tau_0^4, \tau_{j_2}^3\}_{j_1}^2)^1$.

For every rooted tree $\mathbf{t} \in LTS$, there exists a corresponding elementary differential which is a direct generalization of the differential in the deterministic case, see also [21]. The elementary differential is defined recursively for some $x \in \mathbb{R}^n$ by $F(\gamma)(x) = f(x)$, $F(\tau_0)(x) = b(x)$, $F(\tau_j)(x) = \sigma^j(x)$, for single nodes and by

$$F(\mathbf{t})(x) = \begin{cases} f^{(k)}(x) \cdot \big(F(\mathbf{t}_1)(x), \ldots, F(\mathbf{t}_k)(x)\big) & \text{for } \mathbf{t} = (\mathbf{t}_1, \ldots, \mathbf{t}_k), \\ b^{(k)}(x) \cdot \big(F(\mathbf{t}_1)(x), \ldots, F(\mathbf{t}_k)(x)\big) & \text{for } \mathbf{t} = [\mathbf{t}_1, \ldots, \mathbf{t}_k], \\ \sigma^{j\,(k)}(x) \cdot \big(F(\mathbf{t}_1)(x), \ldots, F(\mathbf{t}_k)(x)\big) & \text{for } \mathbf{t} = \{\mathbf{t}_1, \ldots, \mathbf{t}_k\}_j, \end{cases} \tag{14}$$

for a tree $\mathbf{t}$ with more than one node and with $\sigma^j = (\sigma^{i,j})_{1 \le i \le n}$ denoting the jth column of the diffusion matrix σ. Here $f^{(k)}$, $b^{(k)}$ and $\sigma^{j\,(k)}$ define a symmetric k-linear differential operator, and one can choose the sequence of labelled S-trees $\mathbf{t}_1, \ldots, \mathbf{t}_k$ in an arbitrary order. For example, the Ith component of $b^{(k)} \cdot (F(\mathbf{t}_1), \ldots, F(\mathbf{t}_k))$ can be written as

$$\big(b^{(k)} \cdot \big(F(\mathbf{t}_1), \ldots, F(\mathbf{t}_k)\big)\big)^I = \sum_{J_1, \ldots, J_k = 1}^{n} \frac{\partial^k b^I}{\partial x^{J_1} \cdots \partial x^{J_k}} \big(F^{J_1}(\mathbf{t}_1), \ldots, F^{J_k}(\mathbf{t}_k)\big),$$

where the components of vectors are denoted by superscript indices, which are chosen as capitals. As a result of this we get for $\mathbf{t}_I$ and $\mathbf{t}_{II}$ the elementary differentials

$$F(\mathbf{t}_I) = f''\big(b'(\sigma^{j_2}), \sigma^{j_1}\big) = \sum_{J_1, J_2 = 1}^{n} \frac{\partial^2 f}{\partial x^{J_1} \partial x^{J_2}} \left(\sum_{K_1 = 1}^{n} \frac{\partial b^{J_1}}{\partial x^{K_1}} \sigma^{K_1, j_2} \cdot \sigma^{J_2, j_1} \right),$$

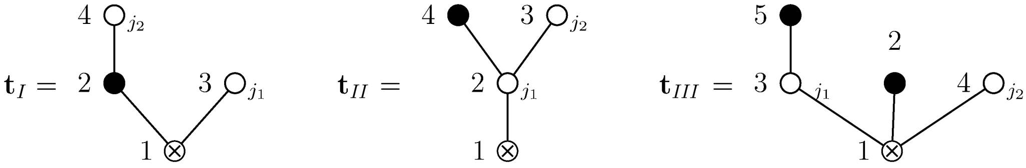

Fig. 1. Some monotonically labelled trees in *LTS*.

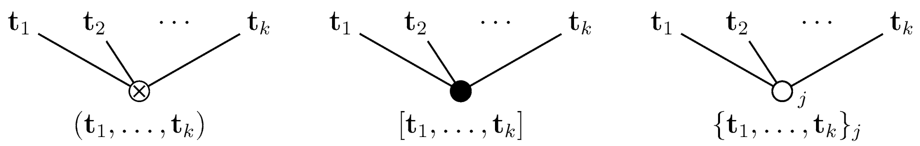

Fig. 2. Writing a coloured S-tree with brackets.

$$F(\mathbf{t}_{II}) = f'\big(\sigma^{j_1 \prime\prime}\big(b, \sigma^{j_2}\big)\big) = \sum_{J_1=1}^{n} \frac{\partial f}{\partial x^{J_1}} \left(\sum_{K_1, K_2=1}^{n} \frac{\partial^2 \sigma^{J_1, j_1}}{\partial x^{K_1}\, \partial x^{K_2}} b^{K_1} \cdot \sigma^{K_2, j_2} \right).$$

Next, we assign recursively to every $\mathbf{t} \in LTS$ a multiple stochastic integral by

$$\mathcal{I}_{\mathbf{t}}\big(g\big(X_s^a\big)\big)_{t_0,t} = \begin{cases} g\big(X_t^a\big) & \text{if } \mathbf{t}''\big(l(\mathbf{t})\big) = \gamma, \\ \int_{t_0}^{t} \mathcal{I}_{\mathbf{t}-}\big(g\big(X_u^a\big)\big)_{t_0,s}\, \mathrm{d}B_s^j & \text{if } \mathbf{t}''\big(l(\mathbf{t})\big) = \tau_j, \end{cases} \tag{15}$$

for $0 \le j \le d$ with $\mathrm{d}B_s^0 = \mathrm{d}s$. Here, $\mathbf{t}-$ denotes the tree which is obtained from $\mathbf{t}$ by removing the last node with label $l(\mathbf{t})$.

3. Main result

We will denote by $C_b^\infty(\mathbb{R}^n; \mathbb{R})$ the space of all infinitely differentiable functions $g \in C^\infty(\mathbb{R}^n; \mathbb{R})$ which have bounded derivatives of all orders, and by $C_P^\infty(\mathbb{R}^n; \mathbb{R})$ the space of all infinitely differentiable functions $g \in C^\infty(\mathbb{R}^n; \mathbb{R})$ for which all partial derivatives have polynomial growth. (Recall that a function $h : \mathbb{R}^n \to \mathbb{R}$ is of polynomial growth, if there exists $c > 0$ and $q \in \mathbb{N}$ such that $|h(x)| \le c(1 + |x|^q)$ for all $x \in \mathbb{R}^n$.)

Moreover set $\mathcal{A}_m = \{0, 1, \dots, d\}^m$ for $m \in \mathbb{N}$, and define the differential operators $\mathcal{D}^0$ and $\mathcal{D}^j$ as

$$\mathcal{D}^0 = \sum_{k=1}^{n} b^k \frac{\partial}{\partial x^k} \quad \text{and} \quad \mathcal{D}^j = \sum_{k=1}^{n} \sigma^{k,j} \frac{\partial}{\partial x^k} \tag{16}$$

for $j = 1, \dots, d$. Finally, set $\mathcal{D}^\alpha = \mathcal{D}^{\alpha_1} \cdots \mathcal{D}^{\alpha_m}$ for a multi-index $\alpha \in \mathcal{A}_m$.

Recall that the family of operators $(P_t, t \in [0, T])$ has been defined by (2). To give the expansion of P_t we will use different sets of assumptions on the function $f : \mathbb{R}^n \to \mathbb{R}$, the drift vector $b = (b^i)_{i=1,\dots,n}$ and the diffusion matrix $\sigma = (\sigma^{i,j})_{i=1,\dots,n,\, j=1,\dots,d}$.

In the case $1/3 < H < 1/2$ we will assume that:

(A1) We have $f, b^i, \sigma^{i,j} \in C_b^\infty(\mathbb{R}^n; \mathbb{R})$ for $i = 1, \dots, n$, $j = 1, \dots, d$. Moreover, $f, b^i, \sigma^{i,j}$, $i = 1, \dots, n$, $j = 1, \dots, d$ are bounded.

When $H > 1/2$ we will work with the following assumptions:

(A2) We have $f \in C_P^\infty(\mathbb{R}^n; \mathbb{R})$ and $b^i, \sigma^{i,j} \in C_b^\infty(\mathbb{R}^n; \mathbb{R})$ for $i = 1, \dots, n$, $j = 1, \dots, d$. Moreover, the following regularity condition for the Malliavin derivative of the solution of Eq. (1) holds:

$$\mathcal{Y} = \max_{i=1,\dots,n} \max_{j=1,\dots,d} \sup_{0 \le u \le s \le T} \mathrm{E}\big|D_u^j X_s^i\big|^4 < \infty.$$

Here D^j denotes the Malliavin derivative with respect to the jth component of the driving fBm, see Section 4.4.

The following theorem gives the expression of the expansion of P_t with respect to t:

Theorem 3.1. (1) *If* $1/3 < H < 1/2$ *and assumption* (A1) *is satisfied, then for any* $m \in \mathbb{N}_0$:

$$\mathrm{P}_t f(a) = \sum_{\substack{\mathbf{t} \in LTS(S) \\ l(\mathbf{t}) \le m+1}} \sum_{j_1, \dots, j_{s(\mathbf{t})}=1}^{d} F(\mathbf{t})(a) \mathrm{E}\big(\mathcal{I}_{\mathbf{t}}(1)_{0,1}\big) t^{\rho(\mathbf{t})} + \mathrm{O}\big(t^{(m+1)H}\big), \quad \text{as } t \to 0. \tag{17}$$

(2) *Let* $H > 1/2$ *and assumption* (A2) *be satisfied. Then, for all* $t \in [0, T]$, *we have*

$$\left| \mathrm{P}_t f(a) - \sum_{\substack{\mathbf{t} \in LTS(S) \\ l(\mathbf{t}) \le m+1}} \sum_{j_1, \dots, j_{s(\mathbf{t})}=1}^{d} F(\mathbf{t})(a) \mathrm{E}\big(\mathcal{I}_{\mathbf{t}}(1)_{0,1}\big) t^{\rho(\mathbf{t})} \right| \le \mathcal{G}_m (1 + \mathcal{Y}) \frac{K^m t^{H(m+1)}}{\sqrt{m!}},$$

$$\text{where } \mathcal{G}_m^2 = \sup_{\alpha\in\mathcal{A}_{m+1}} \sup_{0\le t\le T} \mathrm{E}\big|\mathcal{D}^\alpha f(X_t)\big|^2 + \max_{i=1,\dots,n} \sup_{\alpha\in\mathcal{A}_{m+1}} \sup_{0\le t\le T} \mathrm{E}\left|\frac{\partial}{\partial x_i}\mathcal{D}^\alpha f(X_t)\right|^2 \tag{18}$$

and K is a constant, which depends only on H, T, n and d. In particular, if

$$\mathcal{G}_m = \mathrm{O}\big((m!)^\kappa\big) \tag{19}$$

with $\kappa \in [0, 1/2)$, then we have

$$\mathrm{P}_t f(a) = \sum_{\mathbf{t}\in LTS(S)} \sum_{j_1,\dots,j_{s(\mathbf{t})}=1}^{d} F(\mathbf{t})(a)\mathrm{E}\big(\mathcal{I}_\mathbf{t}(1)_{0,1}\big)t^{\rho(\mathbf{t})}, \quad t\in[0,T]. \tag{20}$$

Remark 3.2. (1) *Here, note that each tree $\mathbf{t}\in LTS(S)$ comprehends the variable indices $j_1,\dots,j_{s(\mathbf{t})}$ which can take the values $1,\dots,d$ although these variables are not mentioned explicitly by writing shortly $\mathbf{t}$ for the whole tree. The variables $j_1,\dots,j_{s(\mathbf{t})}$ correspond to the components of the driving fractional Brownian motion and appear in the second sum in the formulas* (17) *and* (20) *as well as in each tree $\mathbf{t}$ of the summands.*

(2) *The growth condition* (19) *and the regularity condition on the Malliavin derivative, i.e. $\mathcal{Y}<\infty$, are satisfied, e.g., if f is a polynomial and $b^i, \sigma^{i,j}$, $i=1,\dots,n$, $j=1,\dots,d$ are affine functions. Moreover, the growth condition* (19) *for the remainder term is also natural in the case $H=1/2$, i.e. for the asymptotic expansion of Itô stochastic differential equations. Compare, e.g.,* [3] *and chapter* 5 *in* [6].

4. Proof of Theorem 3.1

In the present section we will prove our main result, that is Theorem 3.1. We separate the proof into two parts: firstly, we will show how to use trees for the parametrization of the expansion; secondly, we will control the remainder term, which appears when we expand $P_t f(a)$ with respect to t, according to the value of H.

4.1. *Rooted trees approach*

In this section, we assume that the Hurst index of the fBm B verifies $H > 1/3$. The first step in the proof of the algebraic part of Theorem 3.1 is the following result.

Theorem 4.1. *Let $(X_t^a)_{t\in[0,T]}$ be the solution of* (1) *with initial value $X_0^a = a\in\mathbb{R}^n$. Then for $m\in\mathbb{N}_0$ and $f\in C_b^{m+2}(\mathbb{R}^n;\mathbb{R})$, $b, \sigma^j \in C_b^{m+2}(\mathbb{R}^n;\mathbb{R}^n)$, $1\le j\le d$, we get for $t\in[0,T]$ the expansion $f(X_t^a) = \sum_{\mathbf{t}\in LTS, l(\mathbf{t})-1\le m} \times \sum_{j_1,\dots,j_{s(\mathbf{t})}=1}^d F(\mathbf{t})(a)\mathcal{I}_\mathbf{t}(1)_{0,t} + \mathcal{R}_m(0,t)$ with a truncation term $\mathcal{R}_m(0,t) = \sum_{\mathbf{t}\in LTS, l(\mathbf{t})-1=m+1} \times \sum_{j_1,\dots,j_{s(\mathbf{t})}=1}^d \mathcal{I}_\mathbf{t}(F(\mathbf{t})(X_s^a))_{0,t}$.*

Proof. The proof is very similar to the proof of Theorem 4.2 in [21]. Details are left to the reader. □

Corollary 4.2. *Let $(X_t^a)_{t\in I}$ be the solution of* (1) *with initial value $X_0^a = a\in\mathbb{R}^n$. Then for $m\in\mathbb{N}_0$ and $f\in C_b^{m+2}(\mathbb{R}^n,\mathbb{R})$, $b,\sigma^j\in C_b^{m+2}(\mathbb{R}^n,\mathbb{R}^n)$, $1\le j\le d$, we get for $t\in[0,T]$ the expansion*

$$\mathrm{P}_t f(a) = \sum_{\substack{\mathbf{t}\in LTS(S)\\ l(\mathbf{t})\le m+1}} \sum_{j_1,\dots,j_{s(\mathbf{t})}=1}^{d} F(\mathbf{t})(a)\mathrm{E}\big(\mathcal{I}_\mathbf{t}(1)_{0,1}\big)t^{\rho(\mathbf{t})} + \mathrm{E}\big(\mathcal{R}_m(0,t)\big). \tag{21}$$

Proof. Apply Theorem 4.1 and take the expectation in the formula for $f(X_t^a)$. Moreover, due to the scaling property (12), it follows that $\mathrm{E}(\mathcal{I}_\mathbf{t}(1)_{0,t}) = \mathrm{E}(\mathcal{I}_\mathbf{t}(1)_{0,1})t^{H|\alpha|+m-|\alpha|} = \mathrm{E}(\mathcal{I}_\mathbf{t}(1)_{0,1})t^{\rho(\mathbf{t})}$ holds with $|\alpha| = \sum_{i=1}^m 1_{\{\alpha_i\neq 0\}}$, since we have $\rho(\mathbf{t}) = H|\alpha|+m-|\alpha|$. □

Now we derive the announced controls on the remainder term $\mathrm{E}(\mathcal{R}_m(0,t))$ appearing in (21), according to the value of H and the assumptions on f, b and σ. These estimates will imply easily our Theorem 3.1.

4.2. *Study of the remainder term for* $1/3 < H < 1/2$

We assume in this section that $1/3 < H < 1/2$ and that assumption (A) holds true. Then we will show that for fixed $m \in \mathbb{N}$, we have $\mathrm{E}(\mathcal{R}_m(0,t)) = \mathrm{O}(t^{(m+1)H})$, a fact, which trivially yields (20) in Theorem 3.1. Furthermore, this bound is a direct consequence of the following:

Lemma 4.3. *Let* $g \in \mathrm{C}^2(\mathbb{R}^n, \mathbb{R})$ *be bounded with bounded derivatives and* X *be the unique solution to* (1) *in* $\mathcal{Q}_{\kappa,a}(\mathbb{R}^n)$ *with* $\kappa \in (\frac{1-H}{2}, H)$. *For any* $\alpha_1, \dots, \alpha_r \in \{0, \dots, d\}$ *we have*

$$\mathrm{E}\left|\int_{\Delta^r([0,t])} g(X_{t_r})\,\mathrm{d}B_{t_r}^{\alpha_r}\cdots\mathrm{d}B_{t_1}^{\alpha_1}\right| = \mathrm{O}\big(t^{r-|\alpha|(1-H)}\big), \quad \text{as } t \to 0, \tag{22}$$

where $|\alpha| = \sum_{i=1}^r 1_{\{\alpha_i \neq 0\}}$.

Proof. We only consider the case $r = |\alpha|$, the other cases being easier. We split the proof into three steps.

Step 1: Scaling. For $j \in \{1, \dots, r\}$ and $c > 0$, set $B_u^{\alpha_j,(c)} = c^H B_{u/c}^{\alpha_j}$ and let $X^{(c)}$ denote the solution of (1), where B is replaced by $B^{(c)}$. For fixed t, we have

$$\begin{aligned}\int_{\Delta^r([0,t])} g(X_{t_r})\,\mathrm{d}B_{t_r}^{\alpha_r}\cdots\mathrm{d}B_{t_1}^{\alpha_1} &= \int_0^t \mathrm{d}B_{t_1}^{\alpha_1}\int_0^{t_1}\mathrm{d}B_{t_2}^{\alpha_2}\cdots\int_0^{t_{r-1}}\mathrm{d}B_{t_r}^{\alpha_r}\,g(X_{t_r})\\ &= \int_0^1 \mathrm{d}B_{t\cdot t_1}^{\alpha_1}\int_0^{t_1}\mathrm{d}B_{t\cdot t_2}^{\alpha_2}\cdots\int_0^{t_{r-1}}\mathrm{d}B_{t\cdot t_r}^{\alpha_r}\,g(X_{t\cdot t_r})\\ &\stackrel{\mathcal{L}}{=} t^{rH}\int_0^1 \mathrm{d}B_{t_1}^{\alpha_1,(t)}\int_0^{t_1}\mathrm{d}B_{t_2}^{\alpha_2,(t)}\cdots\int_0^{t_{r-1}}\mathrm{d}B_{t_r}^{\alpha_r,(t)}\,g\big(X_{t\cdot t_r}^{(t)}\big).\end{aligned}$$

Consequently, in order to obtain (22), it suffices to prove that

$$\sup_{t\in[0,T]}\mathrm{E}\left|\int_0^1 \mathrm{d}B_{t_1}^{\alpha_1}\int_0^{t_1}\mathrm{d}B_{t_2}^{\alpha_2}\cdots\int_0^{t_{r-1}}\mathrm{d}B_{t_r}^{\alpha_r}\,g\big(X_{t\cdot t_r}^{(t)}\big)\right| < \infty.$$

Step 2: Fix t and set $z_s = \int_0^s \mathrm{d}B_{t_r}^{\alpha_r}\,g(X_{t\cdot t_r}^{(t)})$ for $s \in [0,1]$. Recall also our notation for norms in the spaces $\mathcal{Q}$ given at Definition 2.3. By (7) and (9) we have

$$\mathcal{N}[z;\mathcal{Q}_{\kappa,0}] \le c_B\big(1+\mathcal{N}\big[g\big(X_{t\cdot}^{(t)}\big);\mathcal{Q}_{\kappa,g(a)}\big]\big) \le c_B c_g\big(1+\mathcal{N}^2\big[X_{t\cdot}^{(t)};\mathcal{Q}_{\kappa,a}\big]\big).$$

Here, $c_B > 1$ is the random constant appearing in (9), whose value will not change from line to line, while c_g denotes a non-random constant depending only on g, whose value can change from one line to another. Set now

$$q_s = \int_0^s \mathrm{d}B_{t_{r-1}}^{\alpha_{r-1}}\int_0^{t_{r-1}}\mathrm{d}B_{t_r}^{\alpha_r}\,g\big(X_{t\cdot t_r}^{(t)}\big) = \int_0^s \mathrm{d}B_{t_{r-1}}^{\alpha_{r-1}}z_{t_{r-1}}, \quad s\in[0,1].$$

Similarly, we have $\mathcal{N}(q,\mathcal{Q}_{\kappa,0}) \le c_B(1+\mathcal{N}(z,\mathcal{Q}_{\kappa,0})) \le c_B{}^2 c_g(1+\mathcal{N}^2(X_{t\cdot}^{(t)},\mathcal{Q}_{\kappa,a}))$. By induction, we easily deduce that

$$\mathcal{N}\left(\int_0^{\cdot}\mathrm{d}B_{t_1}^{\alpha_1}\int_0^{t_1}\mathrm{d}B_{t_2}^{\alpha_2}\cdots\int_0^{t_{r-1}}\mathrm{d}B_{t_r}^{\alpha_r}\,g\big(X_{t\cdot t_r}^{(t)}\big),\mathcal{Q}_{\kappa,0}\right) \le c_B{}^r c_g\big(1+\mathcal{N}^2\big(X_{t\cdot}^{(t)},\mathcal{Q}_{\kappa,a}\big)\big).$$

Since we have $|z_1| \le \|z\|_\kappa \le \mathcal{N}(z,\mathcal{Q}_{\kappa,0})$ for a path z starting from 0, we deduce from the Cauchy–Schwarz inequality that (22) is in fact a consequence of showing $\mathrm{E}(c_B{}^{2r}) < \infty$ and $\sup_{t\in[0,T]}\mathrm{E}|\mathcal{N}^4(X_{t\cdot}^{(t)},\mathcal{Q}_{\kappa,a})| = \sup_{t\in[0,T]}\mathrm{E}|\mathcal{N}^4(X_{t\cdot},\mathcal{Q}_{\kappa,a})| < \infty$.

Step 3: Using that B has moments of all order, we easily obtain by (10) and [9], Corollary 4 (p. 119) that $\mathrm{E}(c_B{}^{2r}) < \infty$. So, let us consider the second condition. We will only prove that $\mathrm{E}|\mathcal{N}^4(X,\mathcal{Q}_{\kappa,a})| < +\infty$, the general case being similar. Recall from the proof of [9], Proposition 7 (p. 113) that X defined on $[0,\tau]$ belongs by definition to the ball $B_M=\{z; z_0=a, \mathcal{N}[z;\mathcal{Q}_{\kappa,a}]\le M\}$, where M and τ verify $M\ge c_{\sigma,B}(1+\tau^{\gamma-\kappa}M^2)$ with $c_{\sigma,B}$ a constant depending only on σ and B. For fixed τ, the inequality $u\ge c_{\sigma,B}(1+\tau^{\gamma-\kappa}u^2)$ admits solutions u iff ${c_{\sigma,B}}^{-2}-4\tau^{\gamma-\kappa}>0$, i.e., iff $\tau^{\gamma-\kappa}<(4{c_{\sigma,B}}^2)^{-1}$. In this case, the solutions are $u\in[M_-,M_+]$, for $M_\pm=(c_{\sigma,B}^{-1}\pm({c_{\sigma,B}}^{-2}-4\tau^{\gamma-\kappa})^{1/2})/(2\tau^{\gamma-\kappa})$. By choosing for instance $\tau^{\gamma-\kappa}=(8{c_{\sigma,B}}^2)^{-1}$, we obtain that $\mathcal{N}(X_{|[0,\tau]},\mathcal{Q}_{\kappa,a})\le(4+2\sqrt{2})c_{\sigma,B}$. Furthermore, due to the crucial fact that σ and its derivatives are bounded, we can in fact choose the same M for the bound of δX on $[\tau,2\tau]$, $[2\tau,3\tau]$, etc. Using the triangle inequality we deduce:

$$\begin{aligned}\mathcal{N}(X,\mathcal{Q}_{\kappa,a})&\le\mathcal{N}(X_{|[0,\tau]},\mathcal{Q}_{\kappa,a})+\mathcal{N}(X_{|[\tau,2\tau]},\mathcal{Q}_{\kappa,X_\tau})+\cdots+\mathcal{N}(X_{|[\lfloor T\tau^{-1}\rfloor\tau,T]},\mathcal{Q}_{\kappa,X_{\lfloor T\tau^{-1}\rfloor\tau}})\\&\le\big(\lfloor T\tau^{-1}\rfloor+1\big)M.\end{aligned}$$

In other words, we have $\mathcal{N}(X,\mathcal{Q}_{\kappa,a})\le \mathrm{cst}\, {c_{\sigma,B}}^{1+2/(\gamma-\kappa)}$. Thus it follows easily that the expectation $\mathrm{E}|\mathcal{N}^4(X,\mathcal{Q}_{\kappa,a})|$ is finite, and the proof of Lemma 4.3 is finished. □

4.3. *Some properties of iterated integrals in the case $H>1/2$*

In this section, we will assume that the Hurst index of B verifies $H>1/2$. Again, the key point in order to prove the second part of Theorem 3.1 will be to obtain a bound for $\mathrm{E}[\mathcal{R}_m(0,t)]$. Here, we will use estimates based on Malliavin calculus tools and explicit computations of moments of iterated integrals with respect to the fractional Brownian motion.

First, let us give a few facts about the Gaussian structure of fractional Brownian motion and its Malliavin derivative process, following Chapter 1.2 in [16] and Section 2 in [18]. Let $\mathcal{E}$ be the set of step-functions on $[0,T]$ with values in $\mathbb{R}^d$. Consider the Hilbert space $\mathcal{H}$ defined as the closure of $\mathcal{E}$ with respect to the scalar product induced by

$$\big\langle(\mathbf{1}_{[0,t_1]},\dots,\mathbf{1}_{[0,t_d]}),(\mathbf{1}_{[0,s_1]},\dots,\mathbf{1}_{[0,s_d]})\big\rangle_{\mathcal{H}}=\sum_{i=1}^d R_H(t_i,s_i),\qquad s_i,t_i\in[0,T], i=1,\dots,d,$$

where $R_H(t,s)=1/2(s^{2H}+t^{2H}-|t-s|^{2H})$. The scalar product between two elements $\phi,\psi\in\mathcal{E}$ is given by

$$\langle\varphi,\psi\rangle_{\mathcal{H}}=\gamma_H\sum_{i=1}^d\int_0^T\int_0^T\varphi^i(r)\psi^i(u)|r-u|^{2H-2}\,\mathrm{d}r\,\mathrm{d}u \tag{23}$$

with $\gamma_H=H(2H-1)$. The space $\mathcal{H}$ contains $L^{1/H}([0,T];\mathbb{R}^d)$, but its elements can be distributions, see, e.g., [19]. Formula (23) holds also for $\varphi,\psi\in L^{1/H}([0,T];\mathbb{R}^d)$. The mapping $(\mathbf{1}_{[0,t_1]},\dots,\mathbf{1}_{[0,t_d]})\mapsto\sum_{i=1}^d B^i_{t_i}$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(B)$ associated with $B=(B^1,\dots,B^d)$. We denote this isometry by $\varphi\mapsto B(\varphi)$.

Let $\mathcal{S}$ be the set of smooth random variables of the form $F=f(B(\varphi_1),\dots,B(\varphi_k))$ for $\varphi_i\in\mathcal{H}$, $i=1,\dots,k$, and $f\in C^\infty(\mathbb{R}^k,\mathbb{R})$ bounded with bounded derivatives. The derivative operator D of a smooth cylindrical random variable of the above form is defined as the $\mathcal{H}$-valued random variable $DF=\sum_{i=1}^k\frac{\partial f}{\partial x_i}(B(\varphi_1),\dots,B(\varphi_n))\varphi_i$. This operator is closable from $L^p(\Omega)$ into $L^p(\Omega;\mathcal{H})$. As usual, $\mathbb{D}^{1,2}$ denotes the closure of the set of smooth random variables with respect to the norm $\|F\|_{1,2}^2=\mathrm{E}|F|^2+\mathrm{E}\|DF\|_{\mathcal{H}}^2$. In particular, if D^iF denotes the Malliavin derivative of a functional $F\in\mathbb{D}^{1,2}$ with respect to B^i, we have $D^iB^j_t=\delta_{i,j}\mathbf{1}_{[0,t]}$ for $i,j=1,\dots,d$. Moreover, the space $\mathbb{D}^{1,2}_{loc}$ is the set of random variables F for which there exists a sequence $\{\Omega_n,F_n\}_{n\in\mathbb{N}}$ such that $\Omega_n\nearrow\Omega$ for $n\to\infty$ and $F_n=F$ a.s. on Ω_n for all $n\in\mathbb{N}$.

The divergence operator δ is the adjoint of the derivative operator. If a random variable $u\in L^2(\Omega;\mathcal{H})$ belongs to $\mathrm{dom}(\delta)$, the domain of the divergence operator, then $\delta(u)$ is defined by the duality relationship $\mathrm{E}(F\delta(u))=\mathrm{E}\langle DF,u\rangle_{\mathcal{H}}$, for every $F\in\mathbb{D}^{1,2}$. Moreover, if $u\in\mathrm{dom}(\delta)$ and $F\in\mathbb{D}^{1,2}$ such that $Fu\in L^2(\Omega;\mathcal{H})$, then we have the following integration by parts formula: $\delta(Fu)=F\delta(u)-\langle DF,u\rangle_{\mathcal{H}}$.

The following proposition is well known. For part (a) see, e.g., [17] and for part (b) and (c), see [18] and [10].

Proposition 4.4. *Let $b^i, \sigma^{i,j} \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$ for $i=1,\dots,n$, $j=1,\dots,d$.*

(a) *Then Eq. (1) has a unique solution $X=(X^1,\dots,X^n)$ in the Young sense in the class of all processes having α-Hölder continuous sample paths with $1-H<\alpha<H$.*

(b) *It holds $\max_{i=1,\dots,n} \mathrm{E}\sup_{0\le t\le T}|X_t^i|^p<\infty$ for all $p\ge 1$.*

(c) *Moreover, we have $X_t^i \in \mathbb{D}_{loc}^{1,2}(\mathcal{H})$ for all $t\in[0,T]$, $i=1,\dots,n$. The Malliavin derivative satisfies almost surely*:

$$D_s^j X_t^i = \sigma^{i,j}(X_s) + \sum_{k=1}^n \int_s^t b_{x_k}^i(X_u) D_s^j X_u^k \,\mathrm{d}u + \sum_{k=1}^n \sum_{l=1}^d \int_s^t \sigma_{x_k}^{i,l}(X_u) D_s^j X_u^k \,\mathrm{d}B_u^l, \quad s\le t,$$

for $j=1,\dots,d$, where $D_s^j X_t^i$ is the jth component of $D_s X_t^i$. If $b^i, \sigma^{i,j}:\mathbb{R}^n\to\mathbb{R}$, $i=1,\dots,n$, $j=1,\dots,d$ are additionally bounded, then

$$\max_{j=1,\dots,d}\max_{i=1,\dots,n}\sup_{0\le s\le t\le T}\mathrm{E}\left|D_s^j X_t^i\right|^p<\infty.$$

Before we turn to the control of the remainder in the case $H>1/2$, we will establish first some properties of iterated integrals with respect to fractional Brownian motion. To do this, we require some additional notations.

For a multi-index $\alpha\in\{0,1,\dots,d\}^k$ with $k\in\mathbb{N}$ denote by $l(\alpha)$ the length of α, i.e., $l(\alpha)=k$. Moreover set $\mathcal{A}_k=\{0,1,\dots,d\}^k$ for $k\in\mathbb{N}$, i.e., $\mathcal{A}_k$ is the set of all multi-indices of length k. Furthermore, define for $\alpha\in\mathcal{A}_k$ the sets $\mathfrak{J}_\alpha=\{j=1,\dots,k\colon \alpha_j\neq 0\}$ and $\mathfrak{J}_{\alpha,i}=\{j=1,\dots,k\colon \alpha_j=i\}$, for $i=1,\dots,d$ and $|\alpha|=|\mathfrak{J}_\alpha|$. Finally for a multi-index $\alpha\in\mathcal{A}_k$ and $j=1,\dots,k$ we denote $\alpha_{-j}=(\alpha_1,\alpha_2,\dots,\alpha_{j-1},\alpha_{j+1},\dots,\alpha_k)$. Recall that for $m\in\mathbb{N}$ and $0\le t_1\le t_2\le T$ we set $\Delta^m([t_1,t_2])=\{(\tau_1,\dots,\tau_m)\in[0,T]^m\colon t_1\le\tau_1\le\tau_2\le\cdots\le\tau_m\le t_2\}$. Moreover, we will use the notation $\int_{\Delta^k([t_1,t_2])}\mathrm{d}B^\alpha=\int_{t_1}^{t_2}\int_{t_1}^{s_{k-1}}\cdots\int_{t_1}^{s_1}\mathrm{d}B_s^{\alpha_1}\,\mathrm{d}B_{s_1}^{\alpha_2}\cdots\mathrm{d}B_{s_k}^{\alpha_k}$ for $\alpha\in\mathcal{A}_k$.

With these notations in hand, the following proposition is shown easily, and its proof will be omitted here. Indeed, part (a) follows immediately by the symmetry of fractional Brownian motion and parts (b) and (c) can be shown analogously to Theorem 11 in [2].

Proposition 4.5. *Let $k\in\mathbb{N}$ and $\alpha\in\mathcal{A}_k$.*

(a) *If $|\alpha|$ is odd, then we have $\mathrm{E}\int_{\Delta^k([0,1])}\mathrm{d}B^\alpha=0$.*

(b) *If $|\alpha|$ is even, then it holds $\mathrm{E}\int_{\Delta^k([0,1])}\mathrm{d}B^\alpha=\frac{(\gamma_H/2)^{|\alpha|/2}}{(|\alpha|/2)!}\sum_{\mathfrak{s}\in\mathfrak{S}_{\mathfrak{J}_\alpha}}\mathcal{V}(\mathfrak{s},\alpha)$ with*

$$\mathcal{V}(\mathfrak{s},\alpha)=\int_{0\le t_1<\cdots<t_k\le 1}\prod_{l=1}^{|\alpha|/2}\delta_{\alpha_{\mathfrak{s}(2l-1)},\alpha_{\mathfrak{s}(2l)}}|t_{\mathfrak{s}(2l)}-t_{\mathfrak{s}(2l-1)}|^{2H-2}\,\mathrm{d}t_1\cdots\mathrm{d}t_k,$$

where $\mathfrak{S}_{\mathfrak{J}_\alpha}$ is the group of all permutations of the set $\mathfrak{J}_\alpha$, $\gamma_H=H(2H-1)$ and $\delta_{i,j}$ is Kronecker's symbol.

(c) *It holds $\mathrm{E}|\int_{\Delta^k([0,1])}\mathrm{d}B^\alpha|^2=\frac{(\gamma_H/2)^{|\alpha|}}{|\alpha|!}\sum_{\mathfrak{s}\in\mathfrak{S}^{\mathfrak{J}_\alpha^2}}\mathcal{W}(\mathfrak{s},\alpha)$ with*

$$\mathcal{W}(\mathfrak{s},\alpha)=\int_{0\le t_1<\cdots<t_k\le 1}\int_{0\le t_{k+1}<\cdots<t_{2k}\le 1}\prod_{l=1}^{|\alpha|}\delta_{\alpha_{\mathfrak{s}(2l-1)},\alpha_{\mathfrak{s}(2l)}}|t_{\mathfrak{s}(2l)}-t_{\mathfrak{s}(2l-1)}|^{2H-2}\,\mathrm{d}t_{k+1}\cdots\mathrm{d}t_{2k}\,\mathrm{d}t_1\cdots\mathrm{d}t_k,$$

where $\mathfrak{S}^{\mathfrak{J}_\alpha^2}$ denotes the group of all permutations of the set $\mathfrak{J}_\alpha^2=\{j=1,\dots,2k\colon j\in\mathfrak{J}_\alpha$ or $j-k\in\mathfrak{J}_\alpha\}$.

Notice that part (a) and (b) of the above proposition yield a representation for the coefficients $\mathrm{E}(\mathcal{I}_{\mathbf{t}}(1)_{0,1})$, since $\mathrm{E}(\mathcal{I}_{\mathbf{t}}(1)_{0,1})=\mathrm{E}\int_{\Delta^{l(\mathbf{t})}([0,1])}\mathrm{d}B^\alpha$ with $\mathbf{t}''(i)=\alpha_i\in\{0,1,\dots,d\}$, $i=1,\dots,l(\mathbf{t})$.

For further computations, we also need the following positivity result for iterated integrals of the fractional Brownian motion.

Proposition 4.6. *Let $m_i\in\mathbb{N}$ for $i=1,\dots,n$ with $n\in\mathbb{N}$. Moreover let $\alpha^{m_i}\in\mathcal{A}_{m_i}$ and $0\le s_i\le t_i\le T$ for $i=1,\dots,n$. It holds $\mathrm{E}[\prod_{i=1}^n\int_{\Delta^{m_i}([s_i,t_i])}\mathrm{d}B^{\alpha^{m_i}}]\ge 0$.*

Proof. Let $t_k^l = k2^{-l}$, $k = 0, 1, \ldots, 2^l$. For $\alpha = 0, \ldots, d$, denote by $B^{l,(\alpha)}$ the piecewise linear interpolation of $B^{(\alpha)}$ with step size 2^{-l}, i.e., $B_t^{l,(\alpha)} = B^{(\alpha)}_{t_k^l} + 2^l(t - t_k^l)(B^{(\alpha)}_{t_{k+1}^l} - B^{(\alpha)}_{t_k^l})$, $t \in [t_k^l, t_{k+1}^l)$. We have $\prod_{i=1}^n \int_{\Delta^{m_i}([s_i,t_i])} \mathrm{d}B^{\alpha^{m_i}} = \lim_{l\to\infty} \prod_{i=1}^n \int_{\Delta^{m_i}([s_i,t_i])} \mathrm{d}B^{l,\alpha^{m_i}}$ almost surely, due to Proposition 2.6. Since $\prod_{i=1}^n \int_{\Delta^{m_i}([s_i,t_i])} \mathrm{d}B^{l,\alpha^{m_i}}$ belongs to a finite Wiener chaos, we also have $\mathrm{E}\prod_{i=1}^n \int_{\Delta^{m_i}([s_i,t_i])} \mathrm{d}B^{\alpha^{m_i}} = \lim_{l\to\infty} \mathrm{E}\prod_{i=1}^n \int_{\Delta^{m_i}([s_i,t_i])} \mathrm{d}B^{l,\alpha^{m_i}}$ according to [4]. Note that

$$\int_{\Delta^{m_i}([s_i,t_i])} \mathrm{d}B^{l,\alpha^{m_i}} = \int_{s_i}^{t_i}\int_{s_i}^{t_{m_i}} \cdots \int_{s_i}^{t_2} \prod_{j=1}^{m_i} Z_{t_j}^{l,(\alpha_j^{m_i})} \mathrm{d}t_1 \cdots \mathrm{d}t_{m_i-1}\,\mathrm{d}t_{m_i},$$

where $Z_t^{l,(\alpha_j^{m_i})} = 2^l(B^{(\alpha_j^{m_i})}_{t_{k+1}^l} - B^{(\alpha_j^{m_i})}_{t_k^l})$, $t \in [t_k^l, t_{k+1}^l)$ for $\alpha_j^{m_i} \neq 0$ and $Z_t^{l,(0)} = 1$ for $t \in [0, T]$. We thus have

$$\prod_{i=1}^n \int_{\Delta^{m_i}([s_i,t_i])} \mathrm{d}B^{l,\alpha^{m_i}} = \int_{\Delta^{m_n}([s_n,t_n])} \cdots \int_{\Delta^{m_1}([s_1,t_1])} \prod_{i=1}^n \prod_{j=1}^{m_i} Z_{t_j^i}^{l,(\alpha_j^{m_i})} \mathrm{d}t_1^{m_1} \cdots \mathrm{d}t_{m_1}^{m_1} \cdots \mathrm{d}t_1^{m_n} \cdots \mathrm{d}t_{m_n}^{m_n}.$$

But the term $\prod_{i=1}^n \prod_{j=1}^{m_i} Z_{t_j^i}^{l,(\alpha_j^{m_i})}$ is a product, which consists only of increments of the independent fractional Brownian motions $B^1, \ldots, B^d$ with Hurst parameter $H > 1/2$ and of the constant factors 1. Since it is well known that the increments of a fractional Brownian motion of Hurst index $H > 1/2$ are positively correlated, and also that we have, for a centered Gaussian vector $(G_1, \ldots, G_{2k})$, that $\mathrm{E}(G_1 \cdots G_{2k}) = \frac{1}{k!2^k}\sum_{\mathfrak{s}\in\mathfrak{S}_{2k}} \prod_{\ell=1}^k \mathrm{E}(G_{\mathfrak{s}(2\ell)}\, G_{\mathfrak{s}(2\ell-1)})$, we clearly deduce that $\mathrm{E}\prod_{i=1}^n \prod_{j=1}^{m_i} Z_{t_j^i}^{l,(\alpha_j^{m_i})} \geq 0$ for all $t_1^{m_1}, \ldots, t_{m_n}^{m_n} \in [0, T]$. Hence we obtain $\mathrm{E}\prod_{i=1}^n \int_{\Delta^{m_i}([s_i,t_i])} \mathrm{d}B^{l,\alpha^{m_i}} \geq 0$ for every $l \in \mathbb{N}$, and the assertion follows. □

Our estimate of the remainder will also require the Malliavin derivative of an iterated integral. Recall then that for a random variable $F \in \mathbb{D}^{1,2}$ we denote by $D^i F$ the ith component of the Malliavin derivative, i.e., $DF = (D^1F, \ldots, D^dF)$. Recall moreover that for $\alpha \in \mathcal{A}_k$, $k \in \mathbb{N}$, we have defined $\mathfrak{J}_\alpha = \{j = 1, \ldots, k\colon\ \alpha_j \neq 0\}$ and $\mathfrak{J}_{\alpha,i} = \{j = 1, \ldots, k : \alpha_j = i\}$, for $i = 1, \ldots, d$ and $\alpha_{-j} = (\alpha_1, \alpha_2, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_k)$. Then the stochastic derivative of a multiple integral can be computed as follows:

Proposition 4.7. *Let $m \in \mathbb{N}$ and $\alpha \in \mathcal{A}_m$. We have, for $i = 1, \ldots, d$:*

$$D_u^i \int_{\Delta^m([s,t])} \mathrm{d}B^\alpha(t_1, \ldots, t_m) = \sum_{j\in\mathfrak{J}_{\alpha,i}} \int_{s\leq t_1\leq\cdots\leq t_{j-1}\leq u\leq t_j\leq\cdots\leq t_{m-1}\leq t} \mathrm{d}B^{\alpha_{-j}}(t_1, \ldots, t_{m-1}).$$

Proof. We can proceed by induction over $l(\alpha)$. Details are left to the reader. □

Now, we will establish an estimate for the second moment of an iterated integral, which will be the key for the control of the remainder $\mathcal{R}_m(0, t)$ in the expansion of $P_t f(a)$.

Proposition 4.8. *Let $m \in \mathbb{N}$ and $\alpha \in \mathcal{A}_m$. There exists a constant $K_1 > 0$, depending only on H and T, such that, for $0 \leq s \leq t \leq T$, we have*

$$\left(\mathrm{E}\left|\int_{\Delta^m([s,t])} \mathrm{d}B^\alpha\right|^2\right)^{1/2} \leq \frac{K_1^m}{\sqrt{m!}}|t - s|^{|\alpha|H+m-|\alpha|}. \tag{24}$$

Proof. The proof is separated into three steps.

(i) By stationary increments (13) of the fractional Brownian motion, it follows $\int_{\Delta^m([s,t])} \mathrm{d}B^\alpha \overset{\mathcal{L}}{=} \int_{\Delta^m([0,t-s])} \mathrm{d}B^\alpha$. Hence we obtain by the scaling property (12) of fractional Brownian motion that $\int_{\Delta^m([s,t])} \mathrm{d}B^\alpha \overset{\mathcal{L}}{=} (t - s)^{H|\alpha|+m-|\alpha|} \times$

$\int_{\Delta^m([0,1])} \mathrm{d}B^{\alpha}$. Now from Proposition 4.5(c) it is obvious that we have

$$\mathrm{E}\left|\int_{\Delta^m([0,1])} \mathrm{d}B^{\alpha}\right|^2 \le \mathrm{E}\left|\int_{\Delta^m([0,1])} \mathrm{d}B^{\widetilde{\alpha}}\right|^2, \tag{25}$$

where $\widetilde{\alpha}$ is given by $\widetilde{\alpha} = (\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_m)$ with $\widetilde{\alpha}_j = 0$ if $j \in \mathfrak{J}_{\alpha,0}$ and $\widetilde{\alpha}_j = 1$ if $j \in \mathfrak{J}_{\alpha}$, i.e., all integrals with respect to $B^{(i)}$, $i = 2, \dots, n$, are replaced by integrals with respect to $B^{(1)}$.

(ii) In the next step, we will replace also the integrals with respect to t by integrals with respect to $B^{(1)}$. More precisely, we will show that

$$\mathrm{E}\left|\int_{\Delta^m([0,1])} \mathrm{d}B^{\widetilde{\alpha}}\right|^2 \le \gamma_H^{|\alpha|-m} \mathrm{E}\left|\int_{\Delta^m([0,1])} \mathrm{d}B^{(1,\dots,1)}\right|^2, \tag{26}$$

with $\gamma_H = H(2H-1)$. To prove (26) assume first that there is only one integral with respect to t, i.e. $|\mathfrak{J}_{\alpha,0}| = 1$. Thus we have $\int_{\Delta^m([0,1])} \mathrm{d}B^{\widetilde{\alpha}} = \int_{\Delta^{k_1}([0,1])} \int_0^s \int_{\Delta^{k_2}([0,s])} \mathrm{d}B^{\widetilde{\alpha}_2} \,\mathrm{d}s \,\mathrm{d}B^{\widetilde{\alpha}_1}$ with $k_1 + k_2 + 1 = m$ and $\widetilde{\alpha} = (\widetilde{\alpha}_2, 0, \widetilde{\alpha}_1)$. By a Fubini-type lemma, we get

$$\int_{\Delta^{k_1}([0,1])} \int_0^s \int_{\Delta^{k_2}([0,s])} \mathrm{d}B^{\widetilde{\alpha}_2} \,\mathrm{d}s \,\mathrm{d}B^{\widetilde{\alpha}_1} = \int_0^1 Y_s \,\mathrm{d}s,$$

where we have set $Y_s = \int_{\Delta^{k_1}([s,1])} \int_{\Delta^{k_2}([0,s])} \mathrm{d}B^{\widetilde{\alpha}_2} \,\mathrm{d}B^{\widetilde{\alpha}_1}$. With this notation in hand, observe that we also have $\int_{\Delta^m([0,1])} \mathrm{d}B^{(1,\dots,1)} = \int_0^1 Y_s \,\mathrm{d}B_s^{(1)}$. Hence, when $|\mathfrak{J}_{\alpha,0}| = 1$, one can recast (26) into

$$\mathrm{E}\left|\int_0^1 Y_s \,\mathrm{d}s\right|^2 \le \gamma_H \mathrm{E}\left|\int_0^1 Y_s \,\mathrm{d}B_s^{(1)}\right|^2. \tag{27}$$

We will now proceed to the estimation of the two terms in (27): first of all, we easily get $\mathrm{E}|\int_0^1 Y_s \,\mathrm{d}s|^2 = \int_0^1 \int_0^1 \mathrm{E}Y_{s_1} Y_{s_2} \,\mathrm{d}s_1 \,\mathrm{d}s_2$. Let us compute now $\mathrm{E}|\int_0^1 Y_s \,\mathrm{d}B^{(1)}(s)|^2$: by the relation between the Young and the divergence integral for fractional Brownian motion, see, e.g. [1] or Proposition 5.2.3 in [16], we have

$$\int_0^1 Y_s \,\mathrm{d}B^{(1)}(s) = \delta^{(1)}(Y 1_{[0,1]}) + \gamma_H \int_0^1 \int_0^1 D^1_{s_1} Y_{s_2} |s_1 - s_2|^{2H-2} \,\mathrm{d}s_1 \,\mathrm{d}s_2,$$

where we use the notation $\delta^{(1)}(Y 1_{[0,1]}) = \delta((Y 1_{[0,1]}, \dots, 0))$. Thus we obtain

$$\begin{aligned}
\mathrm{E}\left|\int_0^1 Y_s \,\mathrm{d}B^{(1)}(s)\right|^2 &= \mathrm{E}\left|\delta^{(1)}(Y 1_{[0,1]})\right|^2 + \gamma_H^2 \mathrm{E}\left|\int_0^1 \int_0^1 D^1_{s_1} Y_{s_2} |s_1 - s_2|^{2H-2} \,\mathrm{d}s_1 \,\mathrm{d}s_2\right|^2 \\
&\quad + 2\gamma_H \mathrm{E}\delta^{(1)}(Y 1_{[0,1]}) \int_0^1 \int_0^1 D^1_{s_1} Y_{s_2} |s_1 - s_2|^{2H-2} \,\mathrm{d}s_1 \,\mathrm{d}s_2 \\
&\ge \mathrm{E}\left|\delta^{(1)}(Y 1_{[0,1]})\right|^2 + 2\gamma_H \mathrm{E}\delta^{(1)}(Y 1_{[0,1]}) \int_0^1 \int_0^1 D^1_{s_1} Y_{s_2} |s_1 - s_2|^{2H-2} \,\mathrm{d}s_1 \,\mathrm{d}s_2.
\end{aligned}$$

Since clearly $\int_0^1 \int_0^1 D^1_{s_1} Y_{s_2} |s_1 - s_2|^{2H-2} \,\mathrm{d}s_1 \,\mathrm{d}s_2 \in \mathbb{D}^{1,2}$, we have, owing to the duality between D and δ, that

$$\begin{aligned}
&\mathrm{E}\left[\delta^{(1)}(Y 1_{[0,1]}) \int_0^1 \int_0^1 D^1_{s_1} Y_{s_2} |s_1 - s_2|^{2H-2} \,\mathrm{d}s_1 \,\mathrm{d}s_2\right] \\
&\quad = \gamma_H \int_0^1 \int_0^1 \int_0^1 \int_0^1 \mathrm{E}\left[Y_{s_1} D^1_{s_2} D^1_{s_3} Y_{s_4}\right] |s_3 - s_4|^{2H-2} |s_1 - s_2|^{2H-2} \,\mathrm{d}s_3 \,\mathrm{d}s_4 \,\mathrm{d}s_1 \,\mathrm{d}s_2.
\end{aligned}$$

By the definition of $Y_s, s \in [0,1]$, and applying Proposition 4.7, we can decompose the product $Y_{s_1} D^1_{s_2} D^1_{s_3} Y_{s_4}$ into a sum of products of iterated integrals, and hence, for any $s_1, s_2, s_3, s_4 \in [0,1]$, we have $\mathrm{E}[Y_{s_1} D^1_{s_2} D^1_{s_3} Y_{s_4}] \ge 0$ by Proposition 4.6. Consequently we obtain $\mathrm{E}|\int_0^1 Y_s \,\mathrm{d}B^{(1)}(s)|^2 \ge \mathrm{E}|\delta^{(1)}(Y 1_{[0,1]})|^2$. Furthermore, invoking [1], we get

$$\begin{aligned}\mathrm{E}\big|\delta^{(1)}(Y 1_{[0,1]})\big|^2 &= \gamma_H \int_0^1\int_0^1 \mathrm{E}Y_{s_1}Y_{s_2}|s_1-s_2|^{2H-2}\,\mathrm{d}s_1\,\mathrm{d}s_2 \\ &\quad + \gamma_H^2 \int_0^1\int_0^1\int_0^1\int_0^1 \mathrm{E}D^1_{\tau_1}Y_{s_1}D^1_{s_2}Y_{\tau_2}|s_1-s_2|^{2H-2}|\tau_1-\tau_2|^{2H-2}\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2.\end{aligned}$$

Besides, according to Proposition 4.6, and thanks to the fact that both $Y_{s_1}Y_{s_2}$ and $D^1_{\tau_1}Y_{s_1}D^1_{s_2}Y_{\tau_2}$ are products of iterated integrals, we obtain that $\mathrm{E}D^1_{\tau_1}Y_{s_1}D^1_{s_2}Y_{\tau_2} \ge 0$ and $\mathrm{E}Y_{s_1}Y_{s_2} \ge 0$ for any $s_1, s_2 \in [0,1]$. Since $|s_1 - s_2|^{2H-2} \ge 1$ when $s_1, s_2 \in [0,1]$, we end up with

$$\mathrm{E}\bigg|\int_0^1 Y_s\,\mathrm{d}B^{(1)}(s)\bigg|^2 \ge \gamma_H \mathrm{E}\bigg|\int_0^1 Y_s\,\mathrm{d}s\bigg|^2,$$

which is the announced relation (27). We have thus proved that

$$\mathrm{E}\bigg|\int_{\Delta^{k_1}([0,1])}\int_0^s\int_{\Delta^{k_2}([0,s])}\mathrm{d}B^{\widetilde{\alpha}_2}\,\mathrm{d}s\,\mathrm{d}B^{\widetilde{\alpha}_1}\bigg|^2 \le \gamma_H^{-1}\mathrm{E}\bigg|\int_{\Delta^{k_1}([0,1])}\int_0^s\int_{\Delta^{k_2}([0,s])}\mathrm{d}B^{\widetilde{\alpha}_2}\,\mathrm{d}B^{(1)}\,\mathrm{d}B^{\widetilde{\alpha}_1}\bigg|^2.$$

Applying this procedure $m - |\alpha|$ times to replace all integrals with respect to t, Eq. (26) is now easily checked.

(iii) Let us conclude our proof: combining (25) and (26) yields

$$\mathrm{E}\bigg|\int_{\Delta^m([0,1])}\mathrm{d}B^{\alpha}\bigg|^2 \le \frac{\gamma_H^{|\alpha|}}{\gamma_H^m}\mathrm{E}\bigg|\int_{\Delta^m([0,1])}\mathrm{d}B^{(1,\dots,1)}\bigg|^2.$$

But clearly $\int_{\Delta^m([0,1])}\mathrm{d}B^{(1,\dots,1)} = (B_1)^m/m!$ and thus we have $\mathrm{E}|\int_{\Delta^m([0,1])}\mathrm{d}B^{(1,\dots,1)}|^2 = (2m)!/(2^m(m!)^3)$. Since $\frac{(2m)!}{2^m(m!)^2} \le 2^m$ the assertion (24) follows.

□

Putting together Propositions 4.7 and 4.8, we also obtain the following estimate for the second moment of the Malliavin derivative of an iterated integral.

Proposition 4.9. *Let $m \in \mathbb{N}$ and $\alpha \in \mathcal{A}_m$. There exists a constant $K_2 > 0$, depending only on H and T, such that we have, for $i = 1, \dots, d$ and all $0 \le s \le t \le T$:*

$$\bigg(\mathrm{E}\bigg|D^i_u\int_{\Delta^m([s,t])}\mathrm{d}B^{\alpha}\bigg|^2\bigg)^{1/2} \le |\mathfrak{I}_{\alpha,i}|\frac{K_2^{m-1}}{\sqrt{(m-1)!}}|t-s|^{(|\alpha|-1)H+m-|\alpha|}. \tag{28}$$

Proof. Thanks to Proposition 4.7 we have that

$$D^i_u\int_{\Delta^m([s,t])}\mathrm{d}B^{\alpha} = \sum_{j\in\mathfrak{I}_{\alpha,i}}\int_{s\le t_1\le\cdots\le t_{j-1}\le u\le t_j\le\cdots\le t_{m-1}\le t}\mathrm{d}B^{\alpha-j}(t_1,\dots,t_{m-1}).$$

Thus it follows

$$\begin{aligned}&\bigg(\mathrm{E}\bigg|D^i_u\int_{\Delta^m([s,t])}\mathrm{d}B^{\alpha}(t_1,\dots,t_m)\bigg|^2\bigg)^{1/2} \\ &\quad\le \sum_{j\in\mathfrak{I}_{\alpha,i}}\bigg(\mathrm{E}\bigg|\int_{s\le t_1\le\cdots\le t_{j-1}\le u\le t_j\le\cdots\le t_{m-1}\le t}\mathrm{d}B^{\alpha-j}(t_1,\dots,t_{m-1})\bigg|^2\bigg)^{1/2}.\end{aligned} \tag{29}$$

Furthermore, it is easily checked that

$$\int_{s\le t_1\le\cdots\le t_{j-1}\le u\le t_j\le\cdots\le t_{m-1}\le t} \mathrm{d}B^{\alpha_{-j}}(t_1,\ldots,t_{m-1}) = \int_{\Delta^{l(\alpha^{j_1})}([s,u])} \mathrm{d}B^{\alpha^{j_1}} \times \int_{\Delta^{l(\alpha^{j_2})}([u,t])} \mathrm{d}B^{\alpha^{j_2}},$$

with $\alpha = (\alpha^{j_1}, i, \alpha^{j_2})$. Since an iterated integral belongs to a finite chaos with respect to B, all its L^p norms are equivalent. See, e.g., Theorem 1.4.1 in [16]. Thus, we obtain from Proposition 4.8 and Hölder's inequality that

$$\begin{aligned}\biggl(\mathrm{E}\biggl|\int_{s\le t_1\le\cdots\le t_{j-1}\le u\le t_j\le\cdots\le t_{m-1}\le t} \mathrm{d}B^{\alpha_{-j}}(t_1,\ldots,t_{m-1})\biggr|^2\biggr)^{1/2} \\ \le c_{2,4}|t-s|^{|\alpha_{-j}|H+m-1-|\alpha_{-j}|}\frac{K_1^{m-1}}{\sqrt{l(\alpha^{j_1})!}\sqrt{l(\alpha^{j_2})!}},\end{aligned} \tag{30}$$

with a constant $c_{2,4} > 0$. Moreover, it is readily seen that $\sqrt{l(\alpha^{j_1})!}\sqrt{l(\alpha^{j_2})!} \ge [m/2]!$, and according to the fact that $(2k)!/(k!)^2 \le 2^{2k}$, we end up with $\frac{1}{\sqrt{l(\alpha^{j_1})!}\sqrt{l(\alpha^{j_2})!}} \le \frac{2^{(m-1)/2}}{\sqrt{(m-1)!}}$. Plugging this inequality into (30) and (29), we obtain

$$\begin{aligned}\biggl(\mathrm{E}\biggl|D_u^i\int_{s\le t_1\le\cdots\le t_m\le t} \mathrm{d}B^{\alpha}(t_1,\ldots,t_m)\biggr|^2\biggr)^{1/2} \\ \le c_{2,4}\frac{(\sqrt{2}K_1)^{m-1}}{\sqrt{(m-1)!}}|\mathfrak{J}_{\alpha,i}||t-s|^{(|\alpha_{-j}|H+m-1-|\alpha_{-j}|)},\end{aligned}$$

and since $|\alpha_{-j}|H+m-1-|\alpha_{-j}| = (|\alpha|-1)H+m-|\alpha|$, our claim (28) follows. □

4.4. *Study of the remainder term for $H > 1/2$*

To avoid notational confusion we will write in the following X_t, $t\in[0,T]$, instead of X_t^a, $t\in[0,T]$, for the solution of the SDE with $X_0 = a$. Moreover, recall that X_t^i, $t\in[0,T]$, denotes the ith component of X. Recall also that the differential operators $\mathcal{D}^0$ and $\mathcal{D}^j$ are defined by (16) and that we have set $\mathcal{D}^\alpha = \mathcal{D}^{\alpha_1}\cdots\mathcal{D}^{\alpha_k}$ for a multi-index $\alpha\in\mathcal{A}_k$.

With the help of the auxiliary results contained in the previous section, we are now able to bound $\mathrm{E}\mathcal{R}_m(0,t)$ in the following way when $H > 1/2$:

Theorem 4.10. *Let $m\in\mathbb{N}$, $H > 1/2$ and assume that assumption* (A2) *holds. Then there exists a constant $K_3 > 0$, depending only on H, T, d and n, such that*

$$\bigl|\mathrm{E}\mathcal{R}_m(0,t)\bigr| \le (\mathcal{U}_{m+1} + \widetilde{\mathcal{U}}_{m+1}\mathcal{Y})\frac{K_3^m t^{H(m+1)}}{\sqrt{m!}}$$

for all $t\in[0,T]$, where

$$\mathcal{U}_m = \sup_{\alpha\in\mathcal{A}_m}\sup_{0\le t\le T}\bigl(\mathrm{E}\bigl|\mathcal{D}^\alpha f(X_t)\bigr|^2\bigr)^{1/2}, \qquad \widetilde{\mathcal{U}}_m = \max_{i=1,\ldots,n}\sup_{\alpha\in\mathcal{A}_m}\sup_{0\le t\le T}\biggl(\mathrm{E}\biggl|\frac{\partial}{\partial x_i}\mathcal{D}^\alpha f(X_t)\biggr|^2\biggr)^{1/2}$$

$$\textit{and}\quad \mathcal{Y} = \max_{i=1,\ldots,n}\max_{j=1,\ldots,d}\sup_{0\le u\le s\le T}\bigl(\mathrm{E}\bigl|D_u^j X_s^i\bigr|^4\bigr)^{1/4}.$$

Notice then that the second part of Theorem 3.1 is an immediate consequence of the above estimate.

Before we can prove Theorem 4.10, we will need the following proposition:

Proposition 4.11. *Let $m \in \mathbb{N}$, $\alpha \in \mathcal{A}_m$, let $g \in C_P^\infty(\mathbb{R}^n;\mathbb{R})$ and let assumption* (A2) *be satisfied. Moreover, set $J_\alpha(s,t) = \int_{\Delta^m([s,t])} \mathrm{d}B^\alpha$. Then it holds, for $t \in [0,T]$ and $j = 1,\dots,d$:*

$$\mathrm{E}\biggl(\int_0^t g(X_s) J_\alpha(s,t)\,\mathrm{d}B_s^j\biggr) = \gamma_H \mathrm{E}\Biggl(\int_0^t \int_0^s \sum_{i=1}^n g_{x_i}(X_s) J_\alpha(s,t) D_u^j X_s^i |s-u|^{2H-2}\,\mathrm{d}u\,\mathrm{d}s\Biggr) + \gamma_H \mathrm{E}\biggl(\int_0^t \int_s^t g(X_s) D_u^j J_\alpha(s,t)|s-u|^{2H-2}\,\mathrm{d}u\,\mathrm{d}s\biggr).$$

Proof. The assertion is a straightforward consequence of Proposition 5.2.3 in [16], Proposition 4.4, the properties of iterated integrals of fractional Brownian motion and the product and chain rule of the Malliavin derivative. □

We are now ready to prove the main result of this section.

Proof of Theorem 4.10. Note that by the proof of Theorem 4.1 we have

$$\mathcal{R}_m(0,t) = \sum_{\alpha \in \mathcal{A}_{m+1}} \int_0^t \int_0^{t_{m+1}} \cdots \int_0^{t_2} \mathcal{D}^\alpha f(X_{t_1})\,\mathrm{d}B_{t_1}^{\alpha_1}\,\mathrm{d}B_{t_2}^{\alpha_2} \cdots \mathrm{d}B_{t_{m+1}}^{\alpha_{m+1}}.$$

(a) We first consider a single integrand. By interchanging the order of integration, which is possible since all integrals are pathwise defined, we have

$$\begin{aligned}
&\int_0^t \int_0^{t_{m+1}} \cdots \int_0^{t_2} \mathcal{D}^\alpha f(X_{t_1})\,\mathrm{d}B_{t_1}^{\alpha_1}\,\mathrm{d}B_{t_2}^{\alpha_2} \cdots \mathrm{d}B_{t_{m+1}}^{\alpha_{m+1}} \\
&\quad = \int_0^t \int_{t_1}^t \int_{t_1}^{t_{m+1}} \cdots \int_{t_1}^{t_3} \mathrm{d}B_{t_2}^{\alpha_2} \cdots \mathrm{d}B_{t_m}^{\alpha_m}\,\mathrm{d}B_{t_{m+1}}^{\alpha_{m+1}} \mathcal{D}^\alpha f(X_{t_1})\,\mathrm{d}B_{t_1}^{\alpha_1} \\
&\quad = \int_0^t \int_{\Delta^m([s,t])} \mathrm{d}B^{\alpha-1} \mathcal{D}^\alpha f(X_s)\,\mathrm{d}B_s^{\alpha_1}.
\end{aligned}$$

Recall that $\mathcal{U}_m = \sup_{\alpha \in \mathcal{A}_m} \sup_{0 \le t \le T} (\mathrm{E}|\mathcal{D}^\alpha f(X_t)|^2)^{1/2}$ and

$$\widetilde{\mathcal{U}}_m = \sup_{i=1,\dots,n} \sup_{\alpha \in \mathcal{A}_m} \sup_{0 \le t \le T} \biggl(\mathrm{E}\biggl|\frac{\partial}{\partial x_i} \mathcal{D}^\alpha f(X_t)\biggr|^2\biggr)^{1/2}.$$

If $\alpha_1 = 0$, we clearly have

$$\begin{aligned}
&\biggl|\mathrm{E}\int_{\Delta^{m+1}([0,t])} \mathcal{D}^\alpha f(X_{t_1})\,\mathrm{d}B^\alpha(t_1,\dots,t_{m+1})\biggr| \\
&\quad \le \mathcal{U}_{m+1} \int_0^t \biggl(\mathrm{E}\biggl|\int_{\Delta^m([s,t])} \mathrm{d}B^{\alpha-1}\biggr|^2\biggr)^{1/2} \mathrm{d}s \le \mathcal{U}_{m+1} K_1^m \frac{t^{Hm+1}}{\sqrt{m!}}.
\end{aligned} \tag{31}$$

If $\alpha_1 \neq 0$ we have, according to Proposition 4.11:

$$\begin{aligned}
&\biggl|\mathrm{E}\int_{\Delta^{m+1}([0,t])} \mathcal{D}^\alpha f(X_{t_1})\,\mathrm{d}B^\alpha(t_1,\dots,t_{m+1})\biggr| \\
&\quad \le \gamma_H \biggl|\int_0^t \int_0^s \sum_{i=1}^n \mathrm{E}\frac{\partial}{\partial x_i} \mathcal{D}^\alpha f(X_s) \int_{\Delta^m([s,t])} \mathrm{d}B^{\alpha-1} D_u^{\alpha_1} X_s^i |s-u|^{2H-2}\,\mathrm{d}u\,\mathrm{d}s\biggr| \\
&\qquad + \gamma_H \biggl|\int_0^t \int_s^t \mathrm{E}\mathcal{D}^\alpha f(X_s) D_u^{\alpha_1} \int_{\Delta^m([s,t])} \mathrm{d}B^{\alpha-1} |s-u|^{2H-2}\,\mathrm{d}u\,\mathrm{d}s\biggr|.
\end{aligned}$$

(Note that we have $D^\alpha f \in C_P^\infty(\mathbb{R}^n;\mathbb{R})$, since $f \in C_P^\infty(\mathbb{R}^n;\mathbb{R})$ and $b^i, \sigma^{i,j} \in C_b^\infty(\mathbb{R}^n;\mathbb{R})$ for $i = 1,\dots,n$, $j = 1,\dots,d$.) Thus it follows

$$\begin{aligned}
&\left|\mathrm{E}\int_{\Delta^{m+1}([0,t])} \mathcal{D}^\alpha f(X_{t_1})\,\mathrm{d}B^\alpha(t_1,\dots,t_{m+1})\right| \\
&\quad\le \sum_{i=1}^n \widetilde{\mathcal{U}}_{m+1}\gamma_H \int_0^t\int_0^s \left(\mathrm{E}\left|D_u^{\alpha_1} X_s^i\right|^4\right)^{1/4}\left(\mathrm{E}\left|\int_{\Delta^m([s,t])}\mathrm{d}B^{\alpha-1}\right|^4\right)^{1/4}|s-u|^{2H-2}\,\mathrm{d}u\,\mathrm{d}s \\
&\qquad + \mathcal{U}_{m+1}\gamma_H \int_0^t\int_s^t \left(\mathrm{E}\left|D_u^{\alpha_1}\int_{\Delta^m([s,t])}\mathrm{d}B^{\alpha-1}\right|^2\right)^{1/2}|s-u|^{2H-2}\,\mathrm{d}u\,\mathrm{d}s.
\end{aligned}$$

So recalling that we have set $\mathcal{Y} = \max_{i=1,\dots,n}\max_{j=1,\dots,d}\sup_{0\le u\le s\le T}(\mathrm{E}|D_u^j X_s^i|^4)^{1/4}$, we get

$$\begin{aligned}
&\left|\mathrm{E}\int_{\Delta^{m+1}([0,t])} \mathcal{D}^\alpha f(X_{t_1})\,\mathrm{d}B^\alpha(t_1,\dots,t_{m+1})\right| \\
&\quad\le n\widetilde{\mathcal{U}}_{m+1}\mathcal{Y}\gamma_H \int_0^t\int_0^s \left(\mathrm{E}\left|\int_{\Delta^m([s,t])}\mathrm{d}B^{\alpha-1}\right|^4\right)^{1/4}|s-u|^{2H-2}\,\mathrm{d}u\,\mathrm{d}s \\
&\qquad + \mathcal{U}_{m+1}\gamma_H \int_0^t\int_s^t \left(\mathrm{E}\left|D_u^{\alpha_1}\int_{\Delta^m([s,t])}\mathrm{d}B^{\alpha-1}\right|^2\right)^{1/2}|s-u|^{2H-2}\,\mathrm{d}u\,\mathrm{d}s.
\end{aligned}$$

Furthermore, invoking again the equivalence of L^p norms for the iterated integral and Proposition 4.8, we obtain

$$\begin{aligned}
&\gamma_H \int_0^t\int_0^s \left(\mathrm{E}\left|\int_{\Delta^m([s,t])}\mathrm{d}B^{\alpha-1}\right|^4\right)^{1/4}|s-u|^{2H-2}\,\mathrm{d}u\,\mathrm{d}s \\
&\quad\le c_{2,4}\frac{K_1^m}{\sqrt{m!}}\gamma_H\int_0^t\int_0^s |t-s|^{Hm}|s-u|^{2H-2}\,\mathrm{d}u\,\mathrm{d}s \le c_{2,4}K_1^m\frac{t^{H(m+2)}}{\sqrt{m!}}.
\end{aligned}$$

By Proposition 4.9, we get similarly

$$\gamma_H \int_0^t\int_s^t \left(\mathrm{E}\left|D_u^{\alpha_1}\int_{\Delta^m([s,t])}\mathrm{d}B^{\alpha-1}\right|^2\right)^{1/2}|s-u|^{2H-2}\,\mathrm{d}u\,\mathrm{d}s \le c_{2,4}K_2^{m-1}\frac{t^{H(m+1)}}{\sqrt{(m-1)!}}.$$

Thus, we have shown for $\alpha_1 \ne 0$ the estimate

$$\begin{aligned}
&\left|\mathrm{E}\int_{\Delta^{m+1}([0,t])} \mathcal{D}^\alpha f(X_{t_1})\,\mathrm{d}B^\alpha(t_1,\dots,t_{m+1})\right| \\
&\quad\le n\widetilde{\mathcal{U}}_{m+1}\mathcal{Y}c_{2,4}K_1^m\frac{t^{H(m+2)}}{\sqrt{m!}} + \mathcal{U}_{m+1}c_{2,4}K_2^{m-1}\frac{t^{H(m+1)}}{\sqrt{(m-1)!}}.
\end{aligned}\tag{32}$$

(b) Now we consider the complete remainder term. We have

$$\begin{aligned}
\left|\mathrm{E}\mathcal{R}_m(0,t)\right| \le{}& \sum_{\alpha\in\mathcal{A}_{m+1},\alpha_1=0}\left|\mathrm{E}\int_{\Delta^{m+1}([0,t])}\mathcal{D}^\alpha f(X_{t_1})\,\mathrm{d}B^\alpha(t_1,\dots,t_{m+1})\right| \\
&+ \sum_{\alpha\in\mathcal{A}_{m+1},\alpha_1\ne 0}\left|\mathrm{E}\int_{\Delta^{m+1}([0,t])}\mathcal{D}^\alpha f(X_{t_1})\,\mathrm{d}B^\alpha(t_1,\dots,t_{m+1})\right|.
\end{aligned}$$

Since $|\mathcal{A}_m| = (d+1)^m$, it follows by (31) and (32), that there exists a constant $K_3 > 0$ depending only on H, T, n and d such that $|\mathrm{E}\mathcal{R}_m(0,t)| \le (\mathcal{U}_{m+1} + \mathcal{Y}\widetilde{\mathcal{U}}_{m+1})K_3^m\frac{t^{H(m+1)}}{\sqrt{m!}}$, which completes the proof. □

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