A PRIORI ESTIMATES FOR ROUGH PDEs WITH APPLICATION TO ROUGH CONSERVATION LAWS

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Abstract. We introduce a general framework to study PDEs driven by rough paths: we develop new a priori estimates based on a rough Gronwall lemma argument for weak solutions to rough PDEs. This will allow us to follow standard PDE strategies to obtain existence and uniqueness results. In particular our approach does not rely on any sort of transformation formula (flow transformation, Feynman–Kac representation formula etc.) and is therefore rather flexible. As an application, we study conservation laws driven by rough paths establishing well–posedness for the corresponding kinetic formulation.

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1. Introduction

Lyons [52] introduced rough paths to give a description of solutions to ordinary differential equation (ODEs) driven by external time varying signals which is robust enough to allow very irregular signals like the sample paths of a Brownian motion. His analysis singles out a rough path as the appropriate topological structure on the input signal with respect to which the solution of an ODE varies in a continuous way. Since its invention, Rough path theory (RPT) has been developed very intensively to provide robust analysis of ODEs and a novel way to define solutions of stochastic differential equations driven by non semimartingale signals (for a comprehensive review see the book of Friz and Victoir [25] and the lecture notes of Lyons, Caruana and Lévy [53] or the more recents ones of Friz and Hairer [24]). RPT can be naturally formulated also in infinite-dimension to analyse ODEs in Banach spaces. This generalisation is, however, not appropriate for the understanding of rough PDEs (RPDEs), i.e. PDEs with irregular perturbations. This is due to two basic facts. First, the notion of rough path encodes in a fundamental way only the nonlinear effects of a time varying signals without possibility to add more dimensions to the parameter space where the signal is allowed to vary in an irregular fashion. Second, in infinite dimension the action of a signal (even finite dimensional) can be described by differential (or more generally unbounded) operators.

Due to these basic difficulties attempts to use RPT to study rough PDEs have been limited by two factors: the first one is the need to look at RPDEs as evolutions in Banach spaces perturbed by one parameter rough signals (in order to keep rough paths as basic objects), the second one is the need to avoid unbounded operators by looking at mild formulations or Feynman–Kac formulas or transforming the equation in order to absorb the rough dependence into better understood objects (e.g. flow of characteristic curves).

These requirements pose strong limitations on the kind of results one is allowed to obtain for RPDEs and the proof strategies are very different from the classical PDE proofs. The most successful approaches to RPDEs do not even allow to characterise the solution directly but only via a transformation to a more standard PDE problem. The need of a mild formulation of a given problem leads usually to very strong structural requirements like for example semilinearity. We list here some pointers to the relevant literature:

- Flow transformations applied to viscosity formulation of fully non-linear RPDEs (including Backward rough differential equations) have been studied in a series of work by Friz and coauthors: Diehl and Friz [17], Friz and Oberhauser [22], Caruana and Friz [6], Diehl, Friz and Oberhauser [18], Caruana, Friz and Oberhauser [7] and finally Friz, Gassiat, Lions and Souganidis [21].
- Rough formulations of evolution heat equation with multiplicative noise (with varying degree of success) have been considered by Gubinelli and Tindel [33], Deya, Gubinelli and Tindel [15], Teichmann [59], Hu and Nualart [43] and Garrido-Atienza, Lu and Schmalfuss [26].
- Mild formulation of rough Burgers equations with spatially irregular noise have been first introduced by Hairer and Weber [38, 37] and Hairer, Maas and Weber [36] leading to the groundbreaking work of Hairer on the Kardar–Parisi–Zhang equation [35].

- Solutions of conservation laws with rough fluxes have been studied via flow transformation by Friz and Gess [23] and via the transformed test function approach by Lions, Perthame and Souganidis [50, 48, 49], Gess and Souganidis [29, 28], Gess, Souganidis and Perthame [27] and Hofmanová [41].

Hairer's regularity structure theory [34] is a wide generalisation of rough path which allows irregular objects parametrized by multidimensional indices. A more conservative approach, useful in many situations but not as general, is the paracontrolled calculus developed by Gubinelli, Imkeller and Perkowski [32, 30]. These techniques go around the first limitation. In order to apply them however the PDEs need usually to have a mild formulation where the unbounded operators are replaced by better behaved quantities and in general by bounded operators in the basic Banach spaces where the theory is set up. Existence and uniqueness of solutions to RPDEs are then consequences of standard fixed-point theorems in the Banach setting.

Standard PDE theory developed tools and strategies to study *weak solutions* to PDEs, that is distributional relations satisfied by the unknown together with its weak derivatives. From a conceptual point of view the wish arises to devise an approach to RPDEs which borrow as much as possible from the variety of tools and techniques of PDE theory. From this point of view various authors started to develop *intrinsic* formulations of RPDEs as which involves relations between certain distributions associated to the unknown and the rough paths associated to the input signal. Let us mention the work of Gubinelli, Tindel and Torrecilla [31] on viscosity solutions to fully non–linear rough PDEs, that of Catellier [8] on rough transport equations (in connection with the regularisation by noise phenomenon), Diehl, Friz and Stannat [16] and finally of Bailleul and Gubinelli [1] on rough transport equations. This last work introduces for the first time apriori estimates for RPDEs, that is estimates which holds for any weak solution of the RPDE (though we should also mention the contribution [55], in which weak formulations are investigated for Young type equations driven by fractional Brownian motions with Hurst parameter $H > 1/2$). These estimates are crucial to derive control on various norms of the solution and obtain existence and uniqueness results bypassing the use of the rough flow of characteristics which has been the main tool of many of the previous works on this subject.

In the present paper we continue the development of general tools for RPDEs. The key problem with rough paths and rough integrals is the absence of very basic and effective tools like Gronwall lemma. Skimming over any book on PDEs shows how fundamental is this tool for any nontrivial result on weak solutions. Our main technical contribution is a strategy to obtain apriori estimates of Gronwall type for RPDE via a *Rough Gronwall lemma*. This result is not very difficult to prove but, as the standard one, it is the cornerstone of various arguments aiming at establishing properties of weak solutions to RPDEs leading to well–posedness. We discuss how to use this new tool using as a first example a toy model of a heat equation with linear transport noise and as a more compelling application we analyse in depth scalar conservation laws with rough fluxes establishing well-posedness in a rather natural way.

Conservation laws and related equations have been paid an increasing attention lately and have become a very active area of research, counting nowadays quite a number of results for deterministic and stochastic setting, that is for conservation laws either of the form

$$\partial_t u + \operatorname{div}(A(u)) = 0 \tag{1.1}$$

(see [4, 5, 44, 46, 51, 47, 56, 57]) or

$$\mathrm{d}u + \operatorname{div}(A(u))\mathrm{d}t = g(x,u)\mathrm{d}W$$

where the Itô stochastic forcing is driven by a finite- or infinite-dimensional Wiener process (see [2, 9, 14, 12, 13, 20, 40, 42, 45, 58, 60]). Degenerate parabolic PDEs were studied in [5, 10] and in the stochastic setting in [3, 11, 39].

Recently, several attempts have already been made to extend rough path techniques to conservation laws as well. First, Lions, Perthame and Souganidis (see [48, 50]) developed a pathwise approach for

$$\mathrm{d}u + \operatorname{div}(A(x,u)) \circ \mathrm{d}W = 0$$

where W is a continuous real-valued signal and $\circ$ stands for the Stratonovich product in the Brownian case, then Friz and Gess (see [23]) studied

$$\mathrm{d}u + \operatorname{div} f(t,x,u)\mathrm{d}t = F(t,x,u)\mathrm{d}t + \Lambda_k(x,u,\nabla u)\mathrm{d}z^k$$

where Λ_k is affine linear in u and ∇u and $z = (z^1,\dots,z^K)$ is a rough driving signal. Gess and Souganidis [29] considered

$$\mathrm{d}u + \operatorname{div}(A(x,u))\mathrm{d}z = 0 \tag{1.2}$$

where $z = (z^1,\dots,z^M)$ is a geometric α-Hölder rough path and in [28] they studied the long-time behavior in the case when z is a Brownian motion. Hofmanová [41] then generalized the method to the case of mixed rough-stochastic model

$$\mathrm{d}u + \operatorname{div}(A(x,u))\mathrm{d}z = g(x,u)\mathrm{d}W,$$

where z is a geometric α-Hölder rough path, W is a Brownian motion and the stochastic integral on the right hand side is understood in the sense of Itô.

It was observed already a long time ago that, in order to find a suitable concept of solution for problems of the form (1.1), on the one hand classical C^1 solutions do not exist in general and, on the other hand, weak or distributional solutions lack uniqueness. The first claim is a consequence of the fact that any smooth solution has to be constant along characteristic lines, which can intersect in finite time (even in the case of smooth data) and shocks can be produced. The second claim demonstrates the inconvenience that often appears in the study of PDEs and SPDEs: the usual way of weakening the equation leads to the occurrence of nonphysical solutions and therefore additional assumptions need to be imposed in order to select the physically relevant ones and to ensure uniqueness. Hence one needs to find some balance that allows to establish existence of a unique (physically reasonable) solution.

Towards this end, Kružkov [46] introduced the notion of entropy solution to (1.1), further developed in the stochastic setting in [2, 5, 20, 45, 60]. Here we pursue the kinetic approach, a concept of solution that was first introduced by Lions, Perthame, Tadmor [47] for deterministic hyperbolic conservation laws and further studied in [4], [10], [44], [51], [47], [57], [56]. This direction also appears in several works on stochastic conservation laws and degenerate parabolic SPDEs, see [11], [14], [12], [13], [40], [39] and in the (rough) pathwise works [28], [29], [41], [48], [50].

Kinetic solutions are more general in the sense that they are well defined even in situations when neither the original conservation law nor the corresponding entropy inequalities can be understood in the sense of distributions. Usually this happens due to lack of integrability of the flux and entropy-flux terms, e.g. $A(u) \notin L^1_{\mathrm{loc}}$. Therefore, further assumptions on initial data or the flux function A are in place in order to overcome this issue and remain in the entropy setting. It will be seen later on that no such restrictions are necessary in the kinetic approach as the equation that is actually being solved – the so-called kinetic formulation, see (4.2) – is in fact linear. In addition, various proofs simplify as methods for linear PDEs are available.

In the present paper, we are concerned with scalar rough conservation laws of the form (1.2), where $z = (z^1, \ldots, z^M)$ can be lifted to a geometric rough path of finite p-variation for $p \in [2, 3)$. We will show how our general tools allows to treat (1.2) along the lines of the standard PDEs proof strategy. Unlike the known results concerning the same problem (see e.g. [29, 48, 50]), our method does not rely on the flow transformation method and so it overcomes the limitations inevitably connected with such a transformation. Namely, we are able to significantly weaken the assumptions on the flux coefficient $A = (A_{ij})$: we assume that $a_{ij} = \partial_\xi A_{ij}$ and $b_j = \operatorname{div}_x A_{\cdot j}$ belong to $W^{3,\infty}$, whereas in [29] the regularity of order $\mathrm{Lip}^{2+\gamma}$ is required for some $\gamma > \frac{1}{\alpha}$ with $\alpha \in (0, 1)$ being the Hölder regularity of the driving signal. For a 2-step rough path, i.e. in the range $\alpha \in (\frac{1}{3}, \frac{1}{2}]$, it therefore means that almost 5 derivatives might be necessary. Nevertheless, let us point out that even the regularity we require is not the optimal one. To be more precise, with a more refined method we expect that one could possibly only assume $W^{\gamma,\infty}$-regularity for the coefficients a, b with $\gamma > p$. Moreover extensions of the method to any $p \geqslant 3$ does not pose any particular problem but only requires heavier notations and longer proofs. For the sake of a cleaner presentation and to convey as effectively as possible the key points of our strategy we will limit here to discuss the first non–trivial case (namely $p \in [2, 3)$) which shows already the general pattern.

Outline of the paper. In Section 2 we fix notations, introduce the notion of unbounded rough driver and establish the main tools used thereafter: a priori estimates for distributional solutions to rough equations and a related rough Gronwall lemma. In order to allow the reader to understand more easily the general strategy we present it via a simple toy model of a rough heat equation with transport noise. Using this toy model we discuss in Section 3 the details of the tensorization method needed to prove bounds on nonlinear functions of the solution. Section 4 introduces the setting for the analysis of conservations laws with rough fluxes. Section 5 uses the tensorization method to obtain estimates leading to reduction, L^1-contraction and finally uniqueness for kinetic solutions. In Section 6 we prove some L^p-apriori bounds on solutions which are stable under rough path topology. These bounds are finally used in Section 7 to prove existence of kinetic solutions.

2. General a priori estimates for rough PDEs

2.1. **Notation.** First of all, let us recall the definition of the increment operator, denoted by δ. If g is a path defined on $[0, T]$ and $s, t \in [0, T]$ then $\delta g_{st} := g_t - g_s$, if g is a 2-index map defined on $[0, T]^2$ then $\delta g_{sut} := g_{st} - g_{su} - g_{ut}$. The norm of the element g in the Banach space E will be written as $\mathcal{N}[g; E]$. For two quantities a and b the relation $a \lesssim_x b$ means $a \leqslant c_x b$, for a constant c_x depending on a multidimensional parameter x.

In the sequel, given an interval I we call *control on I* (and denote it by ω) any superadditive map on $\Delta_I := \{(s, t) \in I^2 : \ s \leqslant t\}$, that is, any map $\omega : \Delta_I \to [0, \infty[$ such that, for all $s \leqslant u \leqslant t$,

$$\omega(s, u) + \omega(u, t) \leqslant \omega(s, t).$$

We will say that a control is *regular* if $\lim_{|t-s|\to 0} \omega(s,t) = 0$. Also, given a control ω on an interval $I = [a,b]$, we will use the notation $\omega(I) := \omega(a,b)$. Given a time interval I, a parameter $p > 0$, a Banach space E and any function $g : I \to E$ we define the norm

$$\mathcal{N}[g; \overline{V}_1^p(I;E)] := \sup_{(t_i)\in\mathcal{P}(I)} \left(\sum_i |g_{t_i} - g_{t_{i+1}}|^p\right)^{\frac{1}{p}},$$

where $\mathcal{P}(I)$ denotes the set of all partitions of the interval I. In this case,

$$\omega_g(s,t) = \mathcal{N}[g; \overline{V}_1^p([s,t];E)]^p$$

defines a control on I, and we denote by $V_1^p(I;E)$ the set of elements $g \in \overline{V}_1^p(I;E)$ for which ω_g is regular on I. We denote by $\overline{V}_2^p(I;E)$ the set of two-index maps $g : I \times I \to E$ with left and right limits in each of the variables and for which there exists a control ω such that

$$|g_{st}| \leqslant \omega(s,t)^{\frac{1}{p}}$$

for all $s, t \in I$. We also define the space $\overline{V}_{2,\mathrm{loc}}^p(I;E)$ of maps $g : I \times I \to E$ such that there exists a countable covering $\{I_k\}_k$ of I satisfying $g \in \overline{V}_2^p(I_k;E)$ for any k. We write $g \in V_2^p(I;E)$ or $g \in V_{2,\mathrm{loc}}^p(I;E)$ if the control can be chosen regular.

Lemma 2.1 (Sewing lemma)**.** *Fix an interval I, a Banach space E and a parameter $\zeta > 1$. Consider a map $h : I^3 \to E$ such that $h \in \operatorname{Im}\delta$ and for every $s < u < t \in I$,*

$$|h_{sut}| \leqslant \omega(s,t)^{\zeta}, \tag{2.1}$$

for some regular control ω on I. Then there exists a unique element $\Lambda h \in V_2^{\frac{1}{\zeta}}(I;E)$ such that $\delta(\Lambda h) = h$ and for every $s < t \in I$,

$$|(\Lambda h)_{st}| \leqslant C_\zeta\, \omega(s,t)^{\zeta}, \tag{2.2}$$

for some universal constant C_ζ.

Proof. The proof follows that given in [24, Lemmma 4.2] for Hölder norms, we only specify the modification needed to handle variation norms. Regarding existence, we recall that since $\delta h = 0$, there exists a 2-index map B such that $\delta B = h$. Let s, $t \in [0,T]$, such that $s < t$, and consider a sequence $\{\pi_n;\, n \geqslant 0\}$ of partitions $\{s = r_0^n < \cdots < r_{k_n+1}^n = t\}$ of $[s,t]$. Assume that $\pi_n \subset \pi_{n+1}$ and $\lim_{n\to\infty} k_n = \infty$. Set

$$M_{st}^{\pi_n} = B_{st} - \sum_{i=0}^{k_n} B_{r_i^n, r_{i+1}^n}.$$

Due to superadditivity of ω it can be seen that there exists $l \in \{1,\ldots,k_n\}$ such that $\omega(r_{l-1}^n, r_{l+1}^n) \leqslant 2\omega(s,t)/k_n$. Now we choose such an index l and transform π_n into $\hat{\pi}$, where $\hat{\pi} = \{r_0^n < r_1^n < \cdots < r_{l-1}^n < r_{l+1}^n < \cdots < r_{k_n+1}^n\}$. Then

$$M_{st}^{\hat{\pi}} = M_{st}^{\pi_n} - (\delta B)_{r_{l-1}^n, r_l^n, r_{l+1}^n} = M_{st}^{\pi_n} - h_{r_{l-1}^n, r_l^n, r_{l+1}^n}$$

and hence

$$|M_{st}^{\hat{\pi}} - M_{st}^{\pi_n}| \leqslant \omega(r_{l-1}^n, r_{l+1}^n)^{\zeta} \leqslant \left[\frac{2\omega(s,t)}{k_n}\right]^{\zeta}.$$

Repeating this operation until we end up with the trivial partition $\hat{\pi}_0 = \{s,t\}$, for which $M_{st}^{\hat{\pi}_0} = 0$ this implies that $M_{st}^{\pi_n}$ converges to some M_{st} satisfying

$$|M_{st}| = \lim_n |M_{st}^{\pi_n}| \leqslant 2^\zeta \omega(s,t)^\zeta \sum_{i=1}^{\infty} i^{-\zeta} \leqslant C_\zeta\, \omega(s,t)^\zeta.$$

□

2.2. **Unbounded rough drivers.** Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In what follows, we call a *scale* any sequence $\left(E_n, \|\cdot\|_n\right)_{n\in\mathbb{N}_0}$ of Banach spaces such that E_{n+1} is continuously embedded into E_n. Besides, for $n \in \mathbb{N}_0$ we denote by E_{-n} the topological dual of E_n.

Definition 2.2. Let $p \in [2,3)$ be given. A continuous unbounded p-rough driver with respect to the scale $\left(E_n, \|\cdot\|_n\right)_{n\in\mathbb{N}_0}$, is a pair $\mathbf{A} = \left(A^1, A^2\right)$ of 2-index maps such that

$$A_{st}^1 \in \mathcal{L}(E_{-n}, E_{-(n+1)}) \quad \text{for} \quad n \in \{0,2\}, \qquad A_{st}^2 \in \mathcal{L}(E_{-n}, E_{-(n+2)}) \quad \text{for} \quad n \in \{0,1\},$$

and there exists a continuous control ω_A on $[0,T]$ such that for every $s,t \in [0,T]$,

$$\begin{aligned} \|A_{st}^1\|_{\mathcal{L}(E_{-n},E_{-(n+1)})}^p &\leqslant \omega_A(s,t) \qquad \text{for} \quad n \in \{0,2\}, \\ \|A_{st}^2\|_{\mathcal{L}(E_{-n},E_{-(n+2)})}^{p/2} &\leqslant \omega_A(s,t) \qquad \text{for} \quad n \in \{0,1\}, \end{aligned}$$

and, in addition, the Chen's relation holds true, that is,

$$\delta A_{sut}^1 = 0, \qquad \delta A_{sut}^2 = A_{ut}^1 A_{su}^1, \qquad \text{for all} \quad 0 \leqslant s < u < t \leqslant T. \tag{2.3}$$

2.3. **Motivation.** In order to present the main ideas of our approach towards rough PDEs, let us consider a toy model of a linear rough heat equation of the form

$$\begin{aligned} \mathrm{d}u &= \Delta u \, \mathrm{d}t + V \cdot \nabla u \, \mathrm{d}z, \qquad x \in \mathbb{R}^N,\ t \in (0,T), \\ u(0) &= u_0, \end{aligned} \tag{2.4}$$

where $V = (V^1, \dots, V^K)$ is a family of sufficiently smooth and bounded vector fields on $\mathbb{R}^N$ and $z = (z^1, \dots, z^K)$ can be lifted to a geometric p-rough path for $p \in [2,3)$. We denote by $\mathbf{Z} = (Z^1, Z^2)$ its rough path lift. Let us insist on the fact that the linearity of the leading order operator does not play any role and the discussion below can be easily adapted to quasilinear elliptic or monotone operators. Setting (using the Einstein summation convention)

$$A_{st}^1 u := Z_{st}^{1,k}\, V^k \cdot \nabla u, \qquad A_{st}^2 u := Z_{st}^{2,jk}\, V^k \cdot \nabla(V^j \cdot \nabla u) \tag{2.5}$$

we observe that $\mathbf{A} = (A^1, A^2)$ defines an unbounded p-rough driver in the scale $E_n = W^{n,2}(\mathbb{R}^N)$. Therefore, motivated by [1] we are lead to the following formulation of (2.4) which should be satisfied for every $0 \leqslant s \leqslant t \leqslant T$ and every test function $\varphi \in E_3$:

$$\delta u(\varphi)_{st} = \int_s^t u_r(\Delta\varphi)\mathrm{d}r + u_s(A_{st}^{1,*}\varphi) + u_s(A_{st}^{2,*}\varphi) + u_{st}^\natural(\varphi) \tag{2.6}$$

where $u^\natural$ is an E_{-3}-valued 2-index map such that for every $\varphi \in E_3$ the map $u_{st}^\natural(\varphi)$ possesses sufficient time regularity, namely, it has finite local $\frac{q}{3}$-variation for some $q < 3$ and some regular control. In expression (2.6), let us also label the operators $A^{1,*}$ and $A^{2,*}$ for further use:

$$A_{st}^{1,*}\varphi = -Z_{st}^{1,k}\, \mathrm{div}(V^k\varphi), \quad \text{and} \quad A_{st}^{2,*}\varphi = Z_{st}^{2,jk}\, \mathrm{div}(V^j \mathrm{div}(V^k\varphi)). \tag{2.7}$$

Remark 2.3. This formulation has an intuitive appeal, however note that it should more precisely be understood as saying that u is a solution of the equation if the distribution

$$u^\natural_{st}(\varphi) := \delta u(\varphi)_{st} - \int_s^t u_r(\Delta\varphi)\mathrm{d}r - u_s(A^{1,*}_{st}\varphi) - u_s(A^{2,*}_{st}\varphi)$$

belongs to $V^{\frac{q}{3}}_{2,\mathrm{loc}}([0,T],E_{-3})$. In this paper we will set up the convention that every 2-index map with a $\natural$ superscript denotes an element of $V^{1/\zeta}_{2,\mathrm{loc}}([0,T],E_{-3})$ for some $\zeta>1$.

Before we continue with the presentation of the rough path framework, let us briefly recall how the basic energy estimates are derived in the classical case when z is a smooth path. In that situation, one formally tests (2.4) by $\varphi=u$ and integration by parts leads to

$$\begin{aligned}\|u_t\|^2_{L^2} + 2\int_0^t \|\nabla u_r\|^2_{L^2}\,\mathrm{d}r &= \|u_0\|^2_{L^2} + \int_0^t u_r^2(\operatorname{div} V)\,\mathrm{d}z_r\\ &\leqslant \|u_0\|^2_{L^2} + \|V\|_{W^{1,\infty}}\int_0^t \|u_r\|^2_{L^2}\,\mathrm{d}|z_r|,\end{aligned}$$

where L^2 stands for $L^2(\mathbb{R}^N)$. Hence the Gronwall lemma applies and we obtain

$$\|u_t\|^2_{L^2} + 2\int_0^t \|\nabla u_r\|^2_{L^2}\,\mathrm{d}r \leqslant \mathrm{e}^{\|V\|_{W^{1,\infty}}\|z\|_{1\text{-var}}}\|u_0\|^2_{L^2}$$

which consequently implies that, in the case of a smooth driver z, weak solutions of (2.4) belong to $C([0,T];L^2(\mathbb{R}^N))\cap L^2(0,T;W^{1,2}(\mathbb{R}^N))$.

Therefore, in order to derive the energy estimate in the rough path framework, one needs to perform two steps. First, one has to justify the "test by u" or in other words one has to apply some Itô formula to the function $u\mapsto\|u\|^2_{L^2}$. Second, one needs a suitable replacement for the Gronwall lemma argument. In the proof of existence, the first step can be easily justified. Namely, one works with a suitable approximate smooth path $\mathbf{Z}^n$ such that the corresponding equation possesses a smooth solution and consequently the equation for u^2 can be derived using classical (deterministic) arguments. Note that such a simplification is no longer possible in the proof of uniqueness where the only formulation and regularity at hand is the one given by the a priori estimate, namely, $C([0,T];L^2(\mathbb{R}^N))\cap L^2(0,T;W^{1,2}(\mathbb{R}^N))$. However, we postpone the discussion of this issue to Section 3.

An important observation is that since we only expect a weak solution to (2.4) to belong to $C([0,T];L^2(\mathbb{R}^N))\cap L^2(0,T;W^{1,2}(\mathbb{R}^N))$, as in the smooth setting, the term

$$\int_0^\cdot \|\nabla u_r\|^2_{L^2}\,\mathrm{d}r,$$

which appears once we test by u, is only expected to be continuous and of finite variation but not Hölder continuous. This is the reason why it is necessary to work in the p-variation framework, even if the driver $\mathbf{A}$ is taken to be unbounded $\frac{1}{p}$-Hölder rough driver according to [1, Definition 5].

In order to understand (somehow heuristically) the rough path mechanism, let us assume that we are able to derive rigorously the equation for u^2. This should yield the following dynamics:

$$\begin{aligned}\delta u^2(\varphi)_{st} = -2\int_s^t |\nabla u_r|^2(\varphi)\,\mathrm{d}r - 2\int_s^t (u\nabla u_r)(\nabla\varphi)\,\mathrm{d}r \\ + u_s^2(A_{st}^{1,*}\varphi) + u_s^2(A_{st}^{2,*}\varphi) + u_{st}^{2,\natural}(\varphi)\end{aligned} \tag{2.8}$$

for some new remainder $u^{2,\natural}$. Remark that since u^2 is expected to belong to $L^1(\mathbb{R}^N)$, the corresponding scale of test function spaces here is $E_n = W^{n,\infty}(\mathbb{R}^N)$, unlike in (2.6) where we considered $E_n = W^{n,2}(\mathbb{R}^N)$. Choosing $\varphi = 1$ and $s = 0$ leads to

$$\begin{aligned}\|u_t\|_{L^2}^2 + 2\int_0^t \|\nabla u_r\|_{L^2}^2\,\mathrm{d}r &= \|u_0\|_{L^2}^2 + u_0^2(A_{0t}^{1,*}1) + u_0^2(A_{0t}^{2,*}1) + u_{0t}^{2,\natural}(1) \\ &\lesssim \|u_0\|_{L^2}^2\Big(1 + |Z_{0t}^1|\|V\|_{W^{1,\infty}} + |Z_{0t}^2|\|V\|_{W^{2,\infty}}^2\Big) + |u_{0t}^{2,\natural}(1)|.\end{aligned} \tag{2.9}$$

In order to achieve a rough Gronwall lemma type argument, it is now easily seen that we need to estimate the remainder $u^{2,\natural}$ in terms of the left hand side in (2.9). Otherwise stated, our problem is reduced to show that:

$$|u_{0t}^{2,\natural}(1)| \lesssim_t \sup_{0\leqslant r\leqslant t} \|u_r\|_{L^2}^2 + 2\int_0^t \|\nabla u_r\|_{L^2}^2\,\mathrm{d}r$$

in such a way that the proportional constant can be made sufficiently small provided t is small. Such an estimate is an important element of our general approach towards RPDEs and is presented in full detail and generality in the next subsection.

2.4. **A priori estimates and rough Gronwall lemma.** The aim of this subsection is to introduce mathematical tools that allow to derive a priori estimates for a wide class of rough PDEs. To this end and in view of the discussion above, we are concerned with general rough PDEs of the form

$$\mathrm{d}g_t = \mu(\mathrm{d}t) + \mathbf{A}(\mathrm{d}t)g_t. \tag{2.10}$$

where $\mathbf{A} = (A^1, A^2)$ is an unbounded p-rough driver on the scale (E_n) and the drift μ, which possibly also depends on the solution, is continuous of finite variation. To make the connection to the toy model (2.4), eq. (2.10) is understood in the sense of distributions and corresponds to (2.8) in our particular example, i.e. g corresponds to u^2 and the drift $\delta\mu$ is an increment of the integral

$$\int_0^{\cdot} u_r\Delta u_r\,\mathrm{d}r.$$

We now give a rigorous meaning to such an equation.

Definition 2.4. Let $p \in [2,3)$ and fix an interval $I \subseteq [0,T]$. Let $\mathbf{A} = \big(A^1, A^2\big)$ be a continuous unbounded p-rough driver on I with respect to the scale (E_n) and let $\mu \in \overline{V}_1^1(I; E_{-3})$. A path $g : I \to E_{-0}$ is called a solution (on I) of the equation (2.10) provided there exists $q < 3$ and $g^\natural \in V_{2,\mathrm{loc}}^{\frac{q}{3}}(I, E_{-3})$ such that, for every $s,t \in I$, $s < t$,

$$(\delta g)_{st}(\varphi) = (\delta\mu)_{st}(\varphi) + g_s(\{A_{st}^{1,*} + A_{st}^{2,*}\}\varphi) + g_{st}^\natural(\varphi). \tag{2.11}$$

Let us now present our first main result on an a priori estimate for the remainder $g^\natural$ which will be employed to derive the energy estimate for (2.10). We make use of a family of smoothing operators $(J^\eta)_{\eta\in(0,1)}$ acting on the scale $(E_n)_{n\in\mathbb{N}}$ and satisfying, for $k\in\{0,1,2\}$,

$$\|J^\eta-\mathrm{Id}\|_{\mathcal{L}(E_{n+k},E_n)}\lesssim \eta^k,\qquad \|J^\eta\|_{\mathcal{L}(E_n,E_{n+k})}\lesssim\eta^{-k}. \tag{2.12}$$

An important role is played by the E_{-1}-valued 2-index map $g^\sharp$ defined as follows

$$g^\sharp_{st}(\varphi):=\delta g(\varphi)_{st}-g_s(A^{1,*}_{st}\varphi), \tag{2.13}$$

observe that due to (2.11) it is also given by

$$g^\sharp_{st}(\varphi)=(\delta\mu)_{st}(\varphi)+g_s(A^{2,*}_{st}\varphi)+g^\natural_{st}(\varphi).$$

In the following result we will make use of both these expressions depending on the necessary regularity: the former one contains terms that are less regular in time but more regular in space (i.e. they require less regular test functions) whereas the terms in the latter one are more regular in time but less regular in space.

Theorem 2.5. *Let $p\in[2,3)$ and fix an interval $I\subseteq[0,T]$. Let $\mathbf{A}=(A^1,A^2)$ be a continuous unbounded p-rough driver on the scale (E_n) endowed with the family of smoothing operators (J^η). Consider a path $\mu\in\overline{V}^1_1(I;E_{-3})$ such that for every $\varphi\in E_3$, there exists a control ω_μ and a constant $\lambda\in[p,3]$ such that*

$$\sup_{\eta\in(0,1)}\eta^{\lambda-k}|(\delta\mu)_{st}(J^\eta\varphi)|\leqslant\omega_\mu(s,t)\,\|\varphi\|_{E_k}\qquad \textit{for } k=1,2. \tag{2.14}$$

Let g be a solution on I of the equation (2.10), with the following additional hypothesis: g is controlled over the whole interval I, that is we have $g^\natural\in V_2^{\frac{q}{3}}(I;E_{-3})$, and assume

$$q\in\left[\frac{3p\lambda}{2p+\lambda},3\right). \tag{2.15}$$

Moreover let $G=\mathcal{N}[g;L^\infty(s,t;E_{-0})]$, fix $\kappa\in[0,\frac{1}{p})$ such that

$$\frac12\left(\frac3p-\frac3q\right)\geqslant\kappa\geqslant\frac{1}{\lambda-2}\left(\frac3q-1-\frac{3-\lambda}{p}\right) \tag{2.16}$$

and let

$$\omega_I(s,t):=G^{\frac q3}\,\omega_A(s,t)^{\frac q3(\frac3p-2\kappa)}+\omega_\mu(s,t)^{\frac q3}\omega_A(s,t)^{\frac q3(\frac{3-\lambda}{p}+\kappa(\lambda-2))}. \tag{2.17}$$

Then there exists a constant $L>0$ (independent of I) such that if $\omega_A(I)\leqslant L$ then for all $s,t\in I$, $s<t$,

$$\|g^\natural_{st}\|_{E_{-3}}\lesssim_{q,\mathbf{A},I}\omega_I(s,t)^{\frac3q}. \tag{2.18}$$

Proof. Let $\omega_\natural(s,t)$ be a regular control such that $\|g^\natural_{st}\|_{E_{-3}}\le\omega_\natural(s,t)^{\frac3q}$ for any $s,t\in I$. Let $\varphi\in E_3$ be such that $\|\varphi\|_{E_3}\leqslant1$. We first show that

$$(\delta g^\natural(\varphi))_{sut}=(\delta g)_{su}(A^{2,*}_{ut}\varphi)+g^\sharp_{su}(A^{1,*}_{ut}\varphi), \tag{2.19}$$

where $g^\sharp$ was defined in (2.13). Indeed, owing to (2.11), we have

$$g^\natural_{st}(\varphi)=\delta g(\varphi)_{st}-g_s\left([A^{1,*}_{st}+A^{2,*}_{st}](\varphi)\right)-\delta\mu(\varphi)_{st},$$

where we omit the time indices for notational sake. Applying δ on both sides of this identity and recalling Chen's relations (2.3) as well as the fact that $\delta\delta = 0$ we thus get

$$\delta g^\natural_{sut}(\varphi) = (\delta g)_{su}([A^{1,*}_{ut} + A^{2,*}_{ut}](\varphi)) - g_s(A^{1,*}_{su} A^{1,*}_{ut}(\varphi)).$$

Plugging relation (2.13) again into this identity, we end up with our claim (2.19).

The aim now is to bound the terms on the right hand side of (2.19) separately by the allowed quantities G, ω_μ, $\omega_\natural$ and to reach a sufficient time regularity as required by the sewing lemma 2.1. To this end, we make use of the smoothing operators (J^η) and repeatedly apply the equation (2.11) as well as the two equivalent definitions of $g^\sharp$ from (2.13). We obtain

$$\begin{aligned}
\delta g^\natural(\varphi)_{sut} &= (\delta g)_{su}(J^\eta A^{2,*}_{ut}\varphi) + (\delta g)_{su}((\mathrm{Id} - J^\eta)A^{2,*}_{ut}\varphi) \\
&\quad + g^\sharp_{su}(J^\eta A^{1,*}_{ut}\varphi) + g^\sharp_{su}((\mathrm{Id} - J^\eta)A^{1,*}_{ut}\varphi) \\
&= g_s(A^{1,*}_{su} J^\eta A^{2,*}_{ut}\varphi) + g_s(A^{2,*}_{su} J^\eta A^{2,*}_{ut}\varphi) + (\delta\mu)_{su}(J^\eta A^{2,*}_{ut}\varphi) + g^\natural_{su}(J^\eta A^{2,*}_{ut}\varphi) \\
&\quad + (\delta g)_{su}((\mathrm{Id} - J^\eta)A^{2,*}_{ut}\varphi) \\
&\quad + g_s(A^{2,*}_{su} J^\eta A^{1,*}_{ut}\varphi) + (\delta\mu)_{su}(J^\eta A^{1,*}_{ut}\varphi) + g^\natural_{su}(J^\eta A^{1,*}_{ut}\varphi) \\
&\quad + (\delta g)_{su}((\mathrm{Id} - J^\eta)A^{1,*}_{ut}\varphi) - g_s(A^{1,*}_{su}(\mathrm{Id} - J^\eta)A^{1,*}_{ut}\varphi) \\
&= I_1 + \cdots + I_{10}
\end{aligned} \tag{2.20}$$

The use of the smoothing operators reflects the competition between space and time regularity in the various terms in the equation. To be more precise, the only available norm of g is $L^\infty(0, T; E_{-0})$. So on the one hand g does not possess any time regularity (at least at this point of the proof) but on the other hand it does not require any space regularity of the corresponding test functions. In general, the presence of $(\mathrm{Id} - J^\eta)$ allows to apply the first estimate from (2.12) to make use of the additional space regularity in order to compensate for the lack of time regularity. Correspondingly, the second estimate from (2.12) allows to use the additional time regularity in order to compensate for the lack of space regularity.

Now bound the above as follows.

$$\begin{aligned}
|I_1| + |I_2| + |I_6| &\lesssim G\,\omega_A(s,t)^{\frac{3}{p}} + \eta^{-1}\,G\,\omega_A(s,t)^{\frac{4}{p}} + G\,\omega_A(s,t)^{\frac{3}{p}}, \\
|I_3| + |I_7| &\lesssim \eta^{1-\lambda}\,\omega_\mu(s,t)\omega_A(s,t)^{\frac{2}{p}} + \eta^{2-\lambda}\,\omega_\mu(s,t)\omega_A(s,t)^{\frac{1}{p}}, \\
|I_4| + |I_8| &\lesssim \eta^{-2}\,\omega_\natural(s,t)^{\frac{3}{q}}\omega_A(s,t)^{\frac{2}{p}} + \eta^{-1}\,\omega_\natural(s,t)^{\frac{3}{q}}\omega_A(s,t)^{\frac{1}{p}}, \\
|I_5| + |I_9| + |I_{10}| &\lesssim \eta\,G\,\omega_A(s,t)^{\frac{2}{p}} + \eta^2\,G\,\omega_A(s,t)^{\frac{1}{p}} + \eta\,G\,\omega_A(s,t)^{\frac{2}{p}}
\end{aligned}$$

Finally, we choose $\eta \in (0,1)$ to equilibrate the various terms, that is, we set

$$\eta = \omega_A(I)^{-\frac{1}{p}+\kappa}\omega_A(s,t)^{\frac{1}{p}-\kappa}$$

for some $\kappa \in [0, \frac{1}{p})$ to be chosen below. Assuming that $\omega_A(I) \leqslant 1$ we deduce

$$\begin{aligned}
|\delta g^\natural(\varphi)_{sut}| &\lesssim G\,\omega_A(s,t)^{\frac{3}{p}} \\
&\quad + \omega_\mu(s,t)\omega_A(s,t)^{\frac{3-\lambda}{p}+\kappa(\lambda-2)} \\
&\quad + 2\omega_A(I)^{\frac{1}{p}+\kappa}\omega_\natural(s,t)^{\frac{3}{q}}\omega_A(s,t)^{\kappa} \\
&\quad + G\,\omega_A(I)^{-2(\frac{1}{p}+\kappa)}\omega_A(s,t)^{\frac{3}{p}-2\kappa}.
\end{aligned}$$

Note that there are only two terms where we kept track of $\omega_A(I)$, namely, the one that needs to be absorbed to the left hand side eventually, i.e. the one containing $\omega_\natural$, and the one with a negative power. The latter one can be further estimated from above by a constant depending on $\mathbf{A}$ and I if we assume that $I \neq \emptyset$ and the former one will then determine the value of the constant L from the statement of the Theorem. Consequently, recalling the definition (2.17) of ω_I, we obtain

$$|(\delta g^\natural)_{sut}(\varphi)| \lesssim \omega_I(s,t)^{\frac{3}{q}} + 2\omega_A(I)^{\frac{1}{p}+\kappa}\omega_\natural(s,t)^{\frac{3}{q}}\omega_A(s,t)^\kappa,$$

By definition, for a suitable choice of the parameter κ, the mapping ω_I is a regular control. Indeed, the regularity of ω_I easily stems from the continuity of ω_A whereas superadditivity is obtained from [25, Exercise 1.9] by recalling that both ω_A and ω_μ are controls, and choosing $\kappa \in [0, \frac{1}{p})$ such that

$$\frac{q}{3}\left(\frac{3}{p} - 2\kappa\right) \geqslant 1, \qquad \frac{q}{3} + \frac{q}{3}\left(\frac{3-\lambda}{p} + \kappa(\lambda - 2)\right) \geqslant 1.$$

This leads to the condition (2.16) and due to (2.15) such a $\kappa \in [0, \frac{1}{p})$ exists.

Since $\omega_\natural$ is also a regular control, $\delta g^\natural(\varphi)$ satisfies the assumptions of Lemma 2.1 and we can thus conclude that, for all $s < t \in I$, $|g^\natural_{st}(\varphi)| \leqslant \omega'_\natural(s,t)^{\frac{3}{q}}$ where

$$\omega'_\natural(s,t) = C_{p,q}(\omega_I(s,t) + \omega_A(I)^{\frac{q}{3}(\frac{1}{p}+\kappa)}\omega_\natural(s,t)\omega_A(s,t)^{\frac{\kappa q}{3}})$$

is a new control. The transformation $\omega_\natural \mapsto \omega'_\natural$ defines a map in the space of controls and if $\omega_A(I)$ is sufficiently small we can find a fixpoint ω_* of such a map by iteration :

$$\omega_*(s,t) = C_{p,q}(\omega_I(s,t) + \omega_A(I)^{\frac{q}{3}(\frac{1}{p}+\kappa)}\omega_*(s,t)\omega_A(s,t)^{\frac{\kappa q}{3}}).$$

taking $\omega_A(I)$ smaller if necessary we conclude that $\omega_*(s,t) \lesssim_{p,q} \omega_I(s,t)$ and the estimate (2.18) immediately follows. □

Remark 2.6. We point out that the time regularity of the remainder $g^\natural$ is determined not only by the corresponding regularity of the driver $\mathbf{A}$ but is also influenced by the regularity of the drift μ. The parameters λ, κ, p represent the competition of the various terms from (2.11) leading to the regularity of $g^\natural$. To illustrate this phenomenon, let us go back to the heat equation (2.4) and show how Theorem 2.5 applies and yields the desired estimate for the remainder.

Recall that the drift in (2.8) satisfies

$$(\delta\mu)_{st}(\varphi) = -2\int_s^t \int_{\mathbb{R}^N} |\nabla u_r|^2 \varphi \,\mathrm{d}r - 2\int_s^t \int_{\mathbb{R}^N} \nabla u_r \cdot \nabla\varphi \, u_r \,\mathrm{d}r.$$

It can be estimated as follows

$$|(\delta\mu)_{st}(\varphi)| \lesssim \int_s^t \|\nabla u_r\|_{L^2}^2 \,\mathrm{d}r \, \|\varphi\|_{L^\infty} + \left(\int_s^t \|\nabla u_r\|_{L^2}^2 \,\mathrm{d}r\right)^{\frac{1}{2}} \left(\int_s^t \|u_r\|_{L^2}^2 \,\mathrm{d}r\right)^{\frac{1}{2}} \|\varphi\|_{W^{1,\infty}}, \tag{2.21}$$

and hence the assumption (2.14) holds true in the scale $E_n = W^{n,\infty}(\mathbb{R}^N)$. Notice that, according to (2.21), the coefficient λ in (2.14) could be chosen as $\lambda = 0$. However, due to other terms in our expansion, we take $\lambda = p$ in the sequel. With this value in hand, we get

$$\omega_\mu(s,t) = \int_s^t \|\nabla u_r\|_{L^2}^2 \,\mathrm{d}r + \left(\int_s^t \|\nabla u_r\|_{L^2}^2 \,\mathrm{d}r\right)^{\frac{1}{2}} \left(\int_s^t \|u_r\|_{L^2}^2 \,\mathrm{d}r\right)^{\frac{1}{2}}.$$

Accordingly, we may choose $q = p$ and $\kappa = 0$ and deduce that

$$\|u_{st}^{2,\natural}\|_{E_{-3}} \lesssim_{p,\mathbf{A},I} \mathcal{N}[u^2; L^\infty(s,t;E_{-0})]\,\omega_A(s,t)^{\frac{3}{p}} + \omega_\mu(s,t)\omega_A(s,t)^{\frac{3-p}{p}}.$$

In other words, the drift in (2.8) has enough space regularity and consequently we obtain better time regularity for the remainder, namely, is has finite $\frac{p}{3}$-variation as expected for equations without drift (cf. [1]).

With Theorem 2.5 in hand, the following Gronwall type lemma is a key step in order to get an a priori bound for a solution to equation (2.11).

Lemma 2.7 (Rough Gronwall Lemma)**.** *Fix a time horizon $T > 0$ and let $G : [0,T] \to [0,\infty)$ be a path such that for some constants $C, L > 0$, $\kappa \geqslant 1$ and some controls ω_1, ω_2 on $[0,T]$, one has*

$$\delta G_{st} \leqslant C\Big(\sup_{0\leqslant r\leqslant t} G_r\Big)\,\omega_1(s,t)^{\frac{1}{\kappa}} + \omega_2(s,t), \tag{2.22}$$

for every $s < t \in [0,T]$ satisfying $\omega_1(s,t) \leqslant L$. Then it holds

$$\sup_{0\leqslant t\leqslant T} G_t \leqslant 2\exp\Big(\frac{\omega_1(0,T)}{\alpha L}\Big)\cdot\Big\{G_0 + \sup_{0\leqslant t\leqslant T}\Big(\omega_2(0,t)\,\exp\Big(-\frac{\omega_1(0,t)}{\alpha L}\Big)\Big)\Big\},$$

where α is defined as

$$\alpha = \min\left(1,\ \frac{1}{L(2Ce^2)^\kappa}\right). \tag{2.23}$$

Proof. Let us successively set, for every $t \in [0,T]$,

$$G_{\leqslant t} := \sup_{0\leqslant s\leqslant t} G_s, \quad H_t := G_{\leqslant t}\exp\Big(-\frac{\omega_1(0,t)}{\alpha L}\Big) \quad \text{and} \quad H_{\leqslant t} := \sup_{0\leqslant s\leqslant t} H_s. \tag{2.24}$$

Also, let us denote by K the integer such that $\alpha L(K-1) \leqslant \omega_1(0,T) \leqslant \alpha L K$, and define a set of times $t_0 < t_1 < \cdots < t_K$ as follows: $t_0 := 0$, $t_K := T$ and for $k \in \{1,\ldots,K-1\}$, t_k is such that $\omega_1(0,t_k) = \alpha L k$. In particular, $\omega_1(t_k,t_{k+1}) \leqslant \alpha L \leqslant L$ (recall that we have chosen $\alpha \leqslant 1$ in (2.23)). Fix $t \in [t_{k-1},t_k]$, for some $k \in \{1,\ldots,K\}$. We start from the trivial decomposition

$$(\delta G)_{0t} = \sum_{i=0}^{k-2}(\delta G)_{t_i t_{i+1}} + (\delta G)_{t_{k-1}t}.$$

Then on each interval $[t_i,t_{i+1}]$ one can apply the a priori bound (2.22). Taking into account the facts that $\omega_1(t_k,t_{k+1}) \leqslant \alpha L$ and that ω_2 is a super-additive function, we get

$$\begin{aligned}(\delta G)_{0t} &\leqslant C(\alpha L)^{\frac{1}{\kappa}}\sum_{i=0}^{k-2} G_{\leqslant t_{i+1}} + \omega_2(0,t_{k-1}) + C(\alpha L)^{\frac{1}{\kappa}} G_{\leqslant t} + \omega_2(t_{k-1},t)\\ &\leqslant C(\alpha L)^{\frac{1}{\kappa}}\sum_{i=0}^{k-1} G_{\leqslant t_{i+1}} + \omega_2(0,t).\end{aligned} \tag{2.25}$$

Let us bound the term $\sum_{i=0}^{k-1} G_{\leqslant t_{i+1}}$ above. According to our definitions (2.24), we have

$$\sum_{i=0}^{k-1} G_{\leqslant t_{i+1}} = \sum_{i=0}^{k-1} H_{t_{i+1}} \exp\Big(\frac{\omega_1(0,t_{i+1})}{\alpha L}\Big) \leqslant H_{\leqslant T} \sum_{i=0}^{k-1} \exp(i+1) \leqslant \exp(k+1)\, H_{\leqslant T}, \tag{2.26}$$

where we have used the fact that $\omega(0,t_{i+1}) \leqslant \alpha L(i+1)$ for the first inequality. Combining (2.25) and (2.26), we thus get that

$$G_{\leqslant t} \leqslant G_0 + \omega_2(0,t) + C(\alpha L)^{\frac{1}{\kappa}} \exp(k+1)\, H_{\leqslant T}.$$

Now,

$$\begin{aligned} H_t &= G_{\leqslant t} \exp\Big(-\frac{\omega_1(0,t)}{\alpha L}\Big) \\ &\leqslant \{G_0 + \omega_2(0,t)\} \exp\Big(-\frac{\omega_1(0,t)}{\alpha L}\Big) + C(\alpha L)^{\frac{1}{\kappa}} \exp(k+1)\, H_{\leqslant T} \exp\Big(-\frac{\omega_1(0,t_{k-1})}{\alpha L}\Big), \end{aligned}$$

and since $\omega(0,t_{k-1}) = \alpha L(k-1)$, we end up with:

$$H_t \leqslant G_0 + \sup_{0\leqslant s\leqslant T} \Big(\omega_2(0,s)\, \exp\Big(-\frac{\omega_1(0,s)}{\alpha L}\Big)\Big) + Ce^2(\alpha L)^{\frac{1}{\kappa}} H_{\leqslant T}.$$

By taking the supremum over $t\in[0,T]$, we deduce that

$$H_{\leqslant T} \leqslant Ce^2(\alpha L)^{\frac{1}{\kappa}} H_{\leqslant T} + G_0 + \sup_{0\leqslant s\leqslant T} \Big(\omega_2(0,s)\, \exp\Big(-\frac{\omega_1(0,s)}{\alpha L}\Big)\Big)$$

and recalling the definition (2.23) of α, it entails that

$$H_{\leqslant T} \leqslant 2G_0 + 2 \sup_{0\leqslant s\leqslant T} \Big(\omega_2(0,s)\, \exp\Big(-\frac{\omega_1(0,s)}{\alpha L}\Big)\Big).$$

The conclusion is now immediate, since

$$G_{\leqslant T} = \exp\Big(\frac{\omega_1(0,T)}{\alpha L}\Big) H_T \leqslant \exp\Big(\frac{\omega_1(0,T)}{\alpha L}\Big) H_{\leqslant T}.$$

□

Remark 2.8. Finally, we have all in hand to derive the energy estimate for our example (2.4). Indeed, let us start from (2.9), which can be recast as:

$$(\delta\|u\|_{L^2}^2)_{st} + 2\int_s^t \|\nabla u_r\|_{L^2}^2\, \mathrm{d}r \lesssim \|u_s\|_{L^2}^2\, \omega_A(s,t)^{\frac{1}{p}} + |u_{st}^{2,\natural}(1)|.$$

Now invoke Remark 2.6 in order to get

$$\begin{aligned} (\delta\|u\|_{L^2}^2)_{st} + 2\int_s^t \|\nabla u_r\|_{L^2}^2\, \mathrm{d}r &\lesssim \sup_{s\leqslant r\leqslant t} \|u_r\|_{L^2}^2\, \omega_A(s,t)^{\frac{1}{p}} + \omega_\mu(s,t)\omega_A(s,t)^{\frac{3-p}{p}} \\ &\lesssim \sup_{s\leqslant r\leqslant t} \|u_r\|_{L^2}^2 \Big[\omega_A(s,t)^{\frac{1}{p}} + |t-s|\, \omega_A(s,t)^{\frac{3-p}{p}}\Big] + \int_s^t \|\nabla u_r\|_{L^2}^2\, \mathrm{d}r\, \omega_A(s,t)^{\frac{3-p}{p}}. \end{aligned}$$

Hence we can apply the rough Gronwall lemma with

$$G_t := \|u_t\|_{L^2}^2 + 2\int_0^t \|\nabla u_r\|_{L^2}^2\, \mathrm{d}r, \quad \omega_1(s,t) := \omega_A(s,t)^{\frac{\kappa}{p}} + |t-s|^\kappa \omega_A(s,t)^{\frac{(3-p)\kappa}{p}} + \omega_A(s,t)^{\frac{(3-p)\kappa}{p}},$$

$\omega_2(s,t)=0$, and $\kappa=\min(p,p/(3-p))\geqslant 1$, obtaining

$$\sup_{0\leqslant t\leqslant T}\|u_t\|_{L^2}^2+\int_0^T\|\nabla u_t\|_{L^2}^2\,\mathrm{d}t\lesssim\exp\Big(\frac{\omega_1(0,T)}{\alpha L}\Big)\|u_0\|_{L^2}^2.$$

With these a priori estimates one can immediately proceed to the proof of existence of a weak solution to (2.4) which would rely on compactness of suitable approximate solutions.

3. Tensorization and uniqueness

3.1. **Preliminary discussion.** The a priori estimates established in the previous section are the essential tools towards the proof of existence of weak solutions to RPDEs. Nevertheless, when it comes to uniqueness, further difficulties arise and new ideas are required. Let us explain these issues on our toy model (2.4). Since the equation is linear, proving uniqueness is equivalent to showing that if a weak solution vanishes at the initial time, it vanishes for all times $t\geqslant 0$. Therefore, one would like to derive the estimate

$$\|u_t\|_{L^2}^2\lesssim\|u_0\|_{L^2}^2$$

for every weak solution to (2.4), that is, for every $u\in C([0,T];L^2(\mathbb{R}^N))\cap L^2(0,T;W^{1,2}(\mathbb{R}^N))$ satisfying (2.6). It was already mentioned above that the major obstacle consists in the rigorous derivation of the equation for u^2, i.e. (2.8). In the deterministic or stochastic setting, one typically employs a mollifying sequence (ψ_ε) of test functions for (2.6), derives the equation for $(u*\psi_\varepsilon)^2$ and passes to the limit as $\varepsilon\to 0$. In the context of transport equation one is thus lead to the DiPerna-Lions commutator lemma [19], which allows to pass to the limit in the transport terms. As we consider a transport-type noise in (2.4), similar arguments are also required here.

As in [1], our method relies on a tensorization argument together with a refined analysis of the approximation error. To be more specific, starting from (2.6) we write down the equation for the tensor product of distributions $U(x,y):=u^{\otimes 2}(x,y)=u(x)u(y)$. Namely, write

$$\delta U_{st}=\delta u_{st}\otimes u_s+u_s\otimes\delta u_{st}+\delta u_{st}\otimes\delta u_{st},\tag{3.1}$$

then expand the terms δu_{st} above thanks to (2.6). This yields:

$$\begin{aligned}\delta U_{st}=&\int_s^t u_r\otimes(\Delta u_r)\,\mathrm{d}r+\int_s^t(\Delta u_r)\otimes u_r\,\mathrm{d}r\\&+\big(A^1_{st}\otimes\mathbb{I}+\mathbb{I}\otimes A^1_{st}\big)U_s+\big(A^2_{st}\otimes\mathbb{I}+\mathbb{I}\otimes A^2_{st}+A^1_{st}\otimes A^1_{st}\big)U_s+U^\natural_{st},\end{aligned}\tag{3.2}$$

where $\mathbb{I}$ denotes the identity map. Note that in view of (2.6), this equation is rigorously derived and understood in the sense of distributions, that is, it holds true for every test function $\Phi\in C_c^\infty(\mathbb{R}^N\times\mathbb{R}^N)$. The remainder $U^\natural$ has finite $\frac{p}{3}$-variation as a distribution with values in $W^{-3,2}(\mathbb{R}^N)\otimes W^{-3,2}(\mathbb{R}^N)$. This regularity is not enough to apply apriori estimates, we will get back to this problem later on. The important observation is that Chen's relation (2.3) is satisfied for $\mathbf{\Gamma}=(\Gamma^1,\Gamma^2)$ where

$$\Gamma^1_{st}:=A^1_{st}\otimes\mathbb{I}+\mathbb{I}\otimes A^1_{st},\qquad\Gamma^2_{st}:=A^2_{st}\otimes\mathbb{I}+\mathbb{I}\otimes A^2_{st}+A^1_{st}\otimes A^1_{st}.\tag{3.3}$$

Hence (3.2) can be written as

$$\delta U_{st}=\delta\Pi_{st}+\Gamma^1_{st}U_s+\Gamma^2_{st}U_s+U^\natural_{st},\tag{3.4}$$

with

$$\Pi_t = \int_0^t u_r \otimes (\Delta u_r)\,\mathrm{d}r + \int_0^t (\Delta u_r) \otimes u_r\,\mathrm{d}r.$$

The main advantage of the tensorization method is thus the following: equation (3.4) is still linear, and thus can possibly be treated by our a priori estimate methods, as in the case of (2.11). Our ultimate goal now is then to test by

$$\Phi_\varepsilon(x,y) = \varphi\left(\frac{x+y}{2}\right)\psi_\varepsilon(x-y) \tag{3.5}$$

and to derive estimates uniform in ε to be able to pass to the limit as $\varepsilon \to 0$. In other words, this will allow us to pass to the diagonal and obtain the equation for u^2. To this end, we introduce a blow-up transformation T_ε on test functions as follows:

$$T_\varepsilon\Phi(x,y) := \varepsilon^{-N}\Phi\big(x_+ + \frac{x_-}{\varepsilon}, x_+ - \frac{x_-}{\varepsilon}\big),$$

where $x_\pm = \frac{x\pm y}{2}$ are the coordinates parallel and transverse to the diagonal. Note that its adjoint for the L^2-inner product reads

$$T_\varepsilon^*\Phi(x,y) = \Phi(x_+ + \varepsilon x_-, x_+ - \varepsilon x_-) \tag{3.6}$$

and its inverse is given by

$$T_\varepsilon^{-1}\Phi(x,y) = \varepsilon^N\Phi(x_+ + \varepsilon x_-, x_+ - \varepsilon x_-). \tag{3.7}$$

Let $\mathbf{\Gamma}_\varepsilon^* := T_\varepsilon^{-1}\mathbf{\Gamma}^* T_\varepsilon$, $U^\varepsilon := T_\varepsilon^* U$, $U^{\natural,\varepsilon} := T_\varepsilon^* U^\natural$ and $\Pi^\varepsilon := T_\varepsilon^*\Pi$. Then (3.4) evaluated against test functions of the form (3.5) (or otherwise stated, testing against $T_\varepsilon\Phi$) leads to the equation

$$\delta U_{st}^\varepsilon(\Phi) = \delta\Pi_{st}^\varepsilon(\Phi) + U_s^\varepsilon((\Gamma_{\varepsilon,st}^{1,*} + \Gamma_{\varepsilon,st}^{2,*})\Phi) + U_{st}^{\natural,\varepsilon}(\Phi). \tag{3.8}$$

3.2. **Renormalizable drivers.** Let us now continue with the general theory for (2.10). In view of the discussion in the previous subsection, we are concerned with equations of the following form, generalizing (3.8):

$$\delta G_{st}^\varepsilon(\Phi) = \delta M_{st}^\varepsilon(\Phi) + G_s^\varepsilon((\Gamma_{\varepsilon,st}^{1,*} + \Gamma_{\varepsilon,st}^{2,*})\Phi) + G_{st}^{\natural,\varepsilon}(\Phi). \tag{3.9}$$

Remark that depending on the particular problem at hand, G is not always defined as $g \otimes g$ as in the case of the linear equation (2.4). We refer the reader to Subsection 5.1 for the application to kinetic solutions of conservation laws.

It is worth noticing that the structure of equation (3.9) is similar to relation (2.11). In order to apply our general results, we are thus reduced to derive uniform bounds for the set of drivers $\{\mathbf{\Gamma}_\varepsilon\}_{\varepsilon\in(0,1)}$ associated with the unbounded rough driver $\mathbf{A}$. This leads us to the following definition.

Definition 3.1 (Renormalizable driver)**.** Let $\mathbf{A}$ be a continuous unbounded p-rough driver on the scale $(E_n)_n$, and $\mathbf{\Gamma}$ its tensorization defined by (3.3) and acting on $C_c^\infty(\mathbb{R}^N \times \mathbb{R}^N)$. We say that $\mathbf{A}$ is renormalizable in the scale of spaces $(\mathcal{E}_n)_n$ if $\{\mathbf{\Gamma}_\varepsilon\}_{\varepsilon\in(0,1)}$ can be extended to a bounded family of continuous unbounded p-rough drivers on the scale $(\mathcal{E}_n)_n$.

In the context of transport-type rough drivers as in (2.5), this renormalization corresponds to the fact that a commutator lemma argument in the sense of DiPerna-Lions [19] can be performed.

Proposition 3.2. *Let a continuous unbounded p-rough driver $\mathbf{A}$ be defined as in (2.5) and (2.7), where $V = (V^1, \dots, V^K)$ is a family of vector fields on $\mathbb{R}^N$ such that $V \in W^{3,\infty}(\mathbb{R}^N)$ and $\mathbf{Z} = (Z^1, Z^2)$ is a geometric p-rough path for $p \in [2, 3)$. Then it is a renormalizable driver in the scale $(\mathcal{E}_n)$ where*

$$\mathcal{E}_n := \left\{\Phi \in W^{n,\infty}(\mathbb{R}^N \times \mathbb{R}^N);\ \Phi(x, y) = 0 \text{ if } |x - y| \geqslant 2\right\}$$

equipped with the subspace topology of $W^{n,\infty}(\mathbb{R}^N \times \mathbb{R}^N)$.

Remark 3.3. The support condition in the definition of spaces $\mathcal{E}_n$ actually means that the test functions are compactly supported in the x_- direction, which is the key point of the proof below. Note that the test functions we are interested in are of the form (3.5), i.e. the dependence on x_- is only in the mollifier ψ which can be taken compactly supported.

Proof of Proposition 3.2. Recall that the driver $\mathbf{\Gamma}$ was defined in (3.3), and that $A^{1,*}_{st}, A^{2,*}_{st}$ are defined by (2.7). Therefore, the driver $\mathbf{\Gamma}_\varepsilon$ which was defined by $\mathbf{\Gamma}^*_\varepsilon = T_\varepsilon^{-1}\mathbf{\Gamma}^* T_\varepsilon$ can be written as

$$\Gamma^{1,*}_{\varepsilon,st} = Z^{1,i}_{st}\Gamma^{1,i,*}_{V,\varepsilon}, \qquad \Gamma^{2}_{\varepsilon,st} = Z^{2,ij}_{st}\Gamma^{1,j,*}_{V,\varepsilon}\Gamma^{1,i,*}_{V,\varepsilon},$$

where

$$\Gamma^{1,*}_{V,\varepsilon} := -V^+_\varepsilon \cdot \nabla^+ - \varepsilon^{-1}V^-_\varepsilon \cdot \nabla^- - D^+_\varepsilon.$$

In order to read the last two formulae, let us introduce the following notation:

$$\nabla^\pm := \frac{1}{2}(\nabla_x \pm \nabla_y), \qquad D(x) = \operatorname{div}_x V(x),$$

and for a real-valued function Ψ on $\mathbb{R}^N$ let us denote by

$$\Psi^\pm(x, y) := \Psi(x) \pm \Psi(y)$$

its symmetric and antisymmetric lift to $\mathbb{R}^N \times \mathbb{R}^N$. In addition, the term D^+_ε above is understood by writing

$$\Psi^\pm_\varepsilon(x, y) := \Psi(x_+ + \varepsilon x_-) \pm \Psi(x_+ - \varepsilon x_-)$$

for the blow up of $\Psi^\pm$ according to the transformation $\Psi^\pm_\varepsilon = T_\varepsilon^{-1}\Psi^\pm T_\varepsilon$. We also used the fact that

$$\nabla^+ T_\varepsilon = T_\varepsilon \nabla^+, \qquad \nabla^- T_\varepsilon = \varepsilon^{-1} T_\varepsilon \nabla^-.$$

Next, by the Taylor formula we obtain

$$\varepsilon^{-1}V^-_\varepsilon = 2\, x_- \int_0^1 \mathrm{D}V\big((x_+ - \varepsilon x_-) + 2\varepsilon r x_-\big)\, \mathrm{d}r.$$

Hence, since functions in $\mathcal{E}_n$ are compactly supported in the x_- direction, we obtain for $n = 0, 1, 2$

$$\|\Gamma^{1,*}_{V,\varepsilon}\Phi\|_{\mathcal{E}_n} \lesssim (\|V\|_{W^{n,\infty}} + \|\mathrm{D}V\|_{W^{n,\infty}})\|\Phi\|_{\mathcal{E}_{n+1}} \lesssim \|V\|_{W^{n+1,\infty}}\|\Phi\|_{\mathcal{E}_{n+1}},$$

and for $n = 0, 1$

$$\|\Gamma^{2,*}_{V,\varepsilon}\Phi\|_{\mathcal{E}_n} = \|\Gamma^{1,*}_{V,\varepsilon}\Gamma^{1,*}_{V,\varepsilon}\Phi\|_{\mathcal{E}_n} \lesssim \|V\|_{W^{n+1,\infty}}\|\Gamma^{1,*}_{V,\varepsilon}\Phi\|_{\mathcal{E}_{n+1}} \lesssim \|V\|_{W^{n+1,\infty}}\|V\|_{W^{n+2,\infty}}\|\Phi\|_{\mathcal{E}_{n+2}},$$

which holds true uniformly in ε. Consequently, uniformly in ε,

$$\|\Gamma^1_{\varepsilon,st}\|^p_{\mathcal{L}(\mathcal{E}_n,\mathcal{E}_{n-1})} \lesssim_{\|V\|_{W^{3,\infty}}} \omega_Z(s,t), \qquad n \in \{-0, -2\},$$

$$\|\Gamma^2_{\varepsilon,st}\|^{p/2}_{\mathcal{L}(\mathcal{E}_n,\mathcal{E}_{n-2})} \lesssim_{\|V\|_{W^{3,\infty}}} \omega_Z(s,t), \qquad n \in \{-0, -1\},$$

where ω_Z is a control corresponding to the rough path $\mathbf{Z}$, namely,

$$|Z^1_{st}| \leqslant \omega_Z(s,t)^{\frac{1}{p}}, \qquad |Z^2_{st}| \leqslant \omega_Z(s,t)^{\frac{2}{p}}.$$

□

Let us now go back to our general equation (2.10). A natural strategy in order to get uniqueness via the tensorization method is to get an equivalent of Proposition 3.2, and then obtain a uniform estimate for G^ε and M^ε in (3.9). This would enable the application of Theorem 2.5, yielding a uniform estimate for the remainder $U^{\natural,\varepsilon}$. However, there is an additional hurdle in the tensorized case. Indeed, one cannot construct smoothing operators (J^η) on the scale $(\mathcal{E}_n)$ in such a way that the corresponding version of (2.12) holds true on tensorized spaces. In particular, applying a smoothing operator necessarily increases the support of the function and this difficulty has to be overcome using an additional localization. This issue is addressed in the next section.

3.3. **Full localization.** We need to introduce spaces of test functions which are suitably localized. Localization in the x_- variable is needed to deal with the renormalization of the rough driver. This localization will disappear after taking the limit $\varepsilon \to 0$. An additional localization in the x_+ direction is needed to derive preliminary uniform bounds for G^ε and M^ε. This localization (parametrized by $R \geq 1$) will be removed afterwards via a rough Gronwall lemma argument which gives us uniform estimates in R. The localized space we are working with is thus $\mathcal{E}_{R,n} \subset \mathcal{E}_n$, defined by

$$\mathcal{E}_{R,n} := \left\{ \Phi \in W^{n,\infty}(\mathbb{R}^N \times \mathbb{R}^N);\ \Phi(x,y) = 0 \text{ if } \rho_R(x,y) \geq 1 \right\},$$

where $\rho_R(x,y)^2 = |x_+|^2/R^2 + |x_-|^2$.

As a consequence of the proof of Proposition 3.2, we immediately obtain uniform bounds for the family of drivers $\{\mathbf{\Gamma}_\varepsilon\}$ in our toy heat equation (2.4).

Corollary 3.4. *The continuous unbounded p-rough driver $\mathbf{A}$ defined in (2.5) and (2.7) is renormalizable (according to Definition 3.1) in the scale $(\mathcal{E}_{R,n})$ and*

$$\|\Gamma^1_{\varepsilon,st}\|^p_{\mathcal{L}(\mathcal{E}_{R,n},\mathcal{E}_{R,n-1})} \lesssim_{\|V\|_{W^{3,\infty}}} \omega_Z(s,t), \qquad n \in \{-0,-2\},$$

$$\|\Gamma^2_{\varepsilon,st}\|^{p/2}_{\mathcal{L}(\mathcal{E}_{R,n},\mathcal{E}_{R,n-2})} \lesssim_{\|V\|_{W^{3,\infty}}} \omega_Z(s,t), \qquad n \in \{-0,-1\},$$

holds true uniformly in ε and R.

As the next step, let us introduce a suitable cut-off function:

Notation 3.5. *Let $\eta \in (0, \frac{1}{3})$ and let $\theta_\eta \in C_c^\infty(\mathbb{R})$ be such that*

$$0 \leqslant \theta_\eta \leqslant 1, \qquad \operatorname{supp} \theta_\eta \subset B_{1-2\eta} \subset \mathbb{R}, \qquad \theta_\eta \equiv 1 \text{ on } B_{1-3\eta} \subset \mathbb{R},$$

where for $\alpha > 0$ we set $B_\alpha := [-\alpha, \alpha]$. We also require the following condition on θ_η:

$$|\nabla^k \theta_\eta| \lesssim \eta^{-k}, \qquad \text{for } k = 1, 2.$$

Finally, we define

$$\Theta_\eta(x,y) := \Theta_{R,\eta}(x,y) = \theta_\eta(\rho_R(x,y)).$$

With this notation in mind, we have the following auxiliary result.

Lemma 3.6. *Let $\Phi \in \mathcal{E}_{R,k}$, $k = 1, 2$, and θ_η, Θ_η defined in Notation 3.5. Then it holds true that*

$$\|\Theta_\eta \Phi\|_{\mathcal{E}_{R,k}} \lesssim \|\Phi\|_{\mathcal{E}_{R,k}}, \tag{3.10}$$

$$\|(1-\Theta_\eta)\Phi\|_{\mathcal{E}_{R,0}} \lesssim \eta^k \|\Phi\|_{\mathcal{E}_{R,k}}, \tag{3.11}$$

and there exists $\Psi_R \in \mathcal{E}_{R,3}$ with $\sup_{R\in\mathbb{R}} \|\Psi_R\|_{\mathcal{E}_{R,3}} < \infty$ such that for all $x, y \in \mathbb{R}^N$

$$\eta^{3-k}|J^\eta \Theta_\eta \Phi(x,y)| \lesssim \Psi_R(x,y)\|\Phi\|_{\mathcal{E}_{R,k}}, \tag{3.12}$$

where the proportional constants are independent of η and R.

Proof. In order to prove (3.10), we write

$$\nabla(\Theta_\eta\Phi) = (\nabla\Theta_\eta)\Phi + \Theta_\eta(\nabla\Phi),$$

where the second term does not pose any problem. On the other hand, the term $\nabla\Theta_\eta$ diverges as η^{-1} due to the assumptions on θ_η, namely, it holds

$$\nabla_x\Theta_\eta = (\nabla\theta_\eta)(\rho_R(x,y))\nabla_x\rho_R(x,y).$$

But due to the support of Φ we have that for every (x,y) in the region where $\nabla_x\Theta_\eta \neq 0$ there exists $(\tilde{x},\tilde{y})$ outside of support of Φ such that $|(x,y)-(\tilde{x},\tilde{y})| \lesssim \eta$ hence

$$|\Phi(x,y)| = |\Phi(x,y) - \Phi(\tilde{x},\tilde{y})| \lesssim \eta\|\Phi\|_{\mathcal{E}_{R,1}}$$

and consequently (3.10) follows for $k = 1$. If $k = 2$, we have

$$\nabla^2(\Theta_\eta\Phi) = (\nabla^2\Theta_\eta)\Phi + 2\nabla\Theta_\eta \cdot \nabla\Phi + \Theta_\eta\nabla^2\Phi,$$

where the third term does not pose any problem and the second one can be estimated using the reasoning above. For the first one, we observe that $\nabla_x^2\Theta_\eta$ diverges like η^{-2} but for every (x,y) such that $\nabla^2\Theta \neq 0$ there exists $(\tilde{x},\tilde{y})$ that lies outside of support of Φ, satisfies $|(x,y)-(\tilde{x},\tilde{y})| \lesssim \eta$. Resorting to a second order Taylor expansion and invoking the fact that both $\Phi(\tilde{x},\tilde{y})$ and $\nabla\Phi(\tilde{x},\tilde{y})$ are vanishing we get:

$$\begin{aligned}|\Phi(x,y)| &= |\Phi(x,y) - \Phi(\tilde{x},\tilde{y}) - \mathrm{D}\Phi(\tilde{x},\tilde{y})((x,y)-(\tilde{x},\tilde{y}))| \\ &\lesssim |\mathrm{D}^2\Phi(z_x,z_y)||(x,y)-(\tilde{x},\tilde{y})|^2 \lesssim \eta^2\|\Phi\|_{\mathcal{E}_{R,2}}\end{aligned}$$

and relation (3.10) follows.

The same approach leads to (3.11). To be more precise, for (x,y) from the support of $(1-\Theta_\eta)\Phi$ we have using the first and second order Taylor expansion, respectively,

$$|(1-\Theta_\eta)\Phi(x,y)| \lesssim \eta^k\|(1-\Theta_\eta)\Phi\|_{\mathcal{E}_{R,k}} \lesssim \eta^k\|\Phi\|_{\mathcal{E}_{R,k}},$$

where we used (3.10) for the second inequality.

Let us now prove (3.12). First of all, we observe the trivial estimate, for $k = 1, 2$, and $(x,y) \in \mathrm{supp}(J^\eta\Theta_\eta\Phi)$,

$$\eta^{3-k}|J^\eta\Theta_\eta\Phi(x,y)| \leqslant \|J^\eta\Theta_\eta\Phi\|_{L^\infty} \lesssim \|\Phi\|_{\mathcal{E}_0} \lesssim \|\Phi\|_{\mathcal{E}_k}. \tag{3.13}$$

Next, we note that

$$\mathrm{supp}(J^\eta\Theta_\eta\Phi) \subset \left\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N;\, \rho_R(x,y) \leqslant 1-\eta\right\} \tag{3.14}$$

since $R \geq 1$, and denote

$$D_R := \left\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N;\, \rho_R(x,y) \leqslant 1\right\}.$$

Let $d(\cdot,\partial D_R)$ denote the distance to its boundary ∂D_R. Owing to (3.14), it satisfies

$$d((x,y),\partial D_R)\geqslant \eta \qquad \text{for all} \quad (x,y)\in \operatorname{supp}(J^\eta\Theta_\eta\Phi).$$

Therefore, performing a Taylor expansion we obtain for $k=1,2$, and $(x,y)\in\operatorname{supp}(J^\eta\Theta_\eta\Phi)$,

$$\begin{aligned}\eta^{3-k}|J^\eta\Theta_\eta\Phi(x,y)| &\lesssim \eta^{3-k}|d((x,y),\partial D_R)|^k\|J^\eta\Theta_\eta\Phi\|_{W^{k,\infty}}\\ &\lesssim |d((x,y),\partial D_R)|^3\|\Phi\|_{\mathcal{E}_{R,k}}\end{aligned} \tag{3.15}$$

where we also used (3.10) and the fact that $\mathcal{E}_{R,k}$ is embedded in $W^{k,\infty}$. Besides we may put (3.13) and (3.15) together to conclude that there exists $\Psi_R\in\mathcal{E}_{R,3}$ satisfying the conditions stated in this Lemma and, in addition,

$$\min\left\{1,|d((x,y),\partial D_R)|^3\right\}\lesssim \Psi_R(x,y)$$

which completes the proof. For example we can take

$$\Psi_R(x,y)=\begin{cases} d((x,y),\partial D_R)^3, & \text{if } (x,y)\in D_R \text{ and } d((x,y),\partial D_R)\le 1/2,\\ 1, & \text{if } (x,y)\in D_R \text{ and } d((x,y),\partial D_R)\ge 3/4,\end{cases}$$

and complete it with a smooth interpolation in between. □

3.4. **Conclusion.** We can now draw some general conclusions concerning uniqueness for our class of equations, first for the general case (2.10) and then for the heat equation (2.6).

3.4.1. *General strategy for uniqueness.* With the framework of Section 3.2 in mind, our main remaining tasks are to establish a uniform bound for the remainder $G^{\natural,\varepsilon}$ in (3.9), pass to the diagonal and then estimate the resulting equation. These steps typically require a careful analysis of the particular structure of G^ε as well as M^ε in (3.9), and therefore we will not give the full details at this level of generality. Let us only discuss the main ideas of the method here and see Section 5 for a detailed application to rough conservation laws.

We intend to proceed similarly as in Theorem 2.5: letting

$$\begin{aligned}G^{\sharp,\varepsilon}_{st}(\Phi):=T^*_\varepsilon G^\sharp_{st}(\Phi)&=\delta G^\varepsilon_{st}(\Phi)-G^\varepsilon_s(\Gamma^{1,*}_{st}\Phi)\\ &=\delta M^\varepsilon_{st}(\Phi)+G^\varepsilon_s(\Gamma^{2,*}_{st}\Phi)+G^{\natural,\varepsilon}_{st}(\Phi),\end{aligned}$$

it follows from (3.9) that

$$\delta G^{\natural,\varepsilon}_{sut}(\Phi)=(\delta G^\varepsilon)_{su}(\Gamma^{2,*}_{\varepsilon,ut}\Phi)+G^{\sharp,\varepsilon}_{su}(\Gamma^{1,*}_{\varepsilon,ut}\Phi).$$

Now, apart from the smoothing operators (J^η) we also include the localization (Θ_η) defined in Subsection 3.3. Decomposition (2.20) becomes:

$$\begin{aligned}\delta G^{\natural,\varepsilon}(\Phi)&=(\delta G^\varepsilon)(J_\eta(\Theta_\eta\Gamma^{2,*}_\varepsilon\Phi))+(\delta G^\varepsilon)((1-J_\eta)(\Theta_\eta\Gamma^{2,*}_\varepsilon\Phi))+(\delta G^\varepsilon)((1-\Theta_\eta)\Gamma^{2,*}_\varepsilon\Phi)\\ &\quad+G^{\sharp,\varepsilon}(J_\eta(\Theta_\eta\Gamma^{1,*}_\varepsilon\Phi))+G^{\sharp,\varepsilon}((1-J_\eta)(\Theta_\eta\Gamma^{1,*}_\varepsilon\Phi))+G^{\sharp,\varepsilon}((1-\Theta_\eta)\Gamma^{1,*}_\varepsilon\Phi)\\ &=G^\varepsilon(\Gamma^{1,*}_\varepsilon J_\eta(\Theta_\eta\Gamma^{2,*}_\varepsilon\Phi))+G^\varepsilon(\Gamma^{2,*}_\varepsilon J_\eta(\Theta_\eta\Gamma^{2,*}_\varepsilon\Phi))+\delta M^\varepsilon(J_\eta(\Theta_\eta\Gamma^{2,*}_\varepsilon\Phi))\\ &\quad+G^{\natural,\varepsilon}(J_\eta(\Theta_\eta\Gamma^{2,*}_\varepsilon\Phi))+(\delta G^\varepsilon)((1-J_\eta)(\Theta_\eta\Gamma^{2,*}_\varepsilon\Phi))+(\delta G^\varepsilon)((1-\Theta_\eta)\Gamma^{2,*}_\varepsilon\Phi)\\ &\quad+G^\varepsilon(\Gamma^{2,*}_\varepsilon J_\eta(\Theta_\eta\Gamma^{1,*}_\varepsilon\Phi))+\delta M^\varepsilon(J_\eta(\Theta_\eta\Gamma^{1,*}_\varepsilon\Phi))+G^{\natural,\varepsilon}(J_\eta(\Theta_\eta\Gamma^{1,*}_\varepsilon\Phi))\\ &\quad+(\delta G^\varepsilon)((1-J_\eta)(\Theta_\eta\Gamma^{1,*}_\varepsilon\Phi))+G^\varepsilon(\Gamma^{1,*}_\varepsilon(1-J_\eta)(\Theta_\eta\Gamma^{1,*}_\varepsilon\Phi))\\ &\quad+(\delta G^\varepsilon)((1-\Theta_\eta)\Gamma^{1,*}_\varepsilon\Phi)+G^\varepsilon(\Gamma^{1,*}_\varepsilon(1-\Theta_\eta)\Gamma^{1,*}_\varepsilon\Phi).\end{aligned} \tag{3.16}$$

In spite of its size, expression (3.16) turns out to be crucial in order to locate the terms which have to be bounded for our purposes. To be more specific, we need suitable bounds for G^ε, M^ε, $G^{\natural,\varepsilon}$ in the scale of spaces $\mathcal{E}_{R,n}$. To this aim, a first natural idea is to use the tensorized equation (3.9) in order to obtain a preliminary estimate for $G^{\natural,\varepsilon}_{st}$ in $\mathcal{E}^*_{R,3}$. A first difficulty is that, in general, we need to establish that, for fixed ε the remainder $G^{\natural,\varepsilon}$ has the correct space regularity, namely $G^{\natural,\varepsilon} \in V^{\frac{q}{3}}_{2,\mathrm{loc}}(I;\mathcal{E}^*_{R,3})$. This does not come for free from the tensorized equation (as observed above in the case of the remainder $U^\natural$ for the tensorized form of the heat equation). An additional argument is needed which will be detailed in the case of the conservation laws, below. For the moment we will assume that this regularity holds for any $\varepsilon > 0$. At this point the main challenge is to derive a new estimate which is uniform in ε. The following two steps procedure is worth stressing in this context:

- In a first step the preliminary estimates of G^ε and M^ε do not necessarily need to be uniform in ε and R, and we are seeking uniformity (in ε) only for $G^{\natural,\varepsilon}_{st}$.
- In a second step, we shall also get uniformity in R, thanks to an elaboration of our Gronwall lemma 2.7.

The details of this approach will be given in Section 5, and especially Proposition 5.8.

3.4.2. *Heat equation.* The implications of Section 3.2 on the heat equation 2.4 are straightforward. Namely, recall that our main problem is a rigorous derivation of equation (2.8) for u^2. However, with our uniform estimates in ε and R in hand, one can finally pass to the diagonal in equation (3.8), yielding relation (2.8). Then one applies the general a priori estimates from Subsection 2.4 again and concludes that

$$\sup_{0\leqslant t\leqslant T} \|u_t\|^2_{L^2} \lesssim \|u_0\|^2_{L^2}$$

is satisfied for every weak solution in $C([0,T];L^2(\mathbb{R}^N)) \cap L^2(0,T;W^{1,2}(\mathbb{R}^N))$. The uniqueness then follows.

4. Rough conservation laws I: Presentation

Throughout the reminder of the paper, we are interested in a rough path driven scalar conservation law of the form

$$\begin{aligned} \mathrm{d}u + \operatorname{div}\big(A(x,u)\big)\mathrm{d}z = 0, \qquad & t\in(0,T),\, x\in\mathbb{R}^N,\\ u(0) = u_0, \qquad & \end{aligned} \tag{4.1}$$

where $z=(z^1,\ldots,z^K)$ and z can be lifted to a geometric p-rough path, $A:\mathbb{R}^N_x\times\mathbb{R}_\xi\to\mathbb{R}^{N\times K}$. Using the Einstein summation convention, (4.1) rewrites

$$\begin{aligned} \mathrm{d}u + \partial_{x_i}\big(A_{ij}(x,u)\big)\,\mathrm{d}z^j = 0, \qquad & t\in(0,T),\, x\in\mathbb{R}^N,\\ u(0) = u_0. \qquad & \end{aligned}$$

As the next step, let us introduce the kinetic formulation of (4.1) as well as the basic definitions concerning the notion of kinetic solution. We refer the reader to [56] for a detailed exposition. The motivation behind this approach is given by the nonexistence of a strong solution and, on the other hand, the nonuniqueness of weak solutions, even in simple cases. The idea is to establish an additional criterion – the kinetic formulation – which is automatically satisfied by any strong solution to (4.1) (in case it exists) and which permits to ensure the well-posedness. The linear character of the kinetic formulation simplifies also the analysis of the remainder terms and the proof of the apriori estimates. It is well-known that in the case of a smooth driving signal z, the kinetic formulation of (4.1), which describes the time evolution of $f_t(x,\xi) = \mathbf{1}_{u_t(x)>\xi}$, reads as

$$\begin{aligned} \mathrm{d}f + \nabla_x f \cdot a\, \mathrm{d}z - \partial_\xi f\, b\, \mathrm{d}z &= \partial_\xi m, \\ f(0) &= f_0, \end{aligned} \tag{4.2}$$

where the coefficients a, b are given by

$$a = (a_{ij}) = (\partial_\xi A_{ij}) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^{N\times K}, \qquad b = (b_j) = (\operatorname{div}_x A_{\cdot j}) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^K$$

and m is a nonnegative finite measure on $[0,T]\times\mathbb{R}^N_x\times\mathbb{R}_\xi$ which becomes part of the solution. The measure m is called kinetic defect measure as it takes account of possible singularities of u. Indeed, if there was a smooth solution to (4.1) then one can derive (4.2) rigorously with $m \equiv 0$. We say then that u is a kinetic solution to (4.1) provided, roughly speaking, there exists a kinetic measure m such that the pair $(f = \mathbf{1}_{u>\xi}, m)$ solves (4.2) in the sense of distributions on $[0,T) \times \mathbb{R}^N_x \times \mathbb{R}_\xi$.

In the case of a rough driver z, we will give an intrinsic notion of kinetic solution to (4.1). In particular, the kinetic formulation (4.2) will be understood in the framework of unbounded rough drivers presented in the previous sections. The reader can immediately observe that (4.2) fits very naturally into this concept: the left hand side of (4.2) is of the form of a rough transport equation whereas the kinetic measure on the right hand side plays the role of a drift. Nevertheless, one has to be careful since the kinetic measure is not given in advance, it is a part of the solution and has to be constructed within the proof of existence. Besides, in the proof of uniqueness, one has to compare two solutions with possibly different kinetic measures.

The kinetic formulation (4.2) can be rewritten as

$$\mathrm{d}f = \begin{pmatrix} b \\ -a \end{pmatrix} \cdot \begin{pmatrix} \partial_\xi f \\ \nabla_x f \end{pmatrix} \mathrm{d}z + \partial_\xi m$$

or

$$\mathrm{d}f = V \cdot \nabla_{\xi,x} f\, \mathrm{d}z + \partial_\xi m, \tag{4.3}$$

where the family of vector fields V is given by

$$V = (V^1, \dots, V^K) = \begin{pmatrix} b \\ -a \end{pmatrix} = \begin{pmatrix} b_1 & \cdots & b_K \\ -a_{11} & \cdots & -a_{1K} \\ \vdots & \cdots & \vdots \\ -a_{N1} & \cdots & -a_{NK} \end{pmatrix}. \tag{4.4}$$

Note that these vector fields satisfy for $i \in \{1, \dots, K\}$

$$\operatorname{div}_{\xi,x} V_i = \partial_\xi b_i - \operatorname{div}_x a_{\cdot i} = \partial_\xi \operatorname{div}_x A_{\cdot i} - \operatorname{div}_x \partial_\xi A_{\cdot i} = 0. \tag{4.5}$$

Let us now label the assumptions we will use in order to solve equation (4.2) or its equivalent form (4.3), beginning with the assumptions on V:

Hypothesis 4.1. Let V be the family of vector fields defined by relation (4.4). We assume that:

$$V \in W^{3,\infty}(\mathbb{R}^{N+1}) \qquad \text{and} \qquad V(x,0) = 0 \qquad \forall x \in \mathbb{R}^N. \tag{4.6}$$

Notice that the assumption $V(x,0)=0$ is only used for the a priori estimates on solutions of (4.2), so that it won't show up before Section 6. In addition, as in the toy heat model case, we shall also assume that z can be understood as a rough path.

Hypothesis 4.2. The function z admits a lift to a geometric p-rough path $\mathbf{Z}=(Z^1,Z^2)$. The related continuous control is called ω_Z, i.e.

$$|Z^1_{st}|\leqslant \omega_Z(s,t)^{\frac{1}{p}}, \qquad |Z^2_{st}|\leqslant \omega_Z(s,t)^{\frac{2}{p}}.$$

The following notation concerns the rough driver structure related to our equation of interest.

Notation 4.3. *The natural scale of spaces (E_n) where (4.2) will be considered is*

$$E_n = W^{n,1}(\mathbb{R}^{N+1})\cap W^{n,\infty}(\mathbb{R}^{N+1}).$$

Then it can be shown that

$$\begin{aligned} A^1_{st}\varphi &:= Z^{1,i}_{st}\, V^i\cdot\nabla_{\xi,x}\varphi,\\ A^2_{st}\varphi &:= Z^{2,ij}_{st}\, V^j\cdot\nabla_{\xi,x}(V^i\cdot\nabla_{\xi,x}\varphi). \end{aligned} \tag{4.7}$$

defines a continuous unbounded p-rough driver on (E_n) with

$$\begin{aligned} \|A^1_{st}\|^p_{\mathcal{L}(E_n,E_{n-1})} &\lesssim \omega_Z(s,t), \qquad n\in\{-0,-2\},\\ \|A^2_{st}\|^{p/2}_{\mathcal{L}(E_n,E_{n-2})} &\lesssim \omega_Z(s,t), \qquad n\in\{-0,-1\}. \end{aligned} \tag{4.8}$$

As in Section 2.2, it will be useful to have the expression of $A^{1,*}$ and $A^{2,*}$ in mind for our computations. Here it is readily checked that:

$$A^{1,*}_{st}\varphi = -Z^{1,k}_{st}\operatorname{div}_{x,\xi}\left(V^k\varphi\right), \quad \text{and} \quad A^{2,*}_{st}\varphi = Z^{2,jk}_{st}\operatorname{div}_{x,\xi}\left(V^j\operatorname{div}_{x,\xi}\left(V^k\varphi\right)\right). \tag{4.9}$$

The last point to be specified in order to include (4.2) in the framework of unbounded rough drivers, is how to understand the drift term given by the kinetic measure m. To be more precise, in view of Subsection 2.4, one would like to rewrite (4.2) as

$$\delta f_{st} = \delta\mu_{st} + (A^1_{st}+A^2_{st})f_s + f^\natural_{st} \tag{4.10}$$

where $\delta\mu$ stands for the increment of the corresponding kinetic measure term and $f^\natural$ is a suitable remainder. However, already in the smooth setting such a formulation can only be true for a.e. $s,t\in[0,T]$. Indeed, the kinetic measure contains shocks of the kinetic solution and thus it is not absolutely continuous with respect to the Lebesgue measure. The atoms of the kinetic measure correspond precisely to singularities of the solution. Therefore, it makes a difference if we set

$$\mu_t(\varphi) := -m(\mathbf{1}_{[0,t]}\partial_\xi\varphi) \qquad \text{or} \qquad \mu_t := -m(\mathbf{1}_{[0,t)}\partial_\xi\varphi). \tag{4.11}$$

According to the properties of functions with bounded variation, the first one is right-continuous whereas the second one is left-continuous. Furthermore, they coincide everywhere except on a set of times which is at most countable. Note also that the rough integral $f^\flat$ defined by

$$\delta f^\flat_{st} = (A^1_{st}+A^2_{st})f_s + f^\natural_{st}$$

is expected to be continuous in time. Thus, depending on the chosen definition of μ in (4.11) we obtain either right- or left-continuous representative of the class of equivalence f on the left hand side of (4.10). These representatives will be denoted by f^+ and f^- respectively.

For the sake of completeness, recall that in the deterministic (as well as stochastic) setting, the kinetic formulation (4.2) is understood in the sense of distributions on $[0,T)\times\mathbb{R}^{N+1}$. That is, the test functions depend also on time. Nevertheless, for our purposes it seems to be more convenient to consider directly the equation for the increments $\delta f^{\pm}_{st}$. Correspondingly, we include two versions of (4.2) in the definition of kinetic solution, even though this presentation may look slightly redundant at first. Both of these equations will actually be needed in the proof of uniqueness.

Before doing so, we proceed with a reminder of Young measures and the related definition of kinetic function that will eventually lead to the notion of generalized kinetic solution, see Definition 4.6. For further reading we refer the reader e.g. to [12, 41, 54].

In what follows, we denote by $\mathcal{P}_1(\mathbb{R})$ the set of probability measures on $\mathbb{R}$.

Definition 4.4 (Young measure)**.** Let (X,λ) be a σ-finite measure space. A mapping $\nu : X\to\mathcal{P}_1(\mathbb{R})$ is called a Young measure provided it is weakly measurable, that is, for all $\phi\in C_b(\mathbb{R})$ the mapping $z\mapsto\nu_z(\phi)$ from X to $\mathbb{R}$ is measurable. A Young measure ν is said to vanish at infinity if

$$\int_X\int_{\mathbb{R}}|\xi|\,\mathrm{d}\nu_z(\xi)\,\mathrm{d}\lambda(z)<\infty.$$

Definition 4.5 (Kinetic function)**.** Let (X,λ) be a σ-finite measure space. A measurable function $f : X\times\mathbb{R}\to[0,1]$ is called a kinetic function on X if there exists a Young measure ν on X that vanishes at infinity and such that for a.e. $z\in X$ and for all $\xi\in\mathbb{R}$

$$f(z,\xi)=\nu_z(\xi,\infty).$$

Generalized kinetic solution are needed as a intermediate step in the construction of full solutions to (4.2).

Definition 4.6 (Generalized kinetic solution)**.** Let $f_0:\mathbb{R}^{N+1}\to[0,1]$ be a kinetic function. A measurable function $f:[0,T]\times\mathbb{R}^{N+1}\to[0,1]$ is called a generalized kinetic solution to (4.1) with initial datum f_0 provided

(i) there exist $f^{\pm}$, such that $f^+_t=f^-_t=f_t$ for a.e. $t\in[0,T]$, $f^{\pm}_t$ are kinetic functions on $\mathbb{R}^N$ for all $t\in[0,T]$, and the associated Young measures $\nu^{\pm}$ satisfy

$$\sup_{0\leqslant t\leqslant T}\int_{\mathbb{R}^N}\int_{\mathbb{R}}|\xi|\,\mathrm{d}\nu^{\pm}_{t,x}(\xi)\,\mathrm{d}x<\infty, \tag{4.12}$$

(ii) $f^+_0=f^-_0=f_0$,

(iii) there exists a finite Borel measure m on $[0,T]\times\mathbb{R}^{N+1}$,

(iv) there exist $f^{\pm,\natural}\in V^{\frac{q}{3}}_{2,\mathrm{loc}}([0,T];E_{-3})$ for some $q<3$,

such that, recalling our definition (4.9) of $A^{1,*}$ and $A^{2,*}$, we have that

$$\delta f^+_{st}(\varphi) = f^+_s(A^{1,*}_{st}\varphi+A^{2,*}_{st}\varphi)-m(\mathbf{1}_{(s,t]}\partial_\xi\varphi)+f^{+,\natural}_{st}(\varphi), \tag{4.13}$$

$$\delta f^-_{st}(\varphi) = f^-_s(A^{1,*}_{st}\varphi+A^{2,*}_{st}\varphi)-m(\mathbf{1}_{[s,t)}\partial_\xi\varphi)+f^{-,\natural}_{st}(\varphi), \tag{4.14}$$

holds true for all $s<t\in[0,T]$ and all $\varphi\in E_3$.

Finally we state the precise notion of solution we will consider for eq. (4.2):

Definition 4.7 (Kinetic solution)**.** Let $u_0 \in L^1(\mathbb{R}^N)$. Then $u \in L^\infty(0,T;L^1(\mathbb{R}^N))$ is called a kinetic solution to (4.1) with initial datum u_0 if the function $f_t(x,\xi) = \mathbf{1}_{u_t(x)>\xi}$ is a generalized kinetic solution according to Definition 4.6 with initial condition $f_0(x,\xi) = \mathbf{1}_{u_0(x)>\xi}$.

4.1. **The main result.** Our well-posedness result for the conservation law (4.1) reads as follows.

Theorem 4.8. *Let $u_0 \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, and assume our Hypothesis 4.1 and 4.2 are satisfied. Then the following statements hold true:*

(i) *There exists a unique kinetic solution to (4.1) and it belongs to $L^\infty(0,T;L^2(\mathbb{R}^N))$.*
(ii) *Any generalized kinetic solution is actually a kinetic solution, that is, if f is a generalized kinetic solution to equation (4.1) with initial datum $\mathbf{1}_{u_0>\xi}$ then there exists a kinetic solution u to (4.1) with initial datum u_0 such that $f = \mathbf{1}_{u>\xi}$ for a.e. (t,x,ξ).*
(iii) *If u_1, u_2 are kinetic solutions to (4.1) with initial data $u_{1,0}$ and $u_{2,0}$, respectively, then for a.e. $t \in [0,T]$*
$$\|(u_1(t) - u_2(t))^+\|_{L^1} \leqslant \|(u_{1,0} - u_{2,0})^+\|_{L^1}.$$

Remark 4.9. Note that in the definition of a kinetic solution, u is a class of equivalence in the functional space $L^\infty(0,T;L^1(\mathbb{R}^N))$. Consequently, the L^1-contraction property holds true only for a.e. $t \in [0,T]$. However, it can be proved that in the class of equivalence u there exists a representative u^+, defined through $\mathbf{1}_{u^+(t,x)>\xi} = f_t^+(x,\xi)$, which has better continuity properties and in particular it is defined for every $t \in [0,T]$. If u_1^+ and u_2^+ are these representatives associated to u_1 and u_2 respectively, then
$$\|(u_1^+(t) - u_2^+(t))^+\|_{L^1} \leqslant \|(u_{1,0} - u_{2,0})^+\|_{L^1}$$
is satisfied for every $t \in [0,T]$.

4.2. **Conservation laws with smooth drivers.** In order to justify our notion of kinetic solution introduced above, let us show that in the case of a smooth driver z, this notion coincides with the classical notion of kinetic solution. Recall that using the standard theory for conservation laws, one obtains existence of a unique kinetic solution $u \in L^\infty(0,T;L^1(\mathbb{R}^N)) \cap L^\infty(0,T;L^2(\mathbb{R}^N))$ to the problem
$$\partial_t u + \operatorname{div}(A(x,u))\dot{z} = 0, \qquad u(0) = u_0.$$
In other words, there exists a kinetic measure m such that $f = \mathbf{1}_{u>\xi}$ satisfies the corresponding kinetic formulation
$$\begin{aligned} \partial_t f &= \partial_\xi m + V \cdot \nabla_{\xi,x} f\, \dot{z}, \\ f(0) &= f_0 = \mathbf{1}_{u_0>\xi}, \end{aligned}$$
in the sense of distributions over $[0,T) \times \mathbb{R}^{N+1}$, that is, for every $\varphi \in C_c^\infty([0,T) \times \mathbb{R}^{N+1})$ it holds true
$$\int_0^T f_t(\partial_t \varphi_t)\, \mathrm{d}t + f_0(\varphi_0) = \int_0^T f_t(V \cdot \nabla \varphi_t)\, \mathrm{d}z_t + m(\partial_\xi \varphi) \tag{4.15}$$
(recall that $\operatorname{div} V = 0$).

Lemma 4.10. *Classical kinetic solutions are rough kinetic solutions in the sense of Definition 4.7.*

In order to derive from this formulation the equations for time increments needed in Definition 4.6, let us first recall a classical compactness result for Young measures.

Lemma 4.11 (Compactness of Young measures)**.** *Let (X, λ) be a σ-finite measure space such that $L^1(X)$ is separable. Let (ν^n) be a sequence of Young measures on X such that for some $p \in [1, \infty)$*

$$\sup_{n\in\mathbb{N}} \int_X \int_{\mathbb{R}} |\xi|^p \, \mathrm{d}\nu_z^n(\xi) \, \mathrm{d}\lambda(z) < \infty. \tag{4.16}$$

Then there exists a Young measure ν on X satisfying (4.16) and a subsequence, still denoted by (ν^n), such that for all $h \in L^1(X)$ and all $\phi \in C_b(\mathbb{R})$

$$\lim_{n\to\infty} \int_X h(z) \int_{\mathbb{R}} \phi(\xi) \, \mathrm{d}\nu_z^n(\xi) \, \mathrm{d}\lambda(z) = \int_X h(z) \int_{\mathbb{R}} \phi(\xi) \, \mathrm{d}\nu_z(\xi) \, \mathrm{d}\lambda(z)$$

Moreover, if f_n, $n \in \mathbb{N}$, are the kinetic functions corresponding to ν^n, $n \in \mathbb{N}$, such that (4.16) holds true, then there exists a kinetic function f (which correponds to the Young measure ν whose existence was ensured by the first part of the statement) and a subsequence still denoted by (f^n) such that

$$f_n \xrightarrow{w^*} f \quad \text{in} \quad L^\infty(X \times \mathbb{R}).$$

With this result in hand, we are able to obtain the representatives f^+ and f^- of f.

Lemma 4.12. *For fixed $t \in (0, T)$ and $\varepsilon > 0$ define:*

$$f_t^{+,\varepsilon} := \frac{1}{\varepsilon} \int_t^{t+\varepsilon} f_s \, \mathrm{d}s, \qquad f_t^{-,\varepsilon} := \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} f_s \, \mathrm{d}s.$$

Then there exist f^+, f^-, representatives of the class of equivalence f, such that, for every $t \in (0, T)$, f_t^+, f_t^- are kinetic functions on $\mathbb{R}^N$ and, along subsequences,

$$f_t^{+,\varepsilon} \overset{*}{\rightharpoonup} f_t^+, \quad \text{and} \quad f_t^{-,\varepsilon} \overset{*}{\rightharpoonup} f_t^- \qquad \text{in} \qquad L^\infty(\mathbb{R}^{N+1}).$$

Moreover, the corresponding Young measures $\nu_t^{\pm}$ satisfy

$$\sup_{t\in(0,T)} \int_{\mathbb{R}^N} \int_{\mathbb{R}} \left(|\xi| + |\xi|^2\right) \nu_{t,x}^{\pm}(\mathrm{d}\xi) \leqslant \|u\|_{L_t^\infty L_x^1} + \|u\|_{L_t^\infty L_x^2}^2. \tag{4.17}$$

Proof. Both $f^{+,\varepsilon}$ and $f^{-,\varepsilon}$ are kinetic functions on $\mathbb{R}^N$, with associated Young measures given by:

$$\nu_t^{+,\varepsilon} = \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \nu_s \, \mathrm{d}s, \qquad \nu_t^{-,\varepsilon} = \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} \nu_s \, \mathrm{d}s$$

and satisfy, due to the fact that $u \in L^\infty(0, T; L^1(\mathbb{R}^N)) \cap L^\infty(0, T; L^2(\mathbb{R}^N))$,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}} \left(|\xi| + |\xi|^2\right) \nu_{t,x}^{\pm,\varepsilon}(\mathrm{d}\xi) \, \mathrm{d}x \leqslant \operatorname*{esssup}_{t\in[0,T]} \int_{\mathbb{R}^N} \int_{\mathbb{R}} \left(|\xi| + |\xi|^2\right) \nu_{t,x}(\mathrm{d}\xi) \, \mathrm{d}x \leqslant \|u\|_{L_t^\infty L_x^1} + \|u\|_{L_t^\infty L_x^2}^2.$$

Thus, we can apply Lemma 4.11 to deduce the existence of f_t^+, f_t^-, which are kinetic functions on $\mathbb{R}^N$ such that, along a subsequence that possibly depends on t,

$$f_t^{+,\varepsilon} \overset{*}{\rightharpoonup} f_t^+, \qquad f_t^{-,\varepsilon} \overset{*}{\rightharpoonup} f_t^- \qquad \text{in} \qquad L^\infty(\mathbb{R}^{N+1}).$$

Moreover, the associated Young measures $\nu_t^{\pm}$ satisfy (4.17). As a consequence of the Lebesgue differentiation theorem, they also fulfill $f_t^+ = f_t^- = f_t$ for a.e. $t \in (0,T)$. $\square$

Proof of Lemma 4.10. As a consequence of Lemma 4.12, for all $\varphi \in C_c^\infty(\mathbb{R}^{N+1})$,

$$\delta f_{st}^+(\varphi) = -\int_s^t f_r(V\cdot\nabla\varphi)\,\mathrm{d}z_r - m(\mathbf{1}_{(s,t]}\partial_\xi\varphi)$$

$$\delta f_{st}^-(\varphi) = -\int_s^t f_r(V\cdot\nabla\varphi)\,\mathrm{d}z_r - m(\mathbf{1}_{[s,t)}\partial_\xi\varphi)$$

holds true for every $s,t\in(0,T)$. This can be obtained by testing (4.15) by $\psi^{+,\varepsilon}\varphi$ and $\psi^{-,\varepsilon}\varphi$ where $\psi^{+,\varepsilon}$ and $\psi^{-,\varepsilon}$ are a suitable approximations of $\mathbf{1}_{[0,t]}$ and $\mathbf{1}_{[0,t)}$, respectively, such as

$$\psi_r^{+,\varepsilon} := \begin{cases} 1, & \text{if } r\in[0,t],\\ 1-\frac{r-t}{\varepsilon}, & \text{if } r\in[t,t+\varepsilon],\\ 0, & \text{if } r\in[t+\varepsilon,T],\end{cases} \qquad \psi_r^{-,\varepsilon} := \begin{cases} 1, & \text{if } r\in[0,t-\varepsilon],\\ -\frac{s-t}{\varepsilon}, & \text{if } r\in[t-\varepsilon,t],\\ 0, & \text{if } r\in[t,T],\end{cases} \tag{4.18}$$

and passing to the limit in ε. Therefore, we arrive at the equivalent formulation

$$\delta f_{st}^+(\varphi) = f_s^+(A_{st}^{1,*}\varphi) + f_s^+(A_{st}^{2,*}\varphi) + f_{st}^{+,\natural}(\varphi) - m(\mathbf{1}_{(s,t]}\partial_\xi\varphi),$$
$$\delta f_{st}^-(\varphi) = f_s^-(A_{st}^{1,*}\varphi) + f_s^-(A_{st}^{2,*}\varphi) + f_{st}^{-,\natural}(\varphi) - m(\mathbf{1}_{[s,t)}\partial_\xi\varphi),$$

which holds true in the scale (E_n) with $E_n = W^{n,1}(\mathbb{R}^{N+1})\cap W^{n,\infty}(\mathbb{R}^{N+1})$ for remainders $f^{\pm,\natural}$ given by

$$f_{st}^{\pm,\natural}(\varphi) = -f_s^{\pm}(A_{st}^{2,*}\varphi) + \int_s^t (f_r^{\pm}(V\cdot\nabla\varphi) - f_s^{\pm}(V\cdot\nabla\varphi))\mathrm{d}z_r.$$

Where we have replaced f by $f^{\pm}$ in the above Riemann-Stieltjes since $f_t^+ = f_t^- = f_t$ for a.e. $t\in(0,T)$. Plugging into the integral the equation for $f^{\pm}$ we get

$$f_{st}^{\pm,\natural}(\varphi) = -f_s^{\pm}(A_{st}^{2,*}\varphi) - \int_s^t\left[\int_s^r f_w(V\cdot\nabla(V\cdot\nabla\varphi))\,\mathrm{d}z_w + m(\mathbf{1}_{(s,r]}\partial_\xi(V\cdot\nabla\varphi))\right]\mathrm{d}z_r.$$

Inspection of this expression shows that $f^{\pm,\natural}\in V_{2,\mathrm{loc}}^p(E_{-2})$ for any $p\geqslant 1/2$. Moreover, it can be proved, cf. [13, Remark 12] or [41, Lemma 4.3], that the kinetic measures m do not have atoms at $t=0$ and consequently $f_0^+ = f_0$. $\square$

5. Rough conservation laws II: Uniqueness and reduction

Let f^1, f^2 be two generalized kinetic solutions to (4.1). In what follows, we will use the following notation: $\bar f := 1-f$ as well as $\mathbf{x} := (x,\xi)\in\mathbb{R}^{N+1}$, $\mathbf{y} := (y,\zeta)\in\mathbb{R}^{N+1}$.

5.1. **Tensorization.** In this section we mimic the program of Section 3.1, our aim being to derive an equation for a tensorized function related to f^1 and f^2. Namely, consider $F = f^{1,+}\otimes\bar f^{2,+}$ defined on $[0,T]\times\mathbb{R}^{N+1}\times\mathbb{R}^{N+1}$ by

$$F_t(\mathbf{x},\mathbf{y}) := f_t^{1,+}(\mathbf{x})\bar f_t^{2,+}(\mathbf{y}). \tag{5.1}$$

We first derive a rough driver type equation for the tensorized path F. We introduce the scale of spaces $\mathcal{E}_n^{\otimes} = W^{n,1}(\mathbb{R}^{N+1}\times\mathbb{R}^{N+1})\cap W^{n,\infty}(\mathbb{R}^{N+1}\times\mathbb{R}^{N+1})$ with norms

$$\|\Phi\|_{\mathcal{E}_n^{\otimes}} = \|\Phi\|_{W^{n,1}(\mathbb{R}^{N+1}\times\mathbb{R}^{N+1})} + \|\Phi\|_{W^{n,\infty}(\mathbb{R}^{N+1}\times\mathbb{R}^{N+1})}.$$

Proposition 5.1. *Consider f^1, f^2 two generalized kinetic solutions of (4.1). Let F be defined by relation (5.1), and recall that the tensorized drivers Γ^1, Γ^2 are given by:*

$$\Gamma^1_{st} := A^1_{st} \otimes \mathbb{I} + \mathbb{I} \otimes A^1_{st}, \qquad \Gamma^2_{st} := A^2_{st} \otimes \mathbb{I} + \mathbb{I} \otimes A^2_{st} + A^1_{st} \otimes A^1_{st}.$$

Then for all test functions $\Phi \in \mathcal{E}_3^{\otimes}$, the following relation is satisfied:

$$\delta F_{st}(\Phi) = \delta Q_{st}(\Phi) + F_s((\Gamma^{1,*}_{st} + \Gamma^{2,*}_{st})\Phi) + F^\natural_{st}(\Phi), \tag{5.2}$$

where $F^\natural \in V_2^{q/3}(\mathcal{E}_{-3}^{\otimes})$ and where Q is the path defined (in the distributional sense) as:

$$Q_t := Q^1_t - Q^2_t = \int_{[0,t]} \partial_\xi m^1_{dr} \otimes \bar{f}^{2,-}_r - \int_{[0,t]} f^{1,+}_r \otimes \partial_\xi m^2_{dr}. \tag{5.3}$$

Proof. Let us first work out the algebraic form of the equation governing F in a formal way. Namely, according to relations (4.13), the equations describing the dynamics of $f^{1,+}$ and $\bar{f}^{2,+}$ in the distributional sense are given by:

$$\delta f^{1,+}_{st} = A^1_{st} f^{1,+}_s + A^2_{st} f^{1,+}_s + \partial_\xi m^1\left(\mathbf{1}_{(s,t]}\right) + f^{1,+,\natural}_{st} \tag{5.4}$$

$$\delta \bar{f}^{2,+}_{st} = A^1_{st} \bar{f}^{2,+}_s + A^2_{st} \bar{f}^{2,+}_s - \partial_\xi m^2\left(\mathbf{1}_{(s,t]}\right) - f^{2,+,\natural}_{st}. \tag{5.5}$$

In order to derive the equation for F, we tensorize the equation for $f^{1,+}$ with the equation for $\bar{f}^{2,+}$. Similarly to (3.1) we obtain the following relation, understood in the sense of distributions over $\mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$:

$$\delta F_{st} = \delta f^{1,+}_{st} \otimes \bar{f}^{2,+}_s + f^{1,+}_s \otimes \delta \bar{f}^{2,+}_{st} + \delta f^{1,+}_{st} \otimes \delta \bar{f}^{2,+}_{st}.$$

Expanding $\delta f^{1,+}_{st}$ and $\delta \bar{f}^{2,+}_{st}$ above according to (5.4) and (5.5), we end up with:

$$\begin{aligned}\delta F_{st} &= \Gamma^1_{st} F_s + \Gamma^2_{st} F_s \\ &\quad - f^{1,+}_s \otimes \partial_\xi m^2(\mathbf{1}_{(s,t]}) - \partial_\xi m^1(1_{(s,t]}) \otimes \partial_\xi m^2(\mathbf{1}_{(s,t]}) + \partial_\xi m^1(\mathbf{1}_{(s,t]}) \otimes \bar{f}^{2,+}_s + R^1_{st},\end{aligned} \tag{5.6}$$

where all the other terms have been included in the remainder R^1_{st}. More explicitly

$$\begin{aligned} R^1_{st} &= A^2_{st} f^{1,+}_s \otimes A^1_{st} \bar{f}^{2,+}_s + A^1_{st} f^{1,+}_s \otimes A^2_{st} \bar{f}^{2,+}_s + A^2_{st} f^{1,+}_s \otimes A^2_{st} \bar{f}^{2,+}_s \\ &\quad + f^{1,+,\natural}_{st} \otimes \bar{f}^{2,+}_s - f^{1,+}_s \otimes f^{2,+,\natural}_{st} \\ &\quad -(A^1_{st} + A^2_{st}) f^{1,+}_s \otimes \partial_\xi m^2(\mathbf{1}_{(s,t]}) + \partial_\xi m^1(\mathbf{1}_{(s,t]}) \otimes (A^1_{st} + A^2_{st}) \bar{f}^{2,+}_s - \partial_\xi m^1(\mathbf{1}_{(s,t]}) \otimes \partial_\xi m^2(\mathbf{1}_{(s,t]}) \\ &\quad -(A^1_{st} + A^2_{st}) f^{1,+}_s \otimes f^{2,+,\natural}_{st} + f^{1,+,\natural}_{st} \otimes (A^1_{st} + A^2_{st}) \bar{f}^{2,+}_s \\ &\quad -\partial_\xi m^1(\mathbf{1}_{(s,t]}) \otimes f^{2,+,\natural}_{st} - f^{1,+,\natural}_{st} \otimes \partial_\xi m^2(\mathbf{1}_{(s,t]}) - f^{1,+,\natural}_{st} \otimes f^{2,+,\natural}_{st}. \end{aligned}$$

Let us further decompose the term $I := \partial_\xi m^1(\mathbf{1}_{(s,t]}) \otimes \partial_\xi m^2(\mathbf{1}_{(s,t]})$ in (5.6). The integration by parts formula for two general BV functions A and B reads as

$$A_t B_t = A_s B_s + \int_{(s,t]} A_r \mathrm{d}B_r + \int_{(s,t]} B_{r-} \mathrm{d}A_r.$$

Applying this identity to I, we obtain $I = I_1 + I_2$ with

$$I_1 = -\int_{(s,t]} \partial_\xi m^1(\mathbf{1}_{(s,r]}) \otimes \mathrm{d}\partial_\xi m^2_r, \quad \text{and} \quad I_2 = -\int_{(s,t]} \mathrm{d}\partial_\xi m^1_r \otimes \partial_\xi m^2(\mathbf{1}_{(s,r)}).$$

We now handle I_1 and I_2 separately. For the term I_1 we invoke again equation (5.4) describing the dynamics of $f^{1,+}$, which yields

$$\begin{aligned} I_1 &= -\int_{(s,t]} (\delta f^{1,+}_{sr} - (A^1_{sr} + A^2_{sr}) f^{1,+}_s - f^{1,+,\natural}_{sr}) \otimes \mathrm{d}\partial_\xi m^2_r = -\int_{(s,t]} \delta f^{1,+}_{sr} \otimes \mathrm{d}\partial_\xi m^2 + R^2_{st} \\ &= -\int_{(s,t]} f^{1,+}_r \otimes \mathrm{d}\partial_\xi m^2 + f^{1,+}_s \otimes \partial_\xi m^2(\mathbf{1}_{(s,t]}) + R^2_{st}. \end{aligned}$$

Similarly, we let the patient reader check that the equivalent of relation (5.5) for $\delta \bar{f}^-_{st}$, derived from (4.14), leads to

$$I_2 = \int_{(s,t]} \bar{f}^{2,-}_r \otimes \mathrm{d}\partial_\xi m^1_r - \overline{f}^{2,-}_s \otimes \partial_\xi m^1(\mathbf{1}_{(s,t]}) + \partial_\xi m^2(\mathbf{1}_{\{s\}}) \otimes \partial_\xi m^1(\mathbf{1}_{(s,t]}) + R^3_{st}.$$

In addition, observe that $f^{2,+}_s - f^{2,-}_s = \partial_\xi m^2(\mathbf{1}_{\{s\}})$. Hence $\bar{f}^{2,-}_s - \partial_\xi m(\mathbf{1}_{\{s\}}) = \bar{f}^{2,+}_s$ and we obtain

$$I_2 = \int_{(s,t]} \bar{f}^{2,-}_r \otimes \mathrm{d}\partial_\xi m^1_r - \overline{f}^{2,+}_s \otimes \partial_\xi m^1(\mathbf{1}_{(s,t]}) + R^3_{st}.$$

Plugging the relations we have obtained for $I = I_1 + I_2$ into (5.6) and looking for cancellations, we end up with the following expression for δF:

$$\delta F_{st} = \Gamma^1_{st} F_s + \Gamma^2_{st} F_s + \int_{(s,t]} \mathrm{d}\partial_\xi m^1_r \otimes \bar{f}^{2,-}_r - \int_{(s,t]} f^{1,+}_r \otimes \mathrm{d}\partial_\xi m^2 + F^\natural_{st}.$$

with $F^\natural_{st} = R^1_{st} + R^2_{st} + R^3_{st}$. Having the definition (5.3) of Q in mind, this proves equation (5.2) in the distributional sense, for test functions $\Phi \in C^\infty_c(\mathbb{R}^{N+1} \times \mathbb{R}^{N+1})$ since distributions can act in each set of variables separately. We know establish the claimed regularity for $F^\natural$. This will be obtained via an interpolation argument. Let $\Phi \in C^\infty_c(\mathbb{R}^{N+1} \times \mathbb{R}^{N+1})$ and let J^η a standard smoothing operator for this space. Consider

$$F^\natural_{st}(\Phi) = F^\natural_{st}(J^\eta \Phi) + F^\natural_{st}((\mathrm{Id} - J^\eta)\Phi)$$

The first term will be estimated with the decomposition into the various remainder terms $F^\natural_{st}(J^\eta \Phi) = R^1_{st}(J^\eta \Phi) + R^2_{st}(J^\eta \Phi) + R^3_{st}(J^\eta \Phi)$. Close inspection of the precise form of R^i for $i = 1, 2, 3$ shows that the terms which require more than three derivatives from $J^\eta \Phi$ (resulting in negative powers of η) are also more regular in time. On the other hand, $F^\natural_{st}((\mathrm{Id} - J^\eta)\Phi)$ can be estimated directly from the equation (5.2) and while the various terms show less time regularity they also require less than three derivatives from $(\mathrm{Id} - J^\eta)\Phi$ which in turn become positive powers of η. Reasoning as in the proof of the apriori estimates we can obtain a suitable choiche for η which shows that there exists a control $\omega_{\natural\natural}$, depending on the controls for $f^i, m^i, f^{i,\natural}$, $i = 1, 2$, such that

$$F^\natural_{st}(\Phi) \leqslant \omega_{\natural\natural}(s,t)^{3/q} \|\Phi\|_{\mathcal{E}^\otimes_3}. \tag{5.7}$$

To complete the argument we need to go from the distributional to the variational form of the dynamics of F. That is, we need to establish eq. (5.2) for all $\Phi \in \mathcal{E}^\otimes_3$ and not only for $\Phi \in C^\infty_c(\mathbb{R}^{N+1} \times \mathbb{R}^{N+1})$. In order to do so we observe that $C^\infty_c(\mathbb{R}^{N+1} \times \mathbb{R}^{N+1})$ is weakly-$\star$ dense in $\mathcal{E}^\otimes_3$. Choosing a sequence $(\Phi_n)_n \subseteq C^\infty_c(\mathbb{R}^{N+1} \times \mathbb{R}^{N+1})$ weakly-$\star$ converging to $\Phi \in \mathcal{E}^\otimes_3$ we see that all the terms in eq. (5.2) apart from the remainder $F^\natural$ converge. Consequently also the remainder converges and it satisfies the required estimates by (5.7). □

We now introduce the blow up transformation T_ε as in Subsection 3.1, i.e.

$$T_\varepsilon\Phi(\mathbf{x},\mathbf{y}) := \varepsilon^{-N-1}\Phi\left(\mathbf{x}_+ + \frac{\mathbf{x}_-}{\varepsilon}, \mathbf{x}_+ - \frac{\mathbf{x}_-}{\varepsilon}\right),$$

as well as the related transforms $T_\varepsilon^*, T_\varepsilon^{-1}$ given respectively by (3.6) and (3.7). We also set

$$\mathbf{\Gamma}_\varepsilon^* := T_\varepsilon^{-1}\mathbf{\Gamma}^* T_\varepsilon, \qquad F^\varepsilon := T_\varepsilon^* F, \qquad F^{\natural,\varepsilon} := T_\varepsilon^* F^\natural \qquad Q^\varepsilon := T_\varepsilon^* Q \tag{5.8}$$

It is easily checked that those increments satisfy the following equation:

$$\delta F_{st}^\varepsilon(\Phi) = \delta Q_{st}^\varepsilon(\Phi) + F_s^\varepsilon((\Gamma_{\varepsilon,st}^{1,*} + \Gamma_{\varepsilon,st}^{2,*})\Phi) + F_{st}^{\natural,\varepsilon}(\Phi). \tag{5.9}$$

According to Proposition 3.2, the set $\{\mathbf{\Gamma}_\varepsilon\}_{\varepsilon\in(0,1)}$ is a bounded family of continuous unbounded p-rough drivers on the corresponding scale $(\mathcal{E}_n)$ where

$$\mathcal{E}_n := \left\{\Phi \in W^{n,\infty}(\mathbb{R}^{N+1}\times\mathbb{R}^{N+1});\ \Phi(\mathbf{x},\mathbf{y}) = 0 \text{ if } |\mathbf{x}_-| \geqslant 1\right\}.$$

Moreover, owing to Corollary 3.4, uniformly in R, $\{\mathbf{\Gamma}_\varepsilon\}_{\varepsilon\in(0,1)}$ is also a bounded family of continuous unbounded p-rough drivers on the scale $(\mathcal{E}_{R,n})$ where

$$\mathcal{E}_{R,n} := \left\{\Phi \in W^{n,\infty}(\mathbb{R}^{N+1}\times\mathbb{R}^{N+1});\ \Phi(\mathbf{x},\mathbf{y}) = 0 \text{ if } \rho_R(\mathbf{x},\mathbf{y}) \geq 1\right\} \subset \mathcal{E}_n,$$

and $\rho_R(\mathbf{x},\mathbf{y})^2 = |\mathbf{x}_+|^2/R^2 + |\mathbf{x}_-|^2$.

5.2. **Preliminary estimates.** Our aim is now to concentrate our coordinates on the diagonal, that is to send ε to 0 in relation (5.9). As a preliminary step, we shall derive in this section some bounds on all the terms in (5.9). Recall that in our general scheme, this is first done thanks to an additional localization in $\mathbf{x}_+$ (parametrized by R). It is worth noting that the estimates below will not seek uniformity in the parameter R. This lack of uniformity will not convey any additional difficulty, since our procedure consists in sending $\varepsilon \to 0$ first and then $R \to \infty$.

We now state our upper bound for F^ε in (5.9).

Lemma 5.2. *Let F^ε be the increment defined by (5.8). Then for all $0 \leqslant s \leqslant t \leqslant T$ it holds that*

$$\mathcal{N}[F^\varepsilon; L^\infty(s,t;\mathcal{E}_{R,0}^*)] \lesssim \varepsilon R^N + \sup_{s\leqslant r\leqslant t}\int_{\mathbb{R}^N}\int_{\mathbb{R}}|\xi|\,\nu_{r,x}^{1,+}(\mathrm{d}\xi)\mathrm{d}x + \sup_{s\leqslant r\leqslant t}\int_{\mathbb{R}^N}\int_{\mathbb{R}}|\zeta|\,\nu_{r,y}^{2,+}(\mathrm{d}\zeta)\mathrm{d}y, \tag{5.10}$$

where the proportional constant does not depend on ε and R.

Proof. Consider the function $\Upsilon^\varepsilon : \mathbb{R}^2 \to [0,\infty[$ defined as

$$\Upsilon^\varepsilon(\xi,\zeta) := \int_\zeta^\infty \int_{-\infty}^\xi \varepsilon^{-1}\mathbf{1}_{|\xi'-\zeta'|\leqslant 2\varepsilon}\,\mathrm{d}\xi'\mathrm{d}\zeta',$$

whose main interest lies in the relation $(\partial_\zeta\partial_\xi\Upsilon^\varepsilon)(\xi,\zeta) = \varepsilon^{-1}\mathbf{1}_{|\xi-\zeta|\leqslant 2\varepsilon}$. Let us derive some elementary properties of Υ^ε. First, we obviously have:

$$\partial_\xi\Upsilon^\varepsilon(\xi,\zeta) = \int_\zeta^\infty \varepsilon^{-1}\mathbf{1}_{|\xi-\zeta'|\leqslant 2\varepsilon}\,\mathrm{d}\zeta', \tag{5.11}$$

and in particular $\partial_\xi\Upsilon^\varepsilon(\xi,+\infty) = 0$. A simple change of variables also yields:

$$\Upsilon^\varepsilon(\xi,\zeta) \;=\; \int_{\zeta-\xi}^\infty \int_{-\infty}^0 \varepsilon^{-1}\mathbf{1}_{|\xi'-\zeta'|\leqslant 2\varepsilon}\,\mathrm{d}\xi'\mathrm{d}\zeta' = \;\Upsilon^\varepsilon(0,\zeta-\xi).$$

Moreover, writing

$$\Upsilon^\varepsilon(0,\zeta) = \varepsilon \int_{-\infty}^{\zeta/\varepsilon} \int_{-\infty}^{\zeta'} \mathbf{1}_{|\xi'|\leqslant 2}\, \mathrm{d}\xi' \mathrm{d}\zeta'$$

it follows that

$$|\Upsilon^\varepsilon(0,0)| = \varepsilon \int_{-\infty}^{0} \int_{-\infty}^{\zeta'} \mathbf{1}_{|\xi'|\leqslant 2}\, \mathrm{d}\xi' \mathrm{d}\zeta' \lesssim \varepsilon.$$

Finally, using the elementary bound

$$|\partial_\zeta \Upsilon^\varepsilon(\xi,\zeta)| \leqslant 2,$$

which stems from (5.11), we obtain that

$$|\Upsilon^\varepsilon(\xi,\zeta)| = |\Upsilon^\varepsilon(0,\zeta-\xi)| \lesssim \varepsilon + |\zeta-\xi| \lesssim \varepsilon + |\xi| + |\zeta|.$$

Recall that, since both f^1 and f^2 are kinetic solutions, we have $f_r^{1,+}(x,\xi) = \nu_{r,x}^{1,+}((\xi,+\infty))$ and $\bar{f}_r^{2,+}(y,\zeta) = \nu_{r,y}^{2,+}((-\infty,\zeta])$. With the above properties of Υ^ε in mind we thus obtain, for all $t \in [0,T]$ and $x, y \in \mathbb{R}^N$,

$$\begin{aligned}
\int_{\mathbb{R}^2} f_r^{1,+}(x,\xi) \bar{f}_r^{2,+}(y,\zeta) \varepsilon^{-1} 1_{|\xi-\zeta|\leqslant 2\varepsilon}\, \mathrm{d}\xi \mathrm{d}\zeta &= - \int_{\mathbb{R}^2} f_r^{1,+}(x,\xi) \bar{f}_r^{2,+}(y,\zeta) (\partial_\zeta \partial_\xi \Upsilon^\varepsilon)(\xi,\zeta)\, \mathrm{d}\xi \mathrm{d}\zeta \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} f_r^{1,+}(x;\xi) (\partial_\xi \Upsilon^\varepsilon)(\xi,\zeta)\, \mathrm{d}\xi\, \nu_{r,y}^{2,+}(\mathrm{d}\zeta) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \Upsilon^\varepsilon(\xi,\zeta)\, \nu_{r,x}^{1,+}(\mathrm{d}\xi)\, \nu_{r,y}^{2,+}(\mathrm{d}\zeta) \\
&\lesssim \varepsilon + \int_{\mathbb{R}} |\xi|\, \nu_{r,x}^{1,+}(\mathrm{d}\xi) + \int_{\mathbb{R}} |\zeta|\, \nu_{r,y}^{2,+}(\mathrm{d}\zeta). \qquad (5.12)
\end{aligned}$$

We are now ready to bound F^ε in $\mathcal{E}_{R,0}^*$, which is a L^1-type space. Namely, a simple change of variables yield:

$$\begin{aligned}
\|F_r^\varepsilon\|_{\mathcal{E}_{R,0}^*} &\leqslant \int_{\mathbb{R}^{N+1}\times\mathbb{R}^{N+1}} F_r^\varepsilon(\mathbf{x},\mathbf{y})\, \mathbf{1}_{|\mathbf{x}_-|\leqslant 1}\, \mathbf{1}_{|x_+|\leqslant R}\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&\leqslant \int_{\mathbb{R}^{N+1}\times\mathbb{R}^{N+1}} F_r(\mathbf{x},\mathbf{y}) \varepsilon^{-N-1}\, \mathbf{1}_{|\mathbf{x}_-|\leqslant \varepsilon}\, \mathbf{1}_{|x_+|\leqslant R}\, \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&\leqslant \int_{\mathbb{R}^N\times\mathbb{R}^N} \varepsilon^{-N}\, \mathbf{1}_{|x_-|\leqslant \varepsilon}\, \mathbf{1}_{|x_+|\leqslant R} \int_{\mathbb{R}^2} F_t(\mathbf{x},\mathbf{y}) \varepsilon^{-1}\, \mathbf{1}_{|\xi_-|\leqslant \varepsilon}\, \mathrm{d}\xi \mathrm{d}\zeta \mathrm{d}x \mathrm{d}y.
\end{aligned}$$

Hence, thanks to relation (5.12), we get

$$\begin{aligned}
\|F_r^\varepsilon\|_{\mathcal{E}_{R,0}^*} &\lesssim \int_{\mathbb{R}^N\times\mathbb{R}^N} \varepsilon^{-N}\, \mathbf{1}_{|x_-|\leqslant \varepsilon}\, \mathbf{1}_{|x_+|\leqslant R} \Big(\varepsilon + \int_{\mathbb{R}} |\xi|\, \nu_{r,x}^{1,+}(\mathrm{d}\xi) + \int_{\mathbb{R}} |\zeta|\, \nu_{r,y}^{2,+}(\mathrm{d}\zeta) \Big) \mathrm{d}x\mathrm{d}y \\
&\lesssim \varepsilon R^N + \int_{\mathbb{R}^N\times\mathbb{R}^N} \varepsilon^{-N}\, \mathbf{1}_{|x_-|\leqslant \varepsilon} \Big(\int_{\mathbb{R}} |\xi|\, \nu_{r,x}^{1,+}(\mathrm{d}\xi) + \int_{\mathbb{R}} |\zeta|\, \nu_{r,y}^{2,+}(\mathrm{d}\zeta) \Big) \mathrm{d}x\mathrm{d}y \\
&\lesssim \varepsilon R^N + \int_{\mathbb{R}^N} \int_{\mathbb{R}} |\xi|\, \nu_{r,x}^{1,+}(\mathrm{d}\xi) \mathrm{d}x + \int_{\mathbb{R}^N} \int_{\mathbb{R}} |\zeta|\, \nu_{r,y}^{2,+}(\mathrm{d}\zeta) \mathrm{d}y,
\end{aligned}$$

and the estimate (5.10) follows. □

Let us now proceed to an estimation of the drift term Q^ε in (5.9), where we recall that Q is defined by (5.3). To this aim, we set:

$$q_t^1 := \int_{[0,t]} m_{\mathrm{d}r}^1 \otimes \bar{f}_r^{2,-}, \qquad \sigma_t^1 := \int_{[0,t]} m_{\mathrm{d}r}^1 \otimes \nu_r^{2,-}$$

and in parallel

$$q_t^2 := \int_{[0,t]} f_r^{1,+} \otimes m_{\mathrm{d}r}^2, \qquad \sigma_t^2 := \int_{[0,t]} \nu_r^{1,+} \otimes m_{\mathrm{d}r}^2 \ .$$

With these notations, it holds true that

$$Q^1 = (\partial_\xi \otimes \mathbb{I}) q^1 = 2\partial_\xi^+ q^1 - \sigma^1 \qquad \text{where} \qquad \partial_\xi^+ := \frac{1}{2}(\partial_\xi \otimes \mathbb{I} + \mathbb{I} \otimes \partial_\xi), \tag{5.13}$$

and in the same way $Q^2 = 2\partial_\xi^+ q^2 + \sigma^2$. We now bound the increments q^ℓ for $\ell = 1,2$ uniformly.

Lemma 5.3. *For all $0 \leqslant s \leqslant t \leqslant T$ and $\Phi \in \mathcal{E}_{R,k}$, $k = 1,2$, it holds that*

$$\sum_{\ell=1,2} |\delta q_{st}^\ell(\partial_\xi^+ T_\varepsilon \Phi)| \lesssim \omega_m(s,t) \|\partial_\xi^+ \Phi\|_{\mathcal{E}_{R,0}},$$

where $\ell = 1,2$, and the proportional constant does not depend on ε, R and the control ω_m is defined as follows

$$\omega_m(s,t) := \|\bar{f}^2\|_{L^\infty} m^1((s,t] \times \mathbb{R}^{N+1}) + \|f^1\|_{L^\infty} m^2((s,t] \times \mathbb{R}^{N+1}). \tag{5.14}$$

Proof. We shall bound $q^1(\partial_\xi^+ T_\varepsilon \Phi)$ only, the bound on $q^2(\partial_\xi^+ T_\varepsilon \Phi)$ being obtained in a similar way.

Step 1: Bound on q^1. Consider a test function $\Psi \in \mathcal{E}_{R,0}$, and let us first point out that

$$|\delta q_{st}^1(\Psi)| \leqslant \int_{\mathbf{x},\mathbf{y}} \delta q_{st}^1(d\mathbf{x}, \mathbf{y}) |\Psi(\mathbf{x}, \mathbf{y})| d\mathbf{y},$$

so that the change of variable $\mathbf{x}^- = \frac{1}{2}(\mathbf{x} - \mathbf{y})$ and $\mathbf{x}$ unchanged yields

$$\begin{aligned}
|\delta q_{st}^1(\Psi)| &= 2^{N+1} \int_{\mathbf{x},\mathbf{x}_-} \delta q_{st}^1(d\mathbf{x}, \mathbf{x} - 2\mathbf{x}_-) |\Psi(\mathbf{x}, \mathbf{x} - 2\mathbf{x}_-)| d\mathbf{x}_- \\
&\leqslant 2^{N+1} \int_{\mathbf{x}_-} \int_{\mathbf{x}} \delta q_{st}^1(d\mathbf{x}, \mathbf{x} - 2\mathbf{x}_-) \sup_{\mathbf{x}} |\Psi(\mathbf{x}, \mathbf{x} - 2\mathbf{x}_-)| d\mathbf{x}_- \\
&\leqslant 2^{N+1} \left[\sup_{\mathbf{x}_-} \int_{\mathbf{x}} \delta q_{st}^1(d\mathbf{x}, \mathbf{x} - 2\mathbf{x}_-) \right] \left[\int_{\mathbf{x}_-} \sup_{\mathbf{x}} |\Psi(\mathbf{x}, \mathbf{x} - 2\mathbf{x}_-)| d\mathbf{x}_- \right] \\
&= 2^{N+1} \left[\sup_{\mathbf{x}_-} \int_{\mathbf{x}} \delta q_{st}^1(d\mathbf{x}, \mathbf{x} - 2\mathbf{x}_-) \right] \left[\int_{\mathbf{x}_-} \sup_{\mathbf{x}_+} |\Psi(\mathbf{x}_+ + \mathbf{x}_-, \mathbf{x}_+ - \mathbf{x}_-)| d\mathbf{x}_- \right]
\end{aligned} \tag{5.15}$$

Furthermore, we have:

$$\begin{aligned}
\sup_{\mathbf{x}_-} \int_{\mathbf{x}} \delta q_{st}^1(d\mathbf{x}, \mathbf{x} - 2\mathbf{x}_-) &\leqslant \int_{]s,t]} \sup_{\mathbf{x}_-} \int_{\mathbf{x}} m^1(dr, d\mathbf{x}) |\bar{f}_r^2(\mathbf{x} - 2\mathbf{x}_-)| \\
&\leqslant \|\bar{f}^2\|_{L^\infty} \int_{]s,t] \times \mathbb{R}^{N+1}} m^1(dr, d\mathbf{x}) \\
&\leqslant \|\bar{f}^2\|_{L^\infty} m^1((s,t] \times \mathbb{R}^{N+1}).
\end{aligned}$$

Reporting this estimate into (5.15) we get:

$$|\delta q^1_{st}(\Psi)| \lesssim \|\bar{f}^2\|_{L^\infty} m^1((s,t]\times\mathbb{R}^{N+1})\|\Psi\|_{L^1_- L^\infty_+}, \tag{5.16}$$

where we have introduced the intermediate norm

$$\|\Psi\|_{L^1_- L^\infty_+} := \int_{\mathbb{R}^{N+1}} d\mathbf{x}_- \sup_{\mathbf{x}_+} |\Psi(\mathbf{x}_+ + \mathbf{x}_-, \mathbf{x}_+ - \mathbf{x}_-)|. \tag{5.17}$$

Step 2: Simple properties of the $L^1_- L^\infty_+$-norm. We still consider a test function $\Psi \in \mathcal{E}_{R,0}$. Observe that by the basic change of variables $\mathbf{x}_- = \varepsilon^{-1}\mathbf{x}_-$, one has

$$\begin{aligned} \|T_\varepsilon\Psi\|_{L^1_- L^\infty_+} &= \varepsilon^{-N-1}\int_{\mathbb{R}^{N+1}} d\mathbf{x}_- \sup_z |\Psi(z+\varepsilon^{-1}\mathbf{x}_-, z-\varepsilon^{-1}\mathbf{x}_-)| \\ &= \int_{\mathbb{R}^{N+1}} d\mathbf{x}_- \sup_z |\Psi(z+\mathbf{x}_-, z-\mathbf{x}_-)| \;=\; \|\Psi\|_{L^1_- L^\infty_+}\,. \end{aligned} \tag{5.18}$$

In addition, if $\Psi \in \mathcal{E}_{R,0}$, we can use the fact that the support of Ψ is bounded in the $\mathbf{x}_-$ variable (independently of R) in order to get:

$$\|\Psi\|_{L^1_- L^\infty_+} \leqslant \|\Psi\|_{\mathcal{E}_{R,0}}.$$

Step 3: Conclusion. As a last preliminary step, notice that

$$\partial^+_\xi T_\varepsilon = T_\varepsilon \partial^+_\xi,$$

which entails:

$$|\delta q^1_{st}(\partial^+_\xi T_\varepsilon\Phi)| = |\delta q^1_{st}(T_\varepsilon\partial^+_\xi\Phi)|.$$

Then, applying successively (5.16) and (5.18) it follows that

$$\begin{aligned} |\delta q^1_{st}(\partial^+_\xi T_\varepsilon\Phi)| &= |\delta q^1_{st}(T_\varepsilon\partial^+_\xi\Phi)| \\ &\leqslant \|\bar{f}^2\|_{L^\infty} m^1((s,t]\times\mathbb{R}^{N+1})\|\partial^+_\xi\Phi\|_{L^1_- L^\infty_+} \\ &\lesssim \|\bar{f}^2\|_{L^\infty} m^1((s,t]\times\mathbb{R}^{N+1})\|\partial^+_\xi\Phi\|_{\mathcal{E}_{R,0}}, \end{aligned}$$

which is our claim. □

Our localization procedure includes some terms involving the smoothing operator J^η as well as the cutoff Θ_η. The following lemma gives some bounds for those terms.

Lemma 5.4. *Let Q be defined by (5.3) and Q^ε introduced in (5.8). For all $0 \leqslant s \leqslant t \leqslant T$ and $\Phi \in \mathcal{E}_{R,k}$, $k = 1,2$, it holds that*

$$|\delta Q^\varepsilon_{st}(J^\eta\Theta_\eta\Phi)| \;\lesssim\; \omega_m(s,t)\|\Phi\|_{\mathcal{E}_{R,1}} + \eta^{k-3}\,\omega_{\sigma,\varepsilon,R}(s,t)\|\Phi\|_{\mathcal{E}_{R,k}} \tag{5.19}$$

$$\delta Q^\varepsilon_{st}(\Psi_R) \;\lesssim\; -\omega_{\sigma,\varepsilon,R}(s,t) + \omega_m(s,t)\|\partial^+_\xi\Psi_R\|_{\mathcal{E}_{R,0}}. \tag{5.20}$$

where the proportional constants do not depend on ε and R and Ψ_R is the function introduced in Lemma 3.6. In (5.19) and (5.20), we also have ω_m given by (5.14) and the control $\omega_{\sigma,\varepsilon,R}$ defined as follows:

$$\omega_{\sigma,\varepsilon,R}(s,t) := \delta\sigma^1_{st}(T^\varepsilon\Psi_R) + \delta\sigma^2_{st}(T^\varepsilon\Psi_R). \tag{5.21}$$

Proof. Recall that Q is written as $Q^1 - Q^2$ in (5.3). We focus here on the estimate for Q^1. Furthermore, owing to (5.13) we have

$$\delta Q^{1,\varepsilon}_{st}(J^\eta \Theta_\eta \Phi) = T^*_\varepsilon \delta Q^1_{st}(J^\eta \Theta_\eta \Phi) = \delta Q^1_{st}(T_\varepsilon J^\eta \Theta_\eta \Phi) = 2\delta Q^{11,\varepsilon}_{st} - \delta Q^{12,\varepsilon}_{st},$$

where

$$\delta Q^{11,\varepsilon}_{st} = \partial^+_\xi \delta q^1_{st}(T_\varepsilon J^\eta \Theta_\eta \Phi), \quad \text{and} \quad \delta Q^{12,\varepsilon}_{st} = \delta \sigma^1_{st}(T_\varepsilon J^\eta \Theta_\eta \Phi).$$

Now thanks to Lemma 5.3, we have that

$$|\delta Q^{11,\varepsilon}_{st}| = |\delta q^1_{st}(\partial^+_\xi T_\varepsilon J^\eta \Theta_\eta \Phi)| \lesssim \|\bar{f}^2\|_{L^\infty} m^1((s,t] \times \mathbb{R}^{N+1}) \|\partial^+_\xi J^\eta \Theta_\eta \Phi\|_{\mathcal{E}_{R,0}}$$

Moreover, invoking the fact that J^η is a bounded operator in $\mathcal{E}_1$ plus inequality (3.10), we get:

$$\|\partial^+_\xi J^\eta \Theta_\eta \Phi\|_{\mathcal{E}_{R,0}} \lesssim \|J^\eta \Theta_\eta \Phi\|_{\mathcal{E}_{R,1}} \lesssim \|\Phi\|_{\mathcal{E}_{R,1}},$$

which entails the following relation:

$$|\delta Q^{11,\varepsilon}_{st}| \lesssim \|\bar{f}^2\|_{L^\infty} m^1((s,t] \times \mathbb{R}^{N+1}) \|\Phi\|_{\mathcal{E}_{R,1}}.$$

As far as the term $\delta Q^{12,\varepsilon}_{st}$ is concerned, we make use of (3.12) to deduce

$$\eta^{3-k} |\delta Q^{12,\varepsilon}_{st}| = \eta^{3-k} |\delta \sigma^1_{st}(T_\varepsilon J^\eta \Theta_\eta \Phi)| \leqslant \eta^{3-k} \delta \sigma^1_{st}(T_\varepsilon |J^\eta \Theta_\eta \Phi|) \leqslant \|\Phi\|_{\mathcal{E}_{R,k}}\, \omega_{\sigma,\varepsilon,R}(s,t).$$

Putting together our bound on $\delta Q^{11,\varepsilon}_{st}$ and $\delta Q^{12,\varepsilon}_{st}$, we thus get:

$$|\delta Q^1_{st}(T_\varepsilon J^\eta \Theta_\eta \Phi)| \lesssim \omega_m(s,t) \|\Phi\|_{\mathcal{E}_{R,1}} + \eta^{k-3}\, \omega_{\sigma,\varepsilon,R}(s,t) \|\Phi\|_{\mathcal{E}_{R,k}},$$

and along the same lines, we can prove that

$$|\delta Q^2_{st}(T_\varepsilon J^\eta \Theta_\eta \Phi)| \lesssim \omega_m(s,t) \|\Phi\|_{\mathcal{E}_{R,1}} + \eta^{k-3}\, \omega_{\sigma,\varepsilon,R}(s,t) \|\Phi\|_{\mathcal{E}_{R,k}},$$

which achieves the proof of our assertion (5.19).

The second claim (5.20) is obtained as follows: we start from relation (5.13), which yields:

$$\begin{aligned}
\delta Q^\varepsilon_{st}(\Psi_R) = \delta Q_{st}(T_\varepsilon \Psi_R) &= -\delta\sigma^1_{st}(T_\varepsilon \Psi_R) - \delta\sigma^2_{st}(T_\varepsilon \Psi_R) + \delta q^1_{st}(T_\varepsilon \partial^+_\xi \Psi_R) + \delta q^2_{st}(T_\varepsilon \partial^+_\xi \Psi_R) \\
&\leqslant -\omega_{\sigma,\varepsilon,R}(s,t) + \delta q^1_{st}(T_\varepsilon \partial^+_\xi \Psi_R) + \delta q^2_{st}(T_\varepsilon \partial^+_\xi \Psi_R)
\end{aligned}$$

We can now proceed as for (5.19) in order to bound $\delta q^1_{st}(T_\varepsilon \partial^+_\xi \Psi_R)$ and $\delta q^2_{st}(T_\varepsilon \partial^+_\xi \Psi_R)$ above, and this immediately implies (5.20). $\square$

5.3. **Passage to the diagonal.** As mentioned in Section 3.4.1, a crucial step towards uniqueness is to establish uniform (in ε) bounds on the increment $F^{\natural,\varepsilon}$ introduced in (5.9). We now have all in hand to derive those uniform estimates.

Proposition 5.5. *Let $F^{\natural,\varepsilon}$ be the remainder introduced in (5.9). We consider q, κ satisfying:*

$$q \in \left[\frac{9p}{2p+3}, 3\right), \qquad \kappa \in \left[\frac{3-p}{3p}, \frac{3-p}{p}\right].$$

Also recall that ω_m is given by (5.14) and that ω_Z has been introduced in Hypothesis 4.2. Then there exists a constant $L > 0$ such that if $\omega_Z(I) \leqslant L$, the following relation is satisfied for all $s < t \in I$:

$$\begin{aligned}
\|F^{\natural,\varepsilon}_{st}\|_{\mathcal{E}^*_{R,3}} &\lesssim \omega_{I,\varepsilon,R}(s,t)^{\frac{3}{q}} \\
&= \mathcal{N}[F^\varepsilon; L^\infty(s,t;\mathcal{E}^*_{R,0})]\, \omega_Z(s,t)^{\frac{3}{p}-2\kappa} + \omega_m(s,t)\omega_Z(s,t)^{\frac{1}{p}} + \omega_{\sigma,\varepsilon,R}(s,t)\omega_Z(s,t)^{\kappa},
\end{aligned}$$

where the proportional constant is independent of ε *and* R *and where* $\omega_m, \omega_{\sigma,\varepsilon,R}$ *are respectively defined by* (5.14) *and* (5.21).

Proof. Let $\Phi \in \mathcal{E}_{R,3}$, such that $\|\Phi\|_{\mathcal{E}_{R,3}} \leqslant 1$. We can follow the general scheme alluded to in Section 3.4.1, replacing $G^{\natural,\varepsilon}$ by our current remainder $F^{\natural,\varepsilon}$. This yields the following identity:

$$
\begin{aligned}
\delta F^{\natural,\varepsilon}(\Phi) &= (\delta F^{\varepsilon})(J_\eta(\Theta_\eta \Gamma_\varepsilon^{2,*}\Phi)) + (\delta F^{\varepsilon})((1-J_\eta)(\Theta_\eta \Gamma_\varepsilon^{2,*}\Phi)) + (\delta F^{\varepsilon})((1-\Theta_\eta)\Gamma_\varepsilon^{2,*}\Phi) \\
&\quad + F^{\sharp,\varepsilon}(J_\eta(\Theta_\eta \Gamma_\varepsilon^{1,*}\Phi)) + F^{\sharp,\varepsilon}((1-J_\eta)(\Theta_\eta \Gamma_\varepsilon^{1,*}\Phi)) + F^{\sharp,\varepsilon}((1-\Theta_\eta)\Gamma_\varepsilon^{1,*}\Phi) \\
&= F^{\varepsilon}(\Gamma_\varepsilon^{1,*} J_\eta(\Theta_\eta \Gamma_\varepsilon^{2,*}\Phi)) + F^{\varepsilon}(\Gamma_\varepsilon^{2,*} J_\eta(\Theta_\eta \Gamma_\varepsilon^{2,*}\Phi)) + \delta Q^{\varepsilon}(J_\eta(\Theta_\eta \Gamma_\varepsilon^{2,*}\Phi)) \\
&\quad + F^{\natural,\varepsilon}(J_\eta(\Theta_\eta \Gamma_\varepsilon^{2,*}\Phi)) + (\delta F^{\varepsilon})((1-J_\eta)(\Theta_\eta \Gamma_\varepsilon^{2,*}\Phi)) + (\delta F^{\varepsilon})((1-\Theta_\eta)\Gamma_\varepsilon^{2,*}\Phi) \\
&\quad + F^{\varepsilon}(\Gamma_\varepsilon^{2,*} J_\eta(\Theta_\eta \Gamma_\varepsilon^{1,*}\Phi)) + \delta Q^{\varepsilon}(J_\eta(\Theta_\eta \Gamma_\varepsilon^{1,*}\Phi)) + F^{\natural,\varepsilon}(J_\eta(\Theta_\eta \Gamma_\varepsilon^{1,*}\Phi)) \\
&\quad + (\delta F^{\varepsilon})((1-J_\eta)(\Theta_\eta \Gamma_\varepsilon^{1,*}\Phi)) + F^{\varepsilon}(\Gamma_\varepsilon^{1,*}(1-J_\eta)(\Theta_\eta \Gamma_\varepsilon^{1,*}\Phi)) \\
&\quad + (\delta F^{\varepsilon})((1-\Theta_\eta)\Gamma_\varepsilon^{1,*}\Phi) + F^{\varepsilon}(\Gamma_\varepsilon^{1,*}(1-\Theta_\eta)\Gamma_\varepsilon^{1,*}\Phi).
\end{aligned}
$$

We now bound all the terms in this decomposition separately. Towards this aim, denote $M^{\varepsilon} = \mathcal{N}[F^{\varepsilon}; L^{\infty}(s,t;\mathcal{E}_{R,0}^{*})]$ and let $\omega_{\natural,\varepsilon,R}$ be a control such that

$$
\|F^{\natural,\varepsilon}\|_{\mathcal{E}_{R,3}^{*}} \leqslant \omega_{\natural,\varepsilon,R}(s,t)^{\frac{3}{q}}. \tag{5.22}
$$

Note that M^{ε} is finite due to Lemma 5.2, whereas relation (5.22) is obtained thanks to the full localization as discussed at the beginning of Subsection 5.1. Then using Corollary 3.4 and Lemma 3.6, Lemma 5.2 and Lemma 5.4, we deduce that

$$
\begin{aligned}
|\delta F^{\natural,\varepsilon}(\Phi)| &\lesssim M^{\varepsilon}\omega_Z(s,t)^{\frac{3}{p}} + \eta^{-1}M^{\varepsilon}\omega_Z(s,t)^{\frac{4}{p}} + \big(\omega_m(s,t) + \eta^{1-3}\,\omega_{\sigma,\varepsilon,R}(s,t)\big)\omega_Z(s,t)^{\frac{2}{p}} \\
&\quad + \eta^{-2}\omega_{\natural,\varepsilon,R}(s,t)^{\frac{3}{q}}\omega_Z(s,t)^{\frac{2}{p}} + \eta M^{\varepsilon}\omega_Z(s,t)^{\frac{2}{p}} + \eta M^{\varepsilon}\omega_Z(s,t)^{\frac{2}{p}} \\
&\quad + M^{\varepsilon}\omega_Z(s,t)^{\frac{3}{p}} + \big(\omega_m(s,t) + \eta^{2-3}\,\omega_{\sigma,\varepsilon,R}(s,t)\big)\omega_Z(s,t)^{\frac{1}{p}} + \eta^{-1}\omega_{\natural,\varepsilon,R}(s,t)^{\frac{3}{q}}\omega_Z(s,t)^{\frac{1}{p}} \\
&\quad + \eta^{2}M^{\varepsilon}\omega_Z(s,t)^{\frac{1}{p}} + \eta M^{\varepsilon}\omega_Z(s,t)^{\frac{2}{p}} \\
&\quad + \eta^{2}M^{\varepsilon}\omega_Z(s,t)^{\frac{1}{p}} + \eta M^{\varepsilon}\omega_Z(s,t)^{\frac{2}{p}}.
\end{aligned}
$$

Hence setting

$$
\eta = \omega_Z(I)^{-\frac{1}{p}+\kappa}\omega_Z(s,t)^{\frac{1}{p}-\kappa}
$$

and assuming that $\omega_Z(I) \leqslant 1$ we obtain

$$
\begin{aligned}
|\delta F^{\natural,\varepsilon}(\varphi)_{sut}| &\lesssim M^{\varepsilon}\,\omega_Z(s,t)^{\frac{3}{p}-2\kappa} + \omega_m(s,t)\omega_Z(s,t)^{\frac{1}{p}} \\
&\qquad + \omega_{\sigma,\varepsilon,R}(s,t)\omega_A(s,t)^{\kappa} + \omega_{\natural,\varepsilon,R}(s,t)^{\frac{3}{q}}\omega_Z(s,t)^{\kappa}.
\end{aligned}
$$

The conclusion now follows as in Theorem 2.5. □

As a first consequence of Proposition 5.5, we can derive the following bound on the limit of $\omega_{\sigma,\varepsilon,R}(s,t)$ as $\varepsilon \to 0$ and $R \to \infty$.

Lemma 5.6. *Let* $\omega_{\sigma,\varepsilon,R}$ *be the control defined by* (5.21). *There exists a finite measure* μ *on* $[0,T]$ *such that, for all* $0 \leqslant s \leqslant t \leqslant T$,

$$
\limsup_{R\to\infty}\limsup_{\varepsilon\to 0}\,\omega_{\sigma,\varepsilon,R}(s,t) \leqslant \mu([s,t]). \tag{5.23}
$$

Proof. Consider the sequence of measures $(\mu_R^\varepsilon)_{\varepsilon>0}$ on $[0,T]$ defined for every Borel set $B \subset [0,T]$ as

$$\mu_R^\varepsilon(B) := \left(\int_B m_{dr}^1 \otimes \nu_r^{2,-}\right)(T_\varepsilon \Psi_R) + \left(\int_B \nu_r^{1,+} \otimes m_{dr}^2\right)(T_\varepsilon \Psi_R), \tag{5.24}$$

so that $\omega_{\sigma,\varepsilon,R}(s,t) = \mu_R^\varepsilon((s,t])$. By applying equation (5.9) to the test function Ψ_R and using (5.20), we get that for every $s < t \in [0,T]$,

$$\delta F_{st}^\varepsilon(\Psi_R) \leqslant F_s^\varepsilon\big((\Gamma_{\varepsilon,st}^{1,*} + \Gamma_{\varepsilon,st}^{2,*})(\Psi_R)\big) - \omega_{\sigma,\varepsilon,R}(s,t) + F_{st}^{\natural,\varepsilon}(\Psi_R) + \omega_m(s,t)\|\partial_\xi^+ \Psi_R\|.$$

and so

$$\omega_{\sigma,\varepsilon,R}(s,t) \leqslant F_s^\varepsilon(|(1 + \Gamma_{\varepsilon,st}^{1,*} + \Gamma_{\varepsilon,st}^{2,*})\Psi_R|) + |F_{st}^{\natural,\varepsilon}(\Psi_R)| + \omega_m(s,t)\|\partial_\xi^+ \Psi_R\|.$$

Therefore, due to Lemma 5.2, Proposition 5.5 and assumption (4.12), we can conclude that for every interval $I \subset [0,T]$ satisfying $\omega_Z(I) \leqslant L$, one has

$$\omega_{\sigma,\varepsilon,R}(I) \lesssim \varepsilon R + 1 + \omega_m(I)(1 + \omega_Z(I)^{\frac{1}{p}}) + \omega_{\sigma,\varepsilon,R}(I)\omega_Z(I)^\kappa,$$

for some proportional constant independent of ε, R. As a consequence, there exists $0 < L' \leqslant L$ such that for every interval $I \subset [0,T]$ satisfying $\omega_Z(I) \leqslant L'$, it holds

$$\omega_{\sigma,\varepsilon,R}(I) \lesssim \varepsilon R + 1 + \omega_m(I).$$

By uniformity of both L' and the proportional constant, the latter bound immediately yields

$$\omega_{\sigma,\varepsilon,R}(0,T) \lesssim \varepsilon R + 1 + \omega_m(0,T). \tag{5.25}$$

Thus, the sequence $(\mu_R^\varepsilon)_{\varepsilon>0}$ defined by (5.24) is bounded in total variation on $[0,T]$ and accordingly, by Banach-Alaoglu theorem, there exists a subsequence, still denoted by $(\mu_R^\varepsilon)_{\varepsilon>0}$, as well as a finite measure μ_R on $[0,T]$ such that for every $\varphi \in C([0,T])$, one has

$$\mu_R^\varepsilon(\varphi) \to \mu_R(\varphi) \qquad \text{as } \varepsilon \to 0.$$

Moreover, as a straightforward consequence of (5.25), we get

$$\mu_R([0,T]) \lesssim 1 + \omega_m(0,T).$$

Therefore $(\mu_R)_{R\in\mathbb{N}}$ is bounded in total variation and there exists a finite measure μ on $[0,T]$ satisfying

$$\mu([0,T]) \lesssim 1 + \omega_m(0,T),$$

such that, along a subsequence,

$$\mu_R(\varphi) \to \mu(\varphi) \qquad \forall\, \varphi \in C([0,T]) \qquad R \to \infty$$

Finally, due to the properties of BV-functions, for every $R \in \mathbb{N}$, there exists an at most countable set $\mathcal{D}_R$ such that the function $t \mapsto \mu_R(]0,t])$ is continuous on $[0,T] \setminus \mathcal{D}_R$. Furthermore, by Portmanteau theorem, one has

$$\mu_R^\varepsilon(]0,t]) \to \mu_R(]0,t]) \qquad \forall\, t \in [0,T] \setminus \mathcal{D}_R \qquad \varepsilon \to 0.$$

Similarly, there exists a countable set $\mathcal{D}$ such that

$$\mu_R(]0,t]) \to \mu(]0,t]) \qquad \forall\, t \in [0,T] \setminus \mathcal{D} \qquad R \to 0.$$

Fix $s < t \in [0, T]$. Since a countable union of countable sets is countable, we may consider a sequence (s_k), resp. (t_k), of points outside of $\cup_R \mathcal{D}_R \cup \mathcal{D}$ that increase, resp. decrease, to s, resp. t, as k tends to infinity. Then

$$\limsup_{\varepsilon\to 0}\ \omega_{\sigma,\varepsilon,R}(s,t) = \limsup_{\varepsilon\to 0}\ \mu_R^{\varepsilon}(]s,t]) \leqslant \limsup_{\varepsilon\to 0}\ \mu_R^{\varepsilon}(]s_k,t_k]) = \mu_R(]s_k,t_k])$$

and

$$\limsup_{R\to\infty}\limsup_{\varepsilon\to 0}\ \omega_{\sigma,\varepsilon,R}(s,t) \leqslant \limsup_{R\to\infty}\mu_R(]s_k,t_k]) = \mu(]s_k,t_k]).$$

By letting k tend to infinity, we get (5.23), which achieves the proof of the lemma. □

We are now ready to prove our main intermediate result towards uniqueness.

Proposition 5.7. *Consider $\psi \in C_c^\infty(\mathbb{R}^{N+1})$ such that*

$$\psi \geqslant 0, \qquad \operatorname{supp}\psi \subset B_{2/\sqrt{2}}, \qquad \int_{\mathbb{R}^{N+1}} \psi(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1.$$

Let also $\{\varphi_R; R > 0\} \subset C_c^\infty(\mathbb{R}^{N+1})$ be a family of smooth functions such that

$$\varphi_R \geqslant 0, \qquad \operatorname{supp}\varphi_R \subset B_{R/\sqrt{2}}, \qquad \sup_R \|\varphi_R\|_{W^{3,\infty}} \lesssim 1.$$

We define

$$\Phi_R(\mathbf{x},\mathbf{y}) = \varphi_R(\mathbf{x}_+)\psi(2\mathbf{x}_-). \tag{5.26}$$

Then for every $0 \leqslant s \leqslant t \leqslant T$, it holds true that

$$F_t^{\varepsilon}(\Phi_R) \to h_t(\varphi_R), \tag{5.27}$$

$$(\Gamma^1_{\varepsilon,st} + \Gamma^2_{\varepsilon,st})F_s^{\varepsilon}(\Phi_R) \to (A^1_{st} + A^2_{st})h_s(\varphi_R), \tag{5.28}$$

where $h_t = f_t^{1,+}\bar{f}_t^{2,+}$.

Proof. Consider first a function Ψ supported in $D_R \equiv B_{R+1} \times B_{R+1} \subset \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$. Then for all functions v^1, v^2 we have:

$$\begin{aligned}|v^1 \otimes v^2(\Psi)| &= \left|\int_{D_R} v^1(\mathbf{x}_+ + \mathbf{x}_-)v^2(\mathbf{x}_+ - \mathbf{x}_-)\Psi(\mathbf{x}_+ + \mathbf{x}_-, \mathbf{x}_+ - \mathbf{x}_-)\mathrm{d}\mathbf{x}_-\mathrm{d}\mathbf{x}_+\right| \\ &\leqslant \int_{D_R} v^1(\mathbf{x}_+ + \mathbf{x}_-)v^2(\mathbf{x}_+ - \mathbf{x}_-)\mathrm{d}\mathbf{x}_+ \sup_{\mathbf{y}_+}|\Psi(\mathbf{y}_+ + \mathbf{x}_-, \mathbf{y}_+ - \mathbf{x}_-)|\mathrm{d}\mathbf{x}_-.\end{aligned}$$

Recalling our definition (5.17), $|v^1 \otimes v^2(\Psi)|$ can be further estimated in two ways: on the one hand we have

$$|v^1 \otimes v^2(\Psi)| \leqslant \|v^1\|_{L^1(B_{R+1})}\|v^2\|_{L^\infty(B_{R+1})}\|\Psi\|_{L^1_- L^\infty_+},$$

and on the other hand we also get

$$|v^1 \otimes v^2(\Psi)| \leqslant \|v^2\|_{L^1(B_{R+1})}\|v^1\|_{L^\infty(B_{R+1})}\|\Psi\|_{L^1_- L^\infty_+}.$$

In order to apply this general estimate, define a new test function $\Phi_R(\mathbf{x},\mathbf{y}) = \varphi_R(\mathbf{x}_+)\psi(2\mathbf{x}_-)$, and observe that Φ_R is compactly supported in the set D_R. Since $|f_t^{1,+}| \leqslant 1$, $|\bar{f}_t^{2,+}| \leqslant 1$ it follows that $f_t^{1,+}, \bar{f}_t^{2,+} \in L^1(B_{R+1})$ (notice that the localization procedure is crucial for this step). Therefore one may find $g^1, g^2 \in C_c^\infty(\mathbb{R}^{N+1})$ such that $|g^1| \leqslant 1$, $|g^2| \leqslant 1$ and

$$\|f_t^{1,+} - g^1\|_{L^1(B_{R+1})} + \|\bar{f}_t^{2,+} - g^2\|_{L^1(B_{R+1})} \leqslant \delta. \tag{5.29}$$

We now split the difference $F_t^\varepsilon - h_t$ as follows:

$$
\begin{aligned}
&|F_t^\varepsilon(\Phi_R) - h_t(\varphi_R)| \\
&\leqslant |F_t^\varepsilon(\Phi_R) - (g_1 \otimes g_2)^\varepsilon(\Phi_R)| + \left|(g_1 \otimes g_2)^\varepsilon(\Phi_R) - (g^1 g^2)(\varphi_R)\right| + \left|(g^1 g^2)(\varphi_R) - h_t(\varphi_R)\right|.
\end{aligned}
\tag{5.30}
$$

We shall bound the 3 terms of the right hand side above separately. Indeed, owing to (5.29) and the fact that $|g^1| \leqslant 1$ and $|g^2| \leqslant 1$, we have

$$
\begin{aligned}
|F_t^\varepsilon(\Phi_R) - (g^1 \otimes g^2)^\varepsilon(\Phi_R)| &\leqslant |(g^1 \otimes (\bar{f}_t^{2,+} - g^2))(T_\varepsilon \Phi_R)| + |((f_t^{1,+} - g^1) \otimes \bar{f}_t^{2,+})(T_\varepsilon \Phi_R)| \\
&\lesssim \|g^1\|_{L^\infty(B_{R+1})} \|\bar{f}_t^{2,+} - g^2\|_{L^1(B_{R+1})} + \|\bar{f}_t^{2,+}\|_{L^\infty(B_{R+1})} \|f_t^{1,+} - g^1\|_{L^1(B_{R+1})} \lesssim \delta
\end{aligned}
\tag{5.31}
$$

On the other hand, using the continuity of g^1, g^2 we have

$$
\lim_{\varepsilon \to 0} (g^1 \otimes g^2)(T_\varepsilon \Phi_R) = (g^1 g^2)(\varphi_R),
$$

and thus $|(g_1 \otimes g_2)^\varepsilon(\Phi_R) - (g^1 g^2)(\varphi_R)| \leq \delta$ for ε small enough. Moreover, as in (5.31), we have

$$
\begin{aligned}
&|(g^1 g^2)(\varphi_R) - h_t(\varphi_R)| \\
&\qquad \leqslant \|g^1\|_{L^\infty(B_{R+1})} \|\bar{f}_t^{2,+} - g^2\|_{L^1(B_{R+1})} + \|\bar{f}_t^{2,+}\|_{L^\infty(B_{R+1})} \|f_t^{1,+} - g^1\|_{L^1(B_{R+1})} \\
&\qquad \leqslant \delta.
\end{aligned}
$$

Since δ is arbitrary we have established (5.27).

Let us now turn to (5.28). Observe that by Corollary 3.4 we have that $T_\varepsilon \Gamma_\varepsilon^{1,*} \Phi_R$ and $T_\varepsilon \Gamma_\varepsilon^{2,*} \Phi_R$ are bounded uniformly in ε in $L_-^1 L_+^\infty$. Specifically, we have:

$$
\begin{aligned}
\|T_\varepsilon \Gamma_\varepsilon^{1,*} \Phi_R\|_{L_-^1 L_+^\infty} &\leqslant \|\Gamma_\varepsilon^{1,*} \Phi_R\|_{L_-^1 L_+^\infty} = \|\Gamma_\varepsilon^{1,*} \Phi_R\|_{L_-^1(B_1; L_+^\infty(B_R))} \\
&\leqslant \|\Gamma_\varepsilon^{1,*} \Phi_R\|_{\mathcal{E}_{R,0}} \lesssim_V \|\Phi_R\|_{\mathcal{E}_{R,1}}
\end{aligned}
$$

where we used in order the boundedness of T_ε in $L_-^1 L_+^\infty$, the compact support of Φ_R to go from L^1 to L^∞ and finally the renormalizability of $\mathbf{A}$ in the spaces $(\mathcal{E}_{R,n})_n$. The same reasoning applies to $T_\varepsilon \Gamma_\varepsilon^{2,*} \Phi_R$. Similarly to (5.30), in order to establish the limit of $F_s^\varepsilon(\Gamma_\varepsilon^{j,*} \Phi_R)$ for $j = 1, 2$, it is enough to consider $(g^1 \otimes g^2)(T_\varepsilon \Gamma_\varepsilon^{1,*} \Phi_R)$ and $(g^1 \otimes g^2)(T_\varepsilon \Gamma_\varepsilon^{2,*} \Phi_R)$. Now

$$
\begin{aligned}
(g^1 \otimes g^2)(T_\varepsilon \Gamma_\varepsilon^{1,*} \Phi_R) &= (g^1 \otimes g^2)(\Gamma^{1,*} T_\varepsilon \Phi_R) = (\Gamma^1 (g^1 \otimes g^2))(T_\varepsilon \Phi_R) \\
&= (A^1 g^1 \otimes g^2 + g^1 \otimes A^1 g^2)(T_\varepsilon \Phi_R)
\end{aligned}
$$

and hence we end up with:

$$
\lim_{\varepsilon \to \infty} (g^1 \otimes g^2)(T_\varepsilon \Gamma_\varepsilon^{1,*} \Phi_R) = ((A^1 g^1) g^2 + g^1 (A^1 g^2))(\varphi_R) = (g^1 g^2)(A^{1,*} \varphi_R).
$$

Similarly we have (see [1] for more details about these computations):

$$
\begin{aligned}
(g^1 \otimes g^2)(T_\varepsilon \Gamma_\varepsilon^{2,*} \Phi_R) &= (g^1 \otimes g^2)(\Gamma^{2,*} T_\varepsilon \Phi_R) = (\Gamma^2 (g^1 \otimes g^2))(T_\varepsilon \Phi_R) \\
&= (A^2 g^1 \otimes g^2 + g^1 \otimes A^2 g^2 + A^1 g^1 \otimes A^1 g^2)(T_\varepsilon \Phi_R)
\end{aligned}
$$

Therefore we obtain:

$$
\lim_{\varepsilon \to \infty} (g^1 \otimes g^2)(T_\varepsilon \Gamma_\varepsilon^{2,*} \Phi_R) = ((A^2 g^1) g^2 + g^1 (A^2 g^2) + (A^1 g^1)(A^1 g^2))(\varphi_R) = (g^1 g^2)(A^{2,*} \varphi_R).
$$

This finishes the proof of (5.28). □

The following contraction principle is the main result of this section. By considering two equal initial conditions, it yields in particular our desired uniqueness result for generalized solutions of equation (4.1).

Proposition 5.8. *Let f^1 and f^2 be two generalized kinetic solutions of (4.1) with initial conditions f^1_0 and f^2_0. Assume that $f^1_0 \bar{f}^2_0 \in L^1(\mathbb{R}^{N+1})$ then*

$$\sup_{t\in[0,T]} \|f^{1,+}_t \bar{f}^{2,+}_t\|_{L^1(\mathbb{R}^{N+1})} \leq \|f^1_0 \bar{f}^2_0\|_{L^1(\mathbb{R}^{N+1})}.$$

Proof. Our global strategy is to take limits in (5.9) in order to show the comparison principle. We now divide the proof in several steps.

Step 1: Limit in ε. Recall that Φ_R has been defined by (5.26). Applying (5.9) to the test function Φ_R yields:

$$\delta F^{\varepsilon}_{st}(\Phi_R) = \delta Q^{\varepsilon}_{st}(\Phi_R) + F^{\varepsilon}_s((\Gamma^{1,*}_{\varepsilon,st} + \Gamma^{2,*}_{\varepsilon,st})\Phi_R) + F^{\natural,\varepsilon}_{st}(\Phi_R). \tag{5.32}$$

Furthermore, similarly to (5.20), we have that

$$\delta Q^{\varepsilon}_{st}(\Phi_R) \leqslant -\delta\sigma^1_{st}(T_\varepsilon\Phi_R) - \delta\sigma^2_{st}(T_\varepsilon\Phi_R) + \omega_m(s,t)\|\partial^+_\xi\Phi_R\|_{\mathcal{E}_{R,0}} \leqslant \|\partial^+_\xi\Phi_R\|_{\mathcal{E}_{R,0}}\,\omega_m(s,t).$$

We can thus take limits in relation (5.32) thanks to Proposition 5.7 which gives the following bound:

$$\delta h_{st}(\varphi_R) \leqslant (A^1_{st} + A^2_{st})h_s(\varphi_R) + \limsup_{\varepsilon\to 0}\,\omega_{I,\varepsilon,R}(s,t)^{\frac{3}{q}} + \|\partial^+_\xi\Phi_R\|_{\mathcal{E}_{R,0}}\omega_m(s,t). \tag{5.33}$$

where $\omega_{I,\varepsilon,R}$ is the control defined in Proposition 5.5. Application of Lemma 5.2 and Lemma 5.6, give a uniform bound in R on $\limsup_{\varepsilon\to 0}\omega_{I,\varepsilon,R}(s,t)$ in terms of the control $\omega_\natural$ given by

$$\omega_\natural(s,t)^{\frac{3}{q}} = \omega_Z(s,t)^{\frac{3}{p}-2\kappa} + \omega_m(s,t)\omega_Z(s,t)^{\frac{1}{p}} + \mu([s,t])\omega_Z(s,t)^{\kappa}. \tag{5.34}$$

Namely, we have $\limsup_{\varepsilon\to 0}\omega_{I,\varepsilon,R}(s,t) \leqslant \omega_\natural(s,t)$, hence we can recast inequality (5.33) as:

$$\delta h_{st}(\varphi_R) \leqslant (A^1_{st} + A^2_{st})h_s(\varphi_R) + \omega_\natural(s,t)^{\frac{3}{q}} + \|\partial^+_\xi\Phi_R\|_{\mathcal{E}_{R,0}}\,\omega_m(s,t). \tag{5.35}$$

Step 2: Uniform L^1 bounds. We now wish to test the increment δh_{st} against the function $\mathbf{1}(x,\xi) = 1$ in order to get uniform (in t) L^1 bounds on h. This should be obtained by taking the limit $R\to\infty$ in (5.35). However, the difficulty here is the estimation of the term $(A^1_{st} + A^2_{st})h_s(\varphi_R)$, uniformly in R. To circumvent this problem, we want to choose another test function which is easier to estimate but with unbounded support. Namely, instead of the the function φ_R of Proposition 5.5, let us consider a function $\varphi_{R,L}(\mathbf{x}_+) = \varphi(\mathbf{x}_+/R)\psi_L(\mathbf{x}_+)$. In this definition $\psi_L(\mathbf{x}_+) = \psi(\mathbf{x}_+/L)$ with $\psi(\mathbf{x}_+) = (1+|\mathbf{x}_+|^2)^{-M}$ for $M > (N+1)/2$, and φ is a smooth compactly supported function with $\varphi|_{B_{1/4}} = 1$.

With these notations in hand, relation (5.35) is still satisfied for the function $\varphi_{R,L}$:

$$\delta h_{st}(\varphi_{R,L}) \leqslant (A^1_{st} + A^2_{st})h_s(\varphi_{R,L}) + \omega_\natural(s,t)^{\frac{3}{q}} + \|\partial^+_\xi\Phi_{R,L}\|_{\mathcal{E}_{R,0}}\,\omega_m(s,t), \tag{5.36}$$

where $\Phi_{R,L}$ is defined similarly to (5.26). We can now take limits as R goes to infinity in (5.36). That is, since $A^{j,*}_{st}\varphi_{R,L}$ is an element of L^1, uniformly in R and for for $j=1,2$, we have

$$\lim_{R\to\infty}(A^1_{st} + A^2_{st})h_s(\varphi_{R,L}) = (A^1_{st} + A^2_{st})h_s(\psi_L)$$

In addition, recall that if $g = g(\mathbf{x}_+)$ then $\partial_\xi^+ g = g'(\mathbf{x}_+)$, while $\partial_\xi^+ g = 0$ whenever $g = g(\mathbf{x}_-)$. Therefore, it is readily checked that:

$$\partial_\xi^+ \Phi_R(\mathbf{x}, \mathbf{y}) = R^{-1}(\partial_\xi \varphi)(\mathbf{x}_+/R)\psi(\mathbf{x}_+/L)\psi(2\mathbf{x}_-) + L^{-1}\varphi(\mathbf{x}_+/R)(\partial_\xi \psi)(\mathbf{x}_+/L)\psi(2\mathbf{x}_-),$$

and thus $\|\partial_\xi^+ \Phi_R\|_{\mathcal{E}_{R,0}} \lesssim (R^{-1} + L^{-1})$. Hence, invoking our last two considerations, we can take limits as $R \to \infty$ in relation (5.36) to deduce (for s, t close enough to each other)

$$\delta h_{st}(\psi_L) \lesssim (A^1_{st} + A^2_{st})h_s(\psi_L) + \omega_\natural(s,t)^{\frac{3}{q}} + CL^{-1}\omega_m(s,t). \tag{5.37}$$

We are now in a position to apply our Gronwall type Lemma 2.7 to relation (5.37). To this aim, we can highlight the reason to choose ψ_L as a new test function. Indeed, it is easy to show that for this particular test function we have (see [1])

$$|(A^1_{st} + A^2_{st})h_s(\psi_L)| \lesssim_V h_s(\psi_L)\omega_Z(s,t)^{1/p}$$

for some constant depending only on the vector fields V but uniform in L. The other terms on the right hand side of (5.37) are controls. Note that even though $\mu([s,t])$ is not superadditive due to the possible presence of jumps, one has the following property which can be used as a replacement for superadditivity

$$\mu([s,u]) + \mu([u,t]) \leqslant 2\mu([s,t]).$$

Hence a simple modification of Lemma 2.7 (to take into account this small deviation from superadditivity) gives readily

$$\sup_{t\in[0,T]} h_t(\psi_L) \lesssim 1,$$

where the constant is uniform in L and depends only on $\omega_m, \omega_Z, V, \mu$ and $h_0(\psi_L)$. This implies that if $h_0 \in L^1$ we can send $L \to \infty$ and get that

$$\sup_{t\in[0,T]} h_t(\mathbf{1}) \lesssim 1, \tag{5.38}$$

by monotone convergence. Summarizing, we have obtained that h_t is in L^1 uniformly in $t \in [0,T]$.

Step3: Conclusion. Having relation (5.38) in hand, we can go back to equation (5.37) for $h_t(\psi_L)$, and send $L \to \infty$ therein. We first resort to the fact that $\sup_{t\in[0,T]} h_t(\mathbf{1})$ is bounded in order to get that:

$$\lim_{L\to\infty} (A^1_{st} + A^2_{st})h_s(\psi_L) \lesssim (A^1_{st} + A^2_{st})h_s(\mathbf{1}) = 0,$$

where the second identity is due to the fact that $\operatorname{div} V = 0$ (as noted in (4.5)). Thus, the limiting relation for $\delta h_{st}(\psi_L)$ is:

$$\delta h_{st}(\mathbf{1}) \lesssim \omega_\natural(s,t)^{\frac{3}{q}}, \tag{5.39}$$

for all $s, t \in [0,T]$ at a sufficiently small distance from each other. Thus, one may telescope (5.39) on a partition $\{0 = t_0 < t_1 < \cdots < t_n = t\}$ whose mesh vanishes with n. Invoking our expression (5.34) for $\omega_\natural$, this entails:

$$
\begin{aligned}
h_t(\mathbf{1}) - h_0(\mathbf{1}) &= \sum_{i=0}^{n-1} \delta h_{t_i t_{i+1}}(\mathbf{1}) \\
&\lesssim \Big[\sup_{i=0,\dots,n-1} \omega_Z(t_i, t_{i+1}) \Big]^{(\frac{3}{p} - 2\kappa - 1) \wedge \frac{1}{p} \wedge \kappa} \big(\omega_Z(0,t) + \omega_m(0,t) + 2\mu([0,t]) \big).
\end{aligned}
$$

Eventually we send $n \to \infty$ and use the fact that ω_Z is a regular control. This yields:

$$
\| f_t^{1,+} \bar{f}_t^{2,+} \|_{L^1_{x,\xi}} = h_t(\mathbf{1}) \leqslant h_0(\mathbf{1}) = \| f_0^1 \bar{f}_0^2 \|_{L^1_{x,\xi}},
$$

which ends the proof. □

By standard arguments on kinetic equations, Proposition 5.8 implies uniqueness of the solution to equation (4.1), as well as the reduction of a generalized kinetic solution to a kinetic solution and the L^1-contraction property. This is the contents of the following corollary.

Corollary 5.9. *Under the assumptions of Theorem 4.8, uniqueness holds true for equation (4.1). Furthermore, Theorem 4.8 (ii) and (iii) are satisfied.*

Proof. Let us start by the reduction part, that is Theorem 4.8 (ii). Let then f be a generalized kinetic solution to (4.1) with an initial condition at equilibrium: $f_0 = \mathbf{1}_{u_0 > \xi}$. Applying Proposition 5.8 to $f^1 = f^2 = f$ leads to

$$
\sup_{0 \leqslant t \leqslant T} \| f_t^+ \bar{f}_t^+ \|_{L^1_{x,\xi}} \leqslant \| f_0 \bar{f}_0 \|_{L^1_{x,\xi}} = \| \mathbf{1}_{u_0 > \xi} \mathbf{1}_{u_0 \leqslant \xi} \|_{L^1_{x,\xi}} = 0.
$$

Hence $f_t^+(1 - f_t^+) = 0$ for a.e. (x,ξ). Now, the fact that f_t^+ is a kinetic function for all $t \in [0,T)$ gives the conclusion: indeed, by Fubini's Theorem, for any $t \in [0,T)$, there is a set B_t of full measure in $\mathbb{R}^N$ such that, for all $x \in B_t$, $f_t^+(x,\xi) \in \{0,1\}$ for a.e. $\xi \in \mathbb{R}$. Recall that $-\partial_\xi f_t^+(x,\cdot)$ is a probability measure on $\mathbb{R}$ hence, necessarily, there exists $u^+ : [0,T) \times \mathbb{R}^N \to \mathbb{R}$ measurable such that $f_t^+(x,\xi) = \mathbf{1}_{u^+(t,x) > \xi}$ for a.e. (x,ξ) and all $t \in [0,T)$. Moreover, according to (4.12), it holds

$$
\sup_{0 \leqslant t \leqslant T} \int_{\mathbb{R}^N} |u^+(t,x)| \, \mathrm{d}x = \sup_{0 \leqslant t \leqslant T} \int_{\mathbb{R}^N} \int_{\mathbb{R}} |\xi| \, \mathrm{d}\nu_{t,x}^+(\xi) \, \mathrm{d}x < \infty.
$$

Thus u^+ is a kinetic solution and Theorem 4.8(ii) follows.

In order to prove the L^1-contraction property (that is Theorem 4.8(iii)), consider two kinetic solutions u_1, u_2 of equation (4.1) with respective initial conditions $u_{1,0}, u_{2,0}$. Then we have:

$$
(u_1 - u_2)^+ = \int_{\mathbb{R}} \mathbf{1}_{u_1 > \xi} \overline{\mathbf{1}_{u_2 > \xi}} \, \mathrm{d}\xi.
$$

Let u_1^+ and u_2^+ denote the representatives of u_1, u_2 constructed as in the reduction step. Then we apply Proposition 5.8 and obtain

$$
\big\| \big(u_1^+(t) - u_2^+(t) \big)^+ \big\|_{L^1_x} = \| f_t^{1,+} \overline{f}_t^{2,+} \|_{L^1_{x,\xi}} \leqslant \| f_0^{1,+} \overline{f}_0^{2,+} \|_{L^1_{x,\xi}} = \| (u_{1,0}^+ - u_{2,0}^+)^+ \|_{L^1_x}
$$

which completes the proof of Theorem 4.8(iii). Uniqueness is obtained in the same way, by considering two identical initial conditions. □

6. Rough conservation laws III: A priori estimates

In this section we will establish a priori L^q-estimates for kinetic solutions to (4.1). We thus consider a kinetic solution u to (4.1) and let $f_t(x,\xi) = \mathbf{1}_{u_t(x)>\xi}$ be the corresponding kinetic function. Let us also introduce some useful notation for the remainder of the section.

Notation 6.1. *We denote by χ_t the function $\chi_t(x,\xi) = f_t(x,\xi) - \mathbf{1}_{\xi<0}$. We also define the functions $\beta_q, \gamma_q : \mathbb{R}^{N+1} \to \mathbb{R}$, where $q \geqslant 0$, as follows:*

$$\beta_{q+1}(x,\xi) = \xi|\xi|^q, \quad \textit{and} \quad \gamma_q(x,\xi) = \begin{cases} |\xi|^q & \textit{if } q > 0, \\ 1 & \textit{if } q = 0. \end{cases}$$

The interest of the functions β_q, γ_q lies in the following elementary relations, which are labeled here for further use:

$$\partial_\xi \beta_{q+1} = (q+1)\gamma_q, \qquad \partial_\xi \gamma_{q+2} = (q+2)\beta_{q+1},$$

and consequently for $q \geqslant 2$ we have

$$\chi_t(\beta_{q-1}) = \frac{1}{q}\int_{\mathbb{R}^N} |u_t(x)|^q \,\mathrm{d}x, \quad \text{and} \quad |\chi_t|(\mathbf{1}) = \chi_t(\mathrm{sgn}(\xi)) = \int_{\mathbb{R}^N} |u_t(x)| \,\mathrm{d}x. \tag{6.1}$$

With these preliminary notations in mind, our a priori estimate takes the following form.

Theorem 6.2. *Assume Hypothesis 4.1 holds true, and let u be a solution of equation (4.1). Then u satisfies the following relation:*

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{L^1} \leq \|u(0,\cdot)\|_{L^1} \tag{6.2}$$

and, for all $q \geqslant 2$,

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{L^q}^q + (q-1)\delta m_{0T}(\gamma_{q-2}) \lesssim_{\mathbf{A},q} \|u(0,\cdot)\|_{L^q}^q + \|u(0,\cdot)\|_{L^2}^2 + \|u(0,\cdot)\|_{L^1}. \tag{6.3}$$

Remark 6.3. The above result gives a priori estimates for kinetic solutions that depend only on the rough regularity of the driver $\mathbf{A}$, and are therefore well-suited for the proof of existence in the next section. Note that in order to make all the arguments below entirely rigorous, it is necessary to either work at the level of a (smooth) approximation or to introduce an additional cut-off of the employed test functions. Since we will only apply Theorem 6.2 to smooth approximations, we omit the technical details here. For classical solutions it is easy to prove L^q bounds. These bounds will depend on the C^1 norm of the driver and so will not pass to the limit. But using the fact, proved in Lemma 4.10, that classical kinetic solutions are, in particular, rough kinetic solutions, we can justify the steps below and get the uniform estimates claimed in Theorem 6.2.

Proof of Theorem 6.2. Due to relation (6.1), our global strategy will be to test u_t against the functions β_q defined in Notation 6.1. We will split this procedure in several steps.
Step 1: Equation governing χ. Let χ be the function introduced in Notation 6.1, and observe that $\delta\chi(\varphi) = \delta f(\varphi)$. Furthermore we have (in the distributional sense) $\nabla \mathbf{1}_{\xi<0} = 0$ whenever $\xi \neq 0$, and we have assumed $V(x,0) = 0$ in (4.6). Having in mind relation (4.7) defining A^1 and A^2, this easily yields:

$$\chi(A^{1,*}\varphi + A^{2,*}\varphi) = f(A^{1,*}\varphi + A^{2,*}\varphi).$$

Then the function χ solves the rough equation

$$\delta\chi(\varphi) = \delta\partial_\xi m(\varphi) + \chi(A^{1,*}\varphi) + \chi(A^{2,*}\varphi) + \chi^\natural(\varphi) \tag{6.4}$$

where $\chi^\natural = f^\natural$.

Step 2: Considerations on weights. Our aim is to apply equation (6.4) to a test function of the from β_{q-1} for some $q \geqslant 2$. The growth of the test function does not pose particular problems since we can use a scale of spaces of test function with a polynomial weight $w_{q-1}(\xi) = 1+\gamma_{q-1}$. However, in order to obtain useful estimates, we cannot apply directly the Rough Gronwall strategy. Indeed, estimates for $\chi^\natural(\beta_{q-1})$ will in general depend on $m(w_{q-1})$ and on $|\chi|(w_{q-1})$, and we cannot easily control $m(w_{q-1})$.

To avoid this problem we have to inspect more carefully the equation satisfied by $\chi^\natural(\beta_{q-1})$. Applying δ to (6.4) with $\varphi(\xi) = \beta_{q-1}$ we obtain

$$\delta\chi^\natural_{sut}(\beta_{q-1}) = (\delta\chi)_{su}(A^{2,*}_{ut}\beta_{q-1}) + \chi^\sharp_{su}(A^{1,*}_{ut}\beta_{q-1}) \tag{6.5}$$

with the usual notation

$$\chi^\sharp_{su} = (\delta\chi)_{su} - A^1_{su}\chi_s.$$

Note that this point can be made rigorous as explained in Remark 6.3. Moreover, using the fact that the vector fields are conservative, or otherwise stated:

$$A^{1,*}\varphi = -Z^1 V\cdot\nabla\varphi, \qquad A^{2,*}\varphi = Z^2 V\cdot\nabla(V\cdot\nabla\varphi),$$

we have that the test functions on the right hand side of (6.5), i.e. $A^{1,*}_{ut}\beta_{q-1}$ and $A^{2,*}_{ut}\beta_{q-1}$, as well as their derivatives are bounded by the weight w_{q-2} and not just w_{q-1} as we would naively expect. So we can use the scale of spaces with weight w_{q-2} in order to estimate the remainder.

To this end, consider the family $(E^q_n)_{n\in\mathbb{N}_0}$ of weighted spaces given by

$$E^q_n := \left\{ \varphi : \mathbb{R}^{N+1} \to \mathbb{R};\ \|\varphi\|_{E^q_n} := \sum_{0\leqslant k\leqslant n} \left\| \frac{\nabla^k\varphi}{w_{q-2}} \right\|_{L^\infty_{x,\xi}} < \infty \right\}$$

It can be checked that one can define a family of smoothing operators $(J^\eta)_{\eta\in(0,1)}$ on $(E^q_n)_{n\in\mathbb{N}_0}$ such that (2.12) holds true. Indeed, convolution with standard mollifiers does the job.

Step 3: Estimation of $\chi^\natural$ as a distribution. We are now in a position to see relation (6.5) as an equation of the form (2.10) on the scale $(E^q_n)_{n\in\mathbb{N}_0}$, and apply the general a priori estimate of Theorem 2.5 in this context. Indeed, if $\varphi \in E^q_1$ then it holds true that

$$|\delta m_{st}(\partial_\xi\varphi)| \leqslant \delta m_{st}(w_{q-2})\|\varphi\|_{E^q_1} = (\delta m_{st}(\mathbf{1}) + \delta m_{st}(\gamma_{q-2}))\,\|\varphi\|_{E^q_1}.$$

Besides, if $\varphi \in E^q_{n+1}$, $n = 0,1,2$, then

$$\|A^{k,*}_{st}\varphi\|_{E^q_n} \lesssim_{\|V\|_{W^{n,\infty}}} \omega_Z(s,t)^{\frac{k}{p}}\|\varphi\|_{E^q_{n+k}}, \qquad k = 1,2,$$

which implies that $\mathbf{A} = (A^1, A^2)$ is a continuous unbounded p-rough driver on the scale $(E^q_n)_{n\in\mathbb{N}_0}$. Hence we can proceed exactly as in the proof of Theorem 2.5 (recall again that one should consider smooth approximations of the noise Z to do so) and get the same conclusion.

We apply the result here with $\lambda = p$, $\kappa = 0$, $q = p$, which yields

$$\begin{aligned}\|\chi^\natural_{st}\|_{E^q_{-3}} &\lesssim \omega_\natural(s,t)^{\frac{3}{p}} \\ &:= \sup_{r\in[s,t]} |\chi_r|(\mathbf{1})\,\omega_A(s,t)^{\frac{3}{p}} + (\delta m_{st}(\mathbf{1}) + \delta m_{st}(\gamma_{q-2}))\,\omega_A(s,t)^{\frac{3-p}{p}}.\end{aligned} \tag{6.6}$$

In this last relation, we still have to find an accurate bound for $|\chi|(\mathbf{1})$ and $m(\gamma_{q-2})$.
Step 4: Reduction to L^1 estimates. Inserting now the smoothing operators into (6.5), we obtain

$$\begin{aligned}\delta\chi^\natural(\beta_{q-1}) = \delta\chi((1-J^\eta)A^{1,*}\beta_{q-1}) + \delta\chi((1-J^\eta)A^{2,*}\beta_{q-1}) - \chi(A^{1,*}(1-J^\eta)A^{1,*}\beta_{q-1}) \\ +\chi(A^{2,*}J^\eta A^{1,*}\beta_{q-1}) + \chi(A^{1,*}J^\eta A^{2,*}\beta_{q-1}) + \chi(A^{2,*}J^\eta A^{2,*}\beta_{q-1}) \\ +\chi^\natural(J^\eta A^{1,*}\beta_{q-1}) + \chi^\natural(J^\eta A^{2,*}\beta_{q-1}) \\ -\delta m(\partial_\xi J^\eta A^{1,*}\beta_{q-1}) - \delta m(\partial_\xi J^\eta A^{2,*}\beta_{q-1})\end{aligned}$$

As already explained above, the test functions on the right hand side always contain derivatives of β_{q-1}, so that the scale (E^q_n) is sufficient to control the right hand side. Indeed, we may use (6.6) for the remainder as well as the elementary bound (observe that $|\chi|(\gamma_{q-1}) = \chi(\beta_{q-1})$)

$$|\chi(\varphi)| \leqslant |\chi|(w_{q-2})\|\varphi\|_{E^q_0} \leqslant |\chi|(\mathbf{1} + w_{q-1})\|\varphi\|_{E^q_0} = (2|\chi|(\mathbf{1}) + \chi(\beta_{q-1}))\,\|\varphi\|_{E^q_0}$$

to deduce, similarly to the proof of Theorem 2.5, that

$$\begin{aligned}|\delta\chi^\natural_{sut}(\beta_{q-1})| \lesssim \left(\sup_{r\in[s,t]}|\chi_r|(\mathbf{1}) + \sup_{r\in[s,t]}\chi_r(\beta_{q-1})\right)\omega_Z(s,t)^{\frac{3}{p}} \\ + \omega_Z(I)^{\frac{1}{p}}\omega_\natural(s,t)^{\frac{3}{p}} + (\delta m_{st}(\mathbf{1}) + \delta m_{st}(\gamma_{q-2}))\,\omega_A(s,t)^{\frac{3-p}{p}},\end{aligned}$$

provided $\omega_Z(I)$ is sufficiently small and $s,u,t \in I$. Still following the proof of Theorem 2.5, we can now resort to the sewing Lemma 2.1, which gives

$$\begin{aligned}|\chi^\natural_{st}(\beta_{q-1})| \lesssim \left(\sup_{r\in[s,t]}|\chi_r|(\mathbf{1}) + \sup_{r\in[s,t]}\chi_r(\beta_{q-1})\right)\omega_Z(s,t)^{\frac{3}{p}} \\ + (\delta m_{st}(\mathbf{1}) + \delta m_{st}(\gamma_{q-2}))\,\omega_A(s,t)^{\frac{3-p}{p}}.\end{aligned}$$

Finally, (6.4) applied to β_{q-1} reads as

$$\delta\chi(\beta_{q-1}) = \chi(A^{1,*}\beta_{q-1}) + \chi(A^{2,*}\beta_{q-1}) - (q-1)\delta m(\gamma_{q-2}) + \chi^\natural(\beta_{q-1})$$

so that recalling relation (6.1) and applying the Rough Gronwall lemma yields, for any $q \geqslant 2$,

$$\sup_{t\in[0,T]} \chi_t(\beta_{q-1}) + (q-1)\delta m_{0T}(\gamma_{q-2}) \lesssim \chi_0(\beta_{q-1}) + \delta m_{0T}(\mathbf{1}) + \sup_{t\in[0,T]}|\chi_t|(\mathbf{1}). \tag{6.7}$$

In particular, for $q = 2$ we obtain an estimate for $\delta m_{0T}(\mathbf{1})$ in terms of $\sup_{t\in[0,T]}|\chi_t|(\mathbf{1})$ and the initial condition only:

$$\sup_{t\in[0,T]} \chi_t(\beta_1) + \delta m_{0T}(\mathbf{1}) \lesssim \chi_0(\xi) + \sup_{t\in[0,T]}|\chi_t|(\mathbf{1}).$$

Plugging this relation into (6.7) and recalling relation (6.1), we thus end up with:

$$\sup_{t\in[0,T]} \|u_t\|^q_{L^q} + (q-1)\delta m_{0T}(\gamma_{q-2}) \lesssim \chi_0(\beta_{q-1}) + \sup_{t\in[0,T]}|\chi_t|(\mathbf{1}). \tag{6.8}$$

This way, we have reduced the problem of obtaining a priori estimates in L^q to estimates in L^1, and more specifically to an upper bound on $\sup_{t\in[0,T]}|\chi_t|(\mathbf{1})$.
Step 5: L^1 estimates. The first obvious idea in order to estimate $|\chi|(\mathbf{1})$ is to follow the computations of the previous step. However, this strategy requires to test the equation against the singular test function $(x,\xi)\mapsto \mathrm{sgn}(\xi)$. It might be possible to approximate this test function and then pass to the limit. In order to do so one would have to prove that the rough driver behaves well under this limit and that we have uniform estimates.

Without embarking in this strategy, we shall first upper bound u_t in L^1. Namely, observe that the L^1-contraction property established in Section 5 immediately implies the L^1-estimate we need. Indeed we note that under hypothesis (4.6), equation (4.1) with null initial condition possesses a kinetic solution which is constantly zero. Hence the L^1-contraction property applied to $u_1 = u$ and $u_2 \equiv 0$ yields (6.2). Going back to relation (6.1), this also implies:

$$\sup_{t\in[0,T]} |\chi_t|(\mathbf{1}) \leq \|u_0\|_{L^1},$$

which is the required bound for $|\chi_t|(\mathbf{1})$ needed to close the L^q-estimate (6.8). Our claim (6.3) thus follows. □

7. Rough conservation laws IV: Existence

In order to establish the existence part of Theorem 4.8, let us consider $(\mathbf{Z}^n)_{n\in\mathbb{N}}$, a family of rough paths lifts of smooth paths (z^n) which converge to $\mathbf{Z}$ in the rough path sense over the time interval $[0,T]$. We define the approximate drivers $\mathbf{A}^n = (A^{n,1}, A^{n,2})$ as follows

$$\begin{aligned} A^{n,1}_{st}\varphi &:= Z^{n,1,i}_{st}\, V^i\cdot\nabla_{\xi,x}\varphi,\\ A^{n,2}_{st}\varphi &:= Z^{n,2,ij}_{st}\, V^j\cdot\nabla_{\xi,x}(V^i\cdot\nabla_{\xi,x}\varphi).\end{aligned}$$

If $\mathbf{A}$ is a continuous unbounded p-rough driver on a scale (E_n) according to Definition 2.2, then the approximations $\mathbf{A}^n$ are also continuous unbounded p-rough drivers on (E_n) and, in addition, they can be chosen in such a way that the estimate (4.8) holds true uniformly in n, possibly by using a different control ω_Z (see [25, Chapter 9]). Using the standard theory for conservation laws, one obtains existence of a unique kinetic solution u^n to the approximate problem

$$\mathrm{d}u^n + \mathrm{div}(A(x,u^n))\,\mathrm{d}z^n = 0, \qquad u^n(0) = u_0.$$

moreover we denote by $f^n = \mathbf{1}_{u^n>\xi}$ the kinetic function associated to u^n and by m^n the kinetic measure appearing in the kinetic formulation (4.15). We are now ready to prove the existence of a solution to equation (4.1).

Proof of Theorem 4.8 (i). Step 1: A priori bound for the regularized solutions. Due to Lemma 4.10, the classical solutions f^n corresponds to rough kinetic solutions $f^{n,\pm}$ satisfying

$$\begin{aligned} \delta f^{n,+}_{st}(\varphi) &= f^{n,+}_s(A^{n,1,*}_{st}\varphi) + f^{n,+}_s(A^{n,2,*}_{st}\varphi) + f^{n,+,\natural}_{st}(\varphi) - m^n(\mathbf{1}_{(s,t]}\partial_\xi\varphi),\\ \delta f^{n,-}_{st}(\varphi) &= f^{n,-}_s(A^{n,1,*}_{st}\varphi) + f^{n,-}_s(A^{n,2,*}_{st}\varphi) + f^{n,-,\natural}_{st}(\varphi) - m^n(\mathbf{1}_{[s,t)}\partial_\xi\varphi),\end{aligned} \tag{7.1}$$

which holds true in the scale (E_n) with $E_n = W^{n,1}(\mathbb{R}^{N+1})\cap W^{n,\infty}(\mathbb{R}^{N+1})$ for some remainders $f^{n,\pm,\natural}$.

Under our standing assumption on the initial condition, it follows from Theorem 6.2 that the approximate solutions u^n are bounded uniformly in $L^\infty(0,T;L^1\cap L^2(\mathbb{R}^N))$ and the corresponding kinetic measures m^n are uniformly bounded in total variation. Therefore the Young measures $\nu^n=\delta_{u^n=\xi}$ satisfy

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^N}\int_{\mathbb{R}}|\xi|\,\nu^n_{t,x}(\mathrm{d}\xi)+\sup_{t\in[0,T]}\int_{\mathbb{R}^N}\int_{\mathbb{R}}|\xi|^2\,\nu^n_{t,x}(\mathrm{d}\xi)\lesssim\|u_0\|_{L^1}+\|u_0\|^2_{L^2}.$$

Now we simply invoke Theorem 2.5 with $\lambda=p$, $q=p$ and $\kappa=0$ we obtain, since $|f^{n,\pm}|\leqslant 1$,

$$\|f^{n,\pm,\natural}_{st}\|_{E_{-3}}\lesssim\omega_Z(s,t)^{\frac{3}{p}}+m^n(\mathbf{1}_{(s,t]})\,\omega_Z(s,t)^{\frac{3-p}{p}},\tag{7.2}$$

provided $\omega_Z(s,t)\leqslant L$. Notice that this restriction on the distance of s,t induces a covering $\{I_k;\,k\leq M\}$ of the interval $[0,T]$, for a finite $M\in\mathbb{N}$. To be more specific, I_k is just chosen so that:

$$\sup_{s,t\in I_k}\omega_Z(s,t)\leqslant L\qquad\forall k.$$

Thus relation (7.2) is satisfied on each interval I_k.
Step 2: Limit in equation (7.1). By (7) the assumptions of Lemma 4.11 are fulfilled and there exists a kinetic function f on $[0,T]\times\mathbb{R}^N$ such that, along a subsequence,

$$f^n\overset{*}{\rightharpoonup}f\qquad\text{in}\qquad L^\infty([0,T]\times\mathbb{R}^{N+1}),\tag{7.3}$$

and the associated Young measure ν satisfies

$$\operatorname*{esssup}_{t\in[0,T]}\int_{\mathbb{R}^N}\int_{\mathbb{R}}\left(|\xi|+|\xi|^2\right)\nu_{t,x}(\mathrm{d}\xi)\lesssim\|u_0\|_{L^1}+\|u_0\|^2_{L^2}.$$

Moreover by the Banach-Alaoglu theorem there exists a nonnegative bounded Borel measure m on $[0,T]\times\mathbb{R}^{N+1}$ such that, along a subsequence,

$$m^n\overset{*}{\rightharpoonup}m\qquad\text{in}\qquad\mathcal{M}_b([0,T]\times\mathbb{R}^{N+1}).\tag{7.4}$$

Moreover using Lemma 4.12 we have also the existence of the good representatives $f^\pm$ of f. In order to pass to the limit in the equation (7.1) the main difficulty originates in the fact that the only available convergence of $f^{n,+}$ (as well as $f^{n,-}$ and f^n) is weak* in time. Consequently, we cannot pass to the limit pointwise for a fixed time t. In order to overcome this issue, we observe that the first three terms on the right hand sides in (7.1), i.e. the approximation of the Riemann-Stieltjes integral, are continuous in t. The kinetic measure poses problems as it contains jumps, which are directly related to the possible noncontinuity of f^n. Therefore, let us define an auxiliary distribution $f^{n,\flat}$ by

$$f^{n,\flat}_t(\varphi):=f^{n,+}_t(\varphi)+m^n(\mathbf{1}_{[0,t]}\partial_\xi\varphi),$$

and observe that due to (7.1) it can also be written as

$$f^{n,\flat}_t(\varphi)=f^{n,-}_t(\varphi)+m^n(\mathbf{1}_{[0,t)}\partial_\xi\varphi).$$

Then we have

$$\delta f^{n,\flat}_{st}(\varphi)=f^{n,\pm}_s(A^{n,1,*}_{st}\varphi)+f^{n,\pm}_s(A^{n,2,*}_{st}\varphi)+f^{n,\pm,\natural}_{st}(\varphi)\tag{7.5}$$

and due to (7.2), satisfied on each I_k, this yields:

$$|\delta f^{n,\flat}_{st}(\varphi)|\lesssim\left(\omega_Z(s,t)^{\frac{1}{p}}+m(\mathbf{1}_{[0,T]})\,\omega_Z(s,t)^{\frac{3-p}{p}}\right)\|\varphi\|_{E_3}\lesssim\omega_Z(s,t)^{\frac{3-p}{p}}\|\varphi\|_{E_3},\tag{7.6}$$

where the second inequality stems from (7.4).

Owing to the fact that $f^{n,\flat}$ is a path, the local bound (7.6) on each interval I_k can be extended globally on $[0,T]$ by a simple telescopic sum argument. In other words, $(f^{n,\flat}(\varphi))_{n\in\mathbb{N}}$ is equicontinuous and bounded in $V_1^q([0,T];\mathbb{R})$ for $q=\frac{p}{3-p}$. So as a corollary of the Arzelà-Ascoli theorem (cf. [25, Proposition 5.28]), there exists a subsequence, possibly depending on φ, and an element $f^{\flat,\varphi}\in V_1^q([0,T];\mathbb{R})$ such that

$$f^{n,\flat}(\varphi)\to f^{\flat,\varphi} \qquad \text{in} \qquad V_1^{q'}([0,T];\mathbb{R}) \qquad \forall q'>q. \tag{7.7}$$

As the next step, we prove that the limit $f^{\flat,\varphi}$ can be identified to be given by a true distribution $f_t^\flat$ as

$$f_t^{\flat,\varphi}=f_t^+(\varphi)+m(\mathbf{1}_{[0,t]}\partial_\xi\varphi)=f_t^-(\varphi)+m(\mathbf{1}_{[0,t)}\partial_\xi\varphi)=:f_t^\flat(\varphi). \tag{7.8}$$

To this end, let us recall that given $r_1,r_2\geqslant 1$ such that $\frac{1}{r_1}+\frac{1}{r_2}>1$, we can define the Young integral as a bilinear continuous mapping

$$V_1^{r_1}([0,T];\mathbb{R})\times V_1^{r_2}([0,T];\mathbb{R})\to V_1^{r_2}([0,T];\mathbb{R}), \qquad (g,h)\mapsto\int_0^\cdot g\,\mathrm{d}h,$$

see [25, Theorem 6.8]. Let $\psi\in C_c^\infty([0,T))$. Then it follows from the definition of $f^{n,\flat}$, the integration by parts formula for Young integrals and $f_0^{n,\flat}=f_0^{n,+}=f_0$ that

$$\int_0^T f_t^n(\varphi)\psi_t'\,\mathrm{d}t-m^n(\psi\partial_\xi\varphi)+f_0(\varphi)\psi_0=-\int_0^T\psi_t\,\mathrm{d}f_t^{n,\flat}(\varphi).$$

The convergences (7.3) and (7.4) allow now to pass to the limit on the left hand side, whereas by (7.7) we obtain the convergence of the Young integrals on the right hand side. We obtain

$$\int_0^T f_t(\varphi)\psi_t'\,\mathrm{d}t-m(\psi\partial_\xi\varphi)+f_0(\varphi)\psi_0=-\int_0^T\psi_t\,\mathrm{d}f_t^{\flat,\varphi}$$

Now, in order to derive (7.8) we consider again the two sequences of test functions (4.18) and pass to the limit as $\varepsilon\to 0$. Indeed, due to Lemma 4.12 we get the convergence of the first term on the left hand side, the kinetic measure term converges due to dominated convergence theorem and for the right hand side we use the continuity of the Young integral. We deduce

$$\begin{aligned}-f_t^+(\varphi)-m(\mathbf{1}_{[0,t]}\partial_\xi\varphi)+f_0(\varphi)&=-f_t^{\flat,\varphi}+f_0^{\flat,\varphi},\\ -f_t^-(\varphi)-m(\mathbf{1}_{[0,t)}\partial_\xi\varphi)+f_0(\varphi)&=-f_t^{\flat,\varphi}+f_0^{\flat,\varphi},\end{aligned}$$

and (7.8) follows since for $t=0$ we have

$$f_0^{\flat,\varphi}=\lim_{n\to\infty}f_0^{n,\flat}(\varphi)=\lim_{n\to\infty}f_0^{n,\pm}(\varphi)=f_0(\varphi).$$

Now, it only remains to prove that $f_0^+=f_0$. The above formula at time $t=0$ rewrites as

$$f_0^+(\varphi)-f_0(\varphi)=-m(\mathbf{1}_{\{0\}}\partial_\xi\varphi).$$

Hence the claim can be proved following the lines of [41, Lemma 4.3] and we omit the details. For the sake of completeness, let us set $f_0^-:=f_0$ and $f_T^+:=f_T^-$.

Finally we have all in hand to complete the proof of convergence in (7.1). Fix $\varphi \in E_3$ and integrate (7.5) over s as follows

$$\frac{1}{\varepsilon}\int_s^{s+\varepsilon}(\delta f^{n,\flat})_{rt}(\varphi)\,\mathrm{d}r - \frac{1}{\varepsilon}\int_s^{s+\varepsilon} f_r^{n,+}(A_{rt}^{n,1,*}\varphi + A_{rt}^{n,2,*}\varphi)\,\mathrm{d}r = \frac{1}{\varepsilon}\int_s^{s+\varepsilon} f_{rt}^{n,+,\natural}(\varphi)\,\mathrm{d}r,$$
$$\frac{1}{\varepsilon}\int_{s-\varepsilon}^{s}(\delta f^{n,\flat})_{rt}(\varphi)\,\mathrm{d}r - \frac{1}{\varepsilon}\int_{s-\varepsilon}^{s} f_r^{n,-}(A_{rt}^{n,1,*}\varphi + A_{rt}^{n,2,*}\varphi)\,\mathrm{d}r = \frac{1}{\varepsilon}\int_{s-\varepsilon}^{s} f_{rt}^{n,-,\natural}(\varphi)\,\mathrm{d}r.$$

On the left hand side we can successively take the limit as $n \to \infty$ and $\varepsilon \to 0$ (or rather for a suitable subsequence of n and ε depending possibly on φ and s, to be more precise). This leads to the following assertion: for every $s < t \in [0, T]$, the quantities

$$\begin{aligned} f_{st}^{+,\natural}(\varphi) &:= \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{\varepsilon}\int_s^{s+\varepsilon} f_{rt}^{n,+,\natural}(\varphi)\,\mathrm{d}r \\ f_{st}^{-,\natural}(\varphi) &:= \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{\varepsilon}\int_{s-\varepsilon}^{s} f_{rt}^{n,-,\natural}(\varphi)\,\mathrm{d}r \end{aligned} \tag{7.9}$$

are well-defined, finite and satisfy

$$(\delta f^\flat)_{st}(\varphi) = f_s^{\pm}(A_{st}^{1,*}\varphi + A_{st}^{2,*}\varphi) + f_{st}^{\pm,\natural}(\varphi). \tag{7.10}$$

Injecting (7.8) into (7.10) yields that for every $s < t \in [0, T]$,

$$\delta f_{st}^{+}(\varphi) = f_s^{+}(A_{st}^{1,*}\varphi + A_{st}^{2,*}\varphi) - m(\mathbf{1}_{(s,t]}\partial_\xi\varphi) + f_{st}^{+,\natural}(\varphi),$$
$$\delta f_{st}^{-}(\varphi) = f_s^{-}(A_{st}^{1,*}\varphi + A_{st}^{2,*}\varphi) - m(\mathbf{1}_{[s,t)}\partial_\xi\varphi) + f_{st}^{-,\natural}(\varphi),$$

and so it only remains to prove that the remainders $f^{\pm,\natural}$ defined by (7.9) are sufficiently regular. To this end, we first observe that

$$\limsup_{n\to\infty} m^n(\mathbf{1}_{(s,t]}) \leqslant m(\mathbf{1}_{[s,t]}), \qquad \limsup_{n\to\infty} m^n(\mathbf{1}_{[s,t)}) \leqslant m(\mathbf{1}_{[s,t]}) \tag{7.11}$$

holds true for every $s < t \in [0, T]$. Indeed, the weak* convergence of m^n to m, as described by (7.4), allows us to assert that for every t in the (dense) subset $\mathfrak{C}_m$ of continuity points of the function $t \mapsto m(\mathbf{1}_{(0,t]})$, one has $m^n(\mathbf{1}_{(0,t]}) \to m(\mathbf{1}_{(0,t]})$ as $n \to \infty$. Consider now two sequences s_k, t_k in $\mathfrak{C}_m$ such that s_k strictly increases to s and t_k decreases to t, as $k \to \infty$. Hence it holds that

$$\limsup_{n\to\infty}\, m^n(\mathbf{1}_{(s,t]}) \leqslant \limsup_{n\to\infty}\, m^n(\mathbf{1}_{(s_k,t_k]}) = m(\mathbf{1}_{(s_k,t_k]}).$$

Taking the limsup over k yields the first part of (7.11), the second part being similar. Next, we make use of (7.2) and (7.11) to deduce for every $\varphi \in E_3$ and every $s < t \in I_k$,

$$|f_{st}^{\pm,\natural}(\varphi)| \lesssim \|\varphi\|_{E_3}\left(\omega_Z(s,t) + m([s,t])^{\frac{p}{3}}\omega_Z(s,t)^{1-\frac{p}{3}}\right)^{\frac{3}{p}}.$$

We can conclude that $f^{\pm,\natural} \in V_{2,\mathrm{loc}}^{\frac{p}{3}}([0,T]; E_3^*)$, and finally the pair (f, m) is indeed a generalized kinetic solution on the interval $[0, T]$.

Step 3: Conclusion. The reduction Theorem 4.8(ii) now applies and yields the existence of $u^+ : [0,T) \times \mathbb{R}^N \to \mathbb{R}$ such that $\mathbf{1}_{u^+(t,x)>\xi} = f_t^+(x,\xi)$ for a.e. (x,ξ) and every t. Besides, we deduce from (4.17) that $u^+ \in L^\infty(0,T; L^1 \cap L^2(\mathbb{R}^N))$. Hence, the function u^+ is a representative of a class of equivalence u which is a kinetic solution to (4.1). In view of Remark 4.9, this is the representative which then satisfies the L^1-contraction property for every $t \in [0,T]$ and not only almost everywhere. The proof of Theorem 4.8(i) is now complete. $\square$

Conflict of Interest

The authors declare that they have no conflict of interest.

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