Moment estimates for some renormalized parabolic Anderson models

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Ongoing joint work with A. Deya, X. Chen, C. Ouyang

Outline

1. Parabolic Anderson model
2. Main results
3. Feyman-Kac representations
4. Heuristics about exponents

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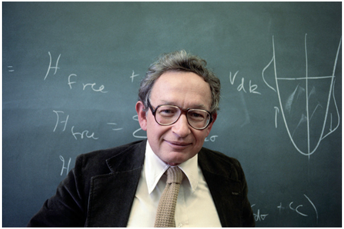
Some history

Philip Anderson:

- Born 1923
- Wide range of achievements
  $\hookrightarrow$ In condensed matter physics
- Nobel prize in 1977
- Still Professor at Princeton



One of Anderson's discoveries:

For particles moving in a disordered media
$\hookrightarrow$ Localized behavior instead of diffusion.

Equation under consideration

Equation:

Stochastic heat equation in $\mathbb{R}^d$, with very rough environment:

$$\partial_t u_t(x) = \frac{1}{2}\Delta u_t(x) + u_t(x)\,\dot{W}_t(x), \tag{1}$$

with

- $t \geq 0$, $x \in \mathbb{R}^d$ (we take $d = 1$ or $d = 2$ to simplify presentation).
- $\dot{W}$ space-time Gaussian noise
- $\dot{W}$ rougher than white in some directions.
- $u_t(x)\,\dot{W}_t(x)$ differential: Stratonovich or Skorohod sense.

Aim:

1. Define and solve the equation
2. Information on moments of the solution

Basic questions

A formal decomposition of PAM: In the equation

$$\partial_t u_t(x) = \frac{1}{2}\Delta u_t(x) + u_t(x)\,\dot{W}(x),$$

we have (here $\dot{W}$ is a spatial noise)

- $\partial_t u_t = \frac{1}{2}\Delta u_t$ implies strong smoothing effect
- $\partial_t u_t = u_t\,\dot{W}$ implies large fluctuations
  $\hookrightarrow$ Formally we would have $u_t(x) = e^{t\dot{W}(x)}$

Basic question 1:
Who wins the above competition? Effect of randomness on u?

Related question 2:
Various aspects of localization

Localization 1: intermittency phenomenon

Equation: $\partial_t u_t(x) = \frac{1}{2}\Delta u_t(x) + \lambda\, u_t(x)\, \dot{W}_t(x)$

Phenomenon: The solution u concentrates its energy in high peaks.

Characterization: through moments
$\hookrightarrow$ Easy possible definition of intermittency: for all $k_1 > k_2 \geq 1$

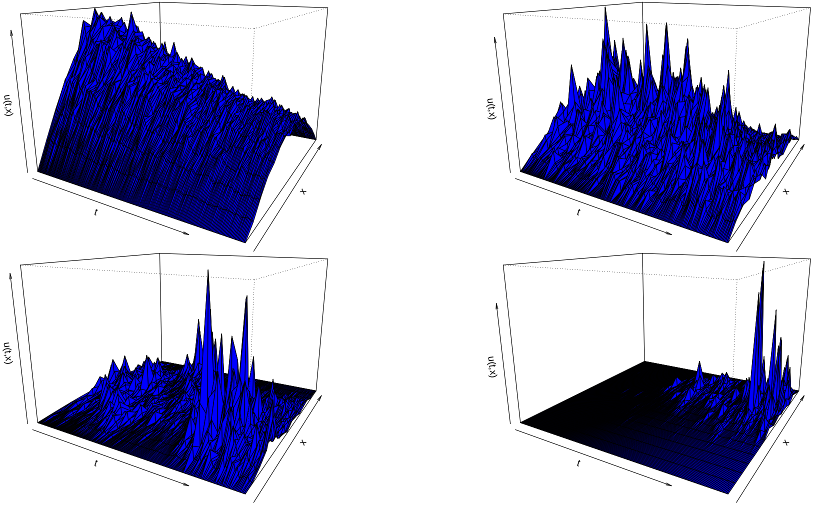
$$\lim_{t\to\infty} \frac{\mathbf{E}^{1/k_1}\left[|u_t(x)|^{k_1}\right]}{\mathbf{E}^{1/k_2}\left[|u_t(x)|^{k_2}\right]} = \infty\,.$$

Results:

- White noise in time: Khoshnevisan, Foondun, Conus, Joseph
- Fractional noise in time: Balan-Conus, Hu-Huang-Nualart-T
- Analysis through Feynman-Kac formula

Intemittency: illustration (by Daniel Conus)

Simulations: for $\lambda = 0.1, 0.5, 1$ and 2.



Localization 2: Eigenfunctions

Equation with spatial noise:

$\partial_t u_t(x) = \frac{1}{2}\Delta u_t(x) + u_t(x)\,\dot{W}(x)$, for $x \in [-M, M]^d$

Fact (discrete case):

The operator $\frac{1}{2}\Delta + \dot{W}(x)$ admits a discrete spectrum (λ_k)

$\hookrightarrow$ Corresponding eigenfunction is v_k

Localization 2:

- The v_k's decay exponentially fast around a center x_k
- This is reflected on λ_k
  $\hookrightarrow$ $\lambda_k \simeq$ principal eigenvalue on a ball centered at x_k

Localization 2: illustration

Image (Filoche-Mayboroda): First eigenvectors for a PAM in $[0,1]^2$

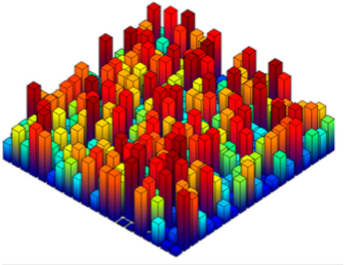


Figure: Discrete random potential

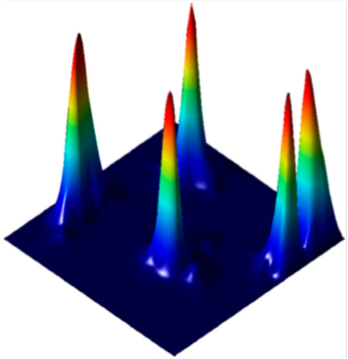


Figure: First five eigenvectors

From spectral localization to $u_t(x)$

Heuristics:

- $u_t(0)$ related to the Laplace transform at $t > 0$
  $\hookrightarrow$ for the spectral measure of $\frac{1}{2}\Delta + \dot{W}$
- Asymptotics of $u_t(0)$ for large t
  $\hookrightarrow$ Information on spectral measure close to 0

Conclusion:

Limiting behavior of $\mathbf{E}[|u_t(0)|^p]$ for large p, t
Related to
Spectral information on $\frac{1}{2}\Delta + \dot{W}$

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Model description

Equation: For $x \in \mathbb{R}$ or $x \in \mathbb{R}^2$ we consider

$$\begin{cases} \partial_t u_t(x) = \frac{1}{2}\Delta u_t(x) + u_t(x)\,\dot{W}_t(x), \\ u_0(x) = 1 \end{cases}$$

Model for the noise: We take

- W fBs with parameters (H_0, H_1, H_2) with some $H_i \in (0, 1/2)$
- $\dot{W}_t(x) = \partial_{t x_1 x_2} W_t(x)$

Description of the noise

Covariance function for W: We have W Gaussian and

$$\mathbf{E}\left[W_t(x)\, W_s(y)\right] = R_0(s,t) \prod_{j=1}^{d} R_j(x_j, y_j),$$

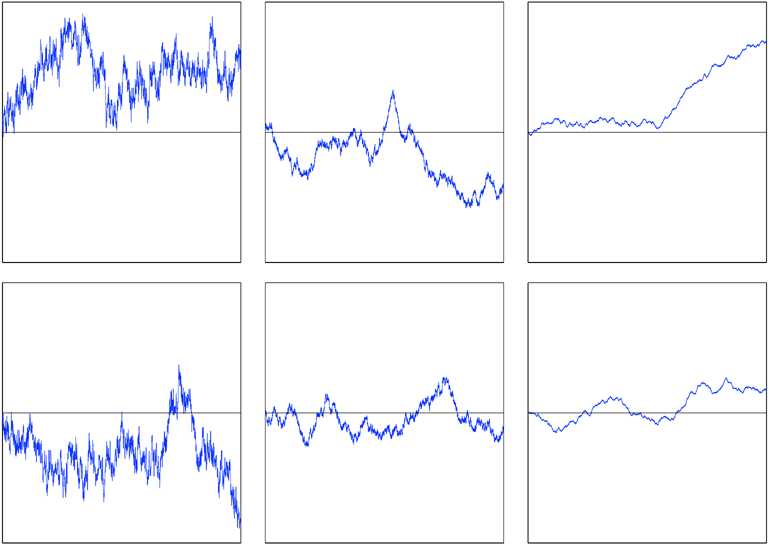
with

$$R_j(u,v) = \frac{1}{2}\left(|u|^{2H_j} + |v|^{2H_j} - |u-v|^{2H_j}\right), \qquad u, v \in \mathbb{R}. \qquad (2)$$

Remarks:

- We have a fBm in each direction
- We are rougher than white noise if $H_j < \frac{1}{2}$

Examples of fBm paths



$H = 0.3$ $\quad$ $H = 0.5$ $\quad$ $H = 0.7$

Description of the noise (2)

Covariance function for $\dot{W}$: We have formally

$$\mathbf{E}\left[\dot{W}_t(x)\,\dot{W}_s(y)\right] = \gamma_0(t-s)\,\prod_{j=1}^{d}\gamma_j(y_j - x_j)$$

with the following distributional relation:

$$\gamma_j(u,v) = \partial_{uv}R(u,v) \text{ `} = \text{'} \ |u-v|^{2H_j-2}. \qquad (3)$$

Remark:

- The covariance γ_j is given in Fourier mode as

$$\gamma_j(x) = \int_{\mathbb{R}} e^{\imath\xi x}|\xi|^{1-2H_j}d\xi$$

Skorohod solution

Skorohod equation: Of the form

$$\begin{cases} \partial_t u_t(x) = \frac{1}{2}\Delta u_t(x) + u_t(x) \diamond \dot{W}_t(x), \\ u_0(x) = 1, \end{cases}$$

where $\diamond$ is the Wick product.

Mild form: Written as

$$u_t(x) = 1 + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) u_s(y)\, d^\diamond W_s(y),$$

where the stochastic integral is a Skorohod integral
$\hookrightarrow$ extension of Itô from Malliavin calculus.

Stratonovich solution

Stratonovich equation: Of the form

$$\begin{cases} \partial_t u_t(x) = \frac{1}{2}\Delta u_t(x) + u_t(x)\dot{W}_t(x), \\ u_0(x) = 1, \end{cases}$$

where the product is the usual product.

Mild form: We have $u = (\text{renormalized}) - \lim_{\varepsilon\to 0} u^\varepsilon$, where

$$u_t^\varepsilon(x) = 1 + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) u_s^\varepsilon(y)\, dW_s^\varepsilon(y), \tag{4}$$

where W^ε is a mollification of W and (4) is an ordinary PDE
$\hookrightarrow$ Regularity structures.

A subcritical zone

Theorem 1.

Let us assume

1. $d = 1$
2. $H_0 > 1/2$ and $H_1 < 1/2$
3. $H_0 + H_1 > \frac{3}{4}$
4. $\frac{3}{2} < 2H_0 + H_1 \leq 2$

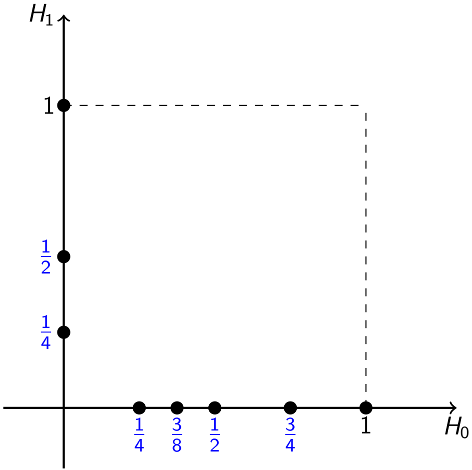
Then we have

- Global exist. and uniqu. for both u and $u^\diamond$
- For all $t \geq 0$, $x \in \mathbb{R}$ and $p \geq 1$ we have

$$\mathbf{E}[|u_t^\diamond(x)|^p] < \infty, \quad \text{and} \quad \mathbf{E}[|u_t(x)|^p] < \infty$$

Subcritical zone: illustration

In the (H_0, H_1) plane:



Subcritical zone: illustration

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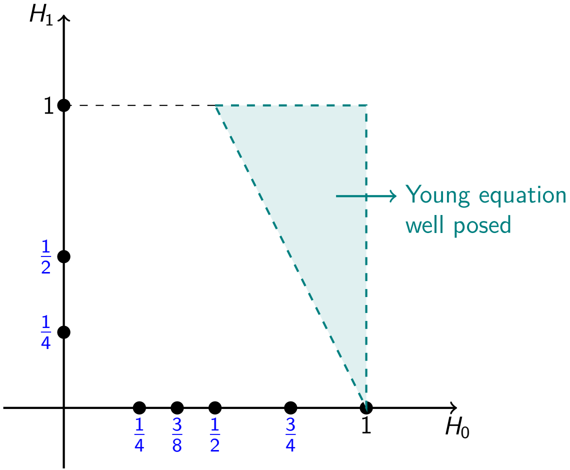

Subcritical zone: illustration

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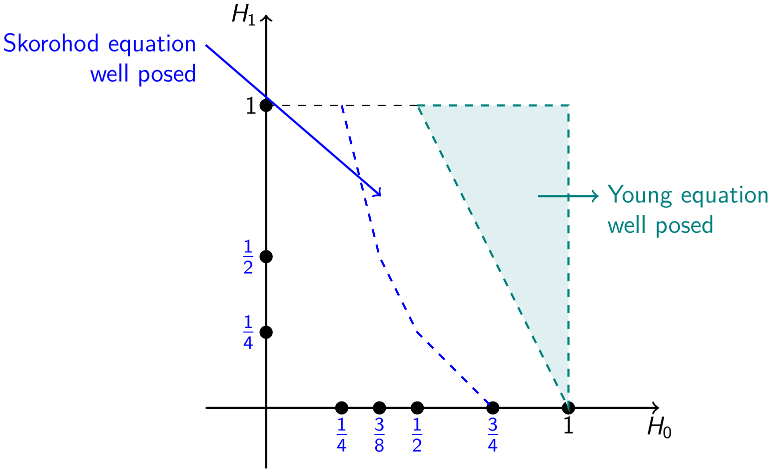
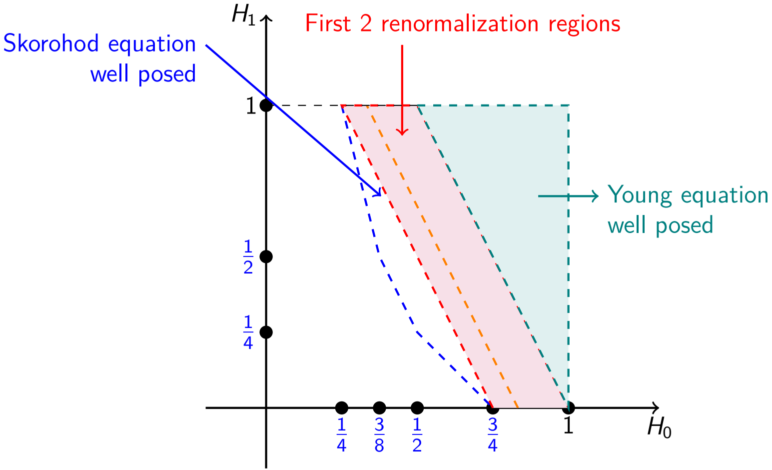


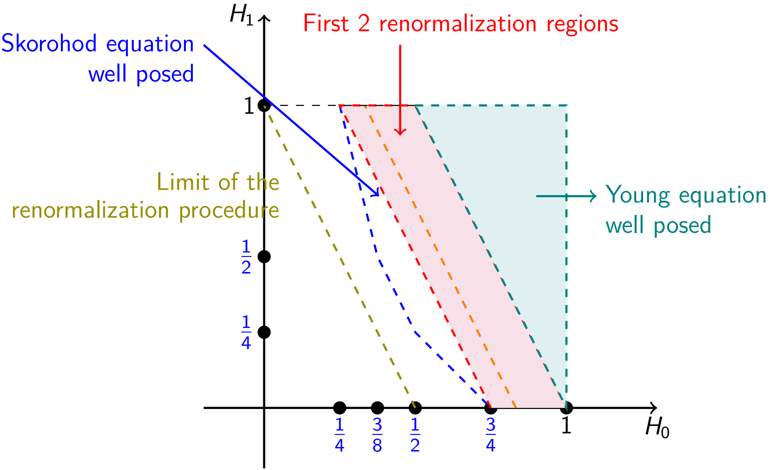
Subcritical zone: illustration

In the (H_0, H_1) plane:



Subcritical zone: illustration

In the (H_0, H_1) plane:



A critical zone

Theorem 2.

Let us assume

1. $d = 2$
2. W does not depend on time: $W = W(x)$
3. $H_1 < 1/2$
4. $H_1 + H_2 = 1$

Then we have

- Local exist. and uniqu. for the Skorohod solution $u^\diamond$
- Global exist. and uniqu. for the Stratonovich solution u

A critical zone (2)

Theorem 3.

Under the same conditions as in Theorem 2 consider $p > 1$
Then

- There exists $\tau_p^\diamond$ such that for all $t > \tau_p^\diamond$, $x \in \mathbb{R}$ we have

$$\mathbf{E}\left[|u_t^\diamond(x)|^p\right] \begin{cases} < \infty, & t < \tau_p^\diamond, \\ = \infty, & t > \tau_p^\diamond. \end{cases}$$

- For $p \geq 2$, exact expression for $\tau_p^\diamond$
- Upper bound for $\tau_p^\diamond$ when $1 < p < 2$
- Finite moments for the Strato solution $u_t(x)$ for small t's

Comments on the results

Previous results on asymptotic behavior of moments:

- $H_0 = \frac{1}{2}$, Itô framework: Khoshnevisan, Conus, Foondun
- Young type cases, $2H_0 + H_1 > 2$:
  Balan-Conus, Hu-Huang-Nualart-T, X. Chen
- Rough Skorohod case: X. Chen

Previous results on renormalization:

- Hairer-Labbé, Deya

Our contribution:

- Existence of moments for renormalized versions of PAM
- Link between renormalized Skorohod and Stratonovich
  $\hookrightarrow$ Through Feyman-Kac representations

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Feynman-Kac for the Skorohod equation

Regularized Feynman-Kac potential:

For $\varepsilon > 0$ and a Brownian motion B, set

$$V_t^{\varepsilon,B}(x) = \int_0^t \int_{\mathbb{R}^2} p_\varepsilon(B_{t-r}^x - y)\, dW_s(y) \tag{5}$$

Regularized Feynman-Kac compensator:

$$\beta_t^{\varepsilon,B} = \int_{[0,t]^2} \int_{\mathbb{R}^d} e^{-\varepsilon|\xi|^2} e^{\imath\langle \xi,\, B_{t-s_1} - B_{t-s_2}\rangle} \gamma_0(s_1 - s_2)\mu(d\xi)$$

where

$$\mu(d\xi) = \prod_{j=1}^d |\xi_j|^{1-2H_j}\, d\xi$$

Feynman-Kac for the Skorohod equation (2)

Limit theorem: We have (subcritical regime)

$$u_t^{\diamond}(x) = L^2(\Omega) - \lim_{\varepsilon \to 0} u_t^{\varepsilon,\diamond}(x),$$

where

$$\begin{aligned} u_t^{\varepsilon,\diamond}(x) &= \mathbf{E}_B\left[e^{V_t^{\varepsilon,B}(x) - \frac{1}{2}\beta_t^{\varepsilon,B}}\right] \\ &= \mathbf{E}_B\left[\exp\left(V_t^{\varepsilon,B}(x) - \frac{1}{2}\mathbf{E}_W\left[\left|V_t^{\varepsilon,B}(x)\right|^2\right]\right)\right]. \end{aligned}$$

Feynman-Kac for the Stratonovich equation

Regularized Feynman-Kac potential:
For $\varepsilon > 0$ and a Brownian motion B, set

$$V_t^{\varepsilon,B}(x) = \int_0^t \int_{\mathbb{R}^2} p_\varepsilon(B_{t-r}^x - y)\, dW_s(y) \tag{6}$$

Regularized Feynman-Kac compensator: Of the form

$$c_\varepsilon t,$$

with

$$c_\varepsilon t \simeq \mathbf{E}_B\left[\beta_t^{\varepsilon,B}\right] \asymp \frac{1}{\varepsilon^{2-2H_0-H_1}}$$

Feynman-Kac for the Stratonovich equation (2)

Limit theorem: We have (subcritical regime)

$$u_t(x) = \text{a.s} - \lim_{\varepsilon \to 0} u_t^\varepsilon(x),$$

where

$$u_t^\varepsilon(x) = \mathbf{E}_B\left[e^{V_t^{\varepsilon,B}(x) - c_\varepsilon t}\right]$$

Comparison between F-K representations

Recall: we have

$$u_t^{\varepsilon}(x) = \mathbf{E}_B\left[e^{V_t^{\varepsilon,B}(x) - c_\varepsilon t}\right]$$
$$u_t^{\varepsilon,\diamond}(x) = \mathbf{E}_B\left[e^{V_t^{\varepsilon,B}(x) - \frac{1}{2}\beta_t^{\varepsilon,B}}\right]$$

Strategy for the comparison: We have

$$\text{Fluctuations}\left(\frac{1}{2}\beta_t^{\varepsilon,B} - c_\varepsilon t\right) \ll \text{Fluctuations}\left(V_t^{\varepsilon,B}(x)\right)$$

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Regularity exponents in parabolic scaling

Parabolic scaling: for $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ set

$$\mathcal{S}_{t,x}\varphi(s,y) = \frac{1}{\delta^3}\varphi\left(\frac{s-t}{\delta^2}, \frac{y-x}{\delta}\right)$$

Distributional exponent: $F \in \mathcal{C}^{-\alpha}$ in parabolic scaling if

$$\int_{\mathbb{R}\times\mathbb{R}} [\mathcal{S}_{t,x}\varphi](s,y)\, F(s,y)\, dsdy \leq c_\varphi\, \delta^{-\alpha}$$

Noise regularity in the $1+1$ case

Wiener integral computation: For a smooth φ we have

$$
\begin{aligned}
&\mathbf{E}\left[\left|\dot{W}(\mathcal{S}_{t,x}\varphi)\right|^2\right] \\
&= \int_{\mathbb{R}^4} [\mathcal{S}_{t,x}\varphi](s_1, y_1)[\mathcal{S}_{t,x}\varphi](s_2, y_2)|s_1 - s_2|^{2H_0-2}|y_1 - y_2|^{2H_1-2}\, dsdy \\
&= \frac{1}{\delta^{2(3-2H_0-H_1)}}\, J(\varphi), \quad \text{with } J(\varphi) \text{ independent of } \delta.
\end{aligned}
$$

Fractional noise irregularity: We have $\dot{W} \in \mathcal{C}^{-\alpha-\varepsilon}$ with

$$\alpha = 3 - 2H_0 - H_1$$

Heuristics for the Young equation region

Recall: We consider

$$\partial_t u_t(x) = \frac{1}{2}\Delta u_t(x) + u_t(x)\,\dot{W}_t(x)$$

Heat semigroup smoothing: We expect $u \in \mathcal{C}^\beta$ with

$$\beta = -\alpha + 2 = 2H_0 + H_1 - 1$$

Definition of $u\dot{W}$: Whenever

$$\beta - \alpha > 0 \quad \Leftrightarrow \quad 2H_0 + H_1 > 2$$