Relativistic stable processes in quasi-ballistic heat conduction

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Outline

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Experimental setting

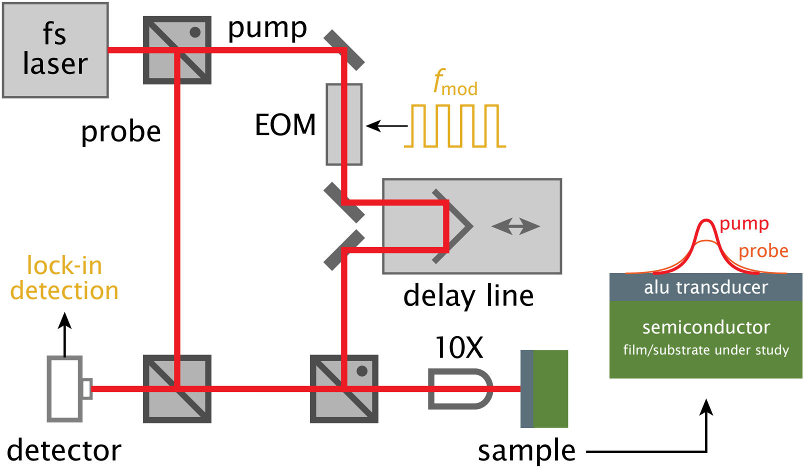
TDTR: Time domain thermoreflectance

- Short heat impulses
- Strong pump pulses get into material
- Weaker probe signals
  $\hookrightarrow$ in order to measure change in reflectance due to heat
- Transform to $-V_{\text{in}}/V_{\text{out}}$

Material used:

$$In_{0.53}Ga_{0.47}As$$

Illustration:Experimental Setting



Classical heat equation

Notation: We set

$$T_0 \equiv \quad \text{Initial temperature distribution}$$

and

$$\partial_t T(t,x) = \frac{\partial}{\partial t} T(t,x)$$

Equation: For $x \in \mathbb{R}^d$ and $d = 1, 2, 3$

$$\partial_t T(t,x) = \frac{1}{2}\Delta T(t,x), \text{ with } T(0,x) = T_0(x).$$

Brownian motion

Definition 1.

Let

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space
- $\{W_t;\, t \geq 0\}$ stochastic process, $\mathbb{R}$-valued

We say that W is a Brownian motion if:

1. $W_0 = 0$ almost surely
2. Let $n \geq 1$ and $0 = t_0 < t_1 < \cdots < t_n$. The increments

$$\delta W_{t_0 t_1}, \delta W_{t_1 t_2}, \ldots, \delta W_{t_{n-1} t_n} \quad \text{are independent}$$

3. For $t > 0$ we have

$$W_t \sim \mathcal{N}(0, t), \quad \text{or} \quad \mathbf{E}\left[e^{\imath \xi W_t}\right] = e^{-\frac{1}{2} t \xi^2}$$

Illustration: chaotic path

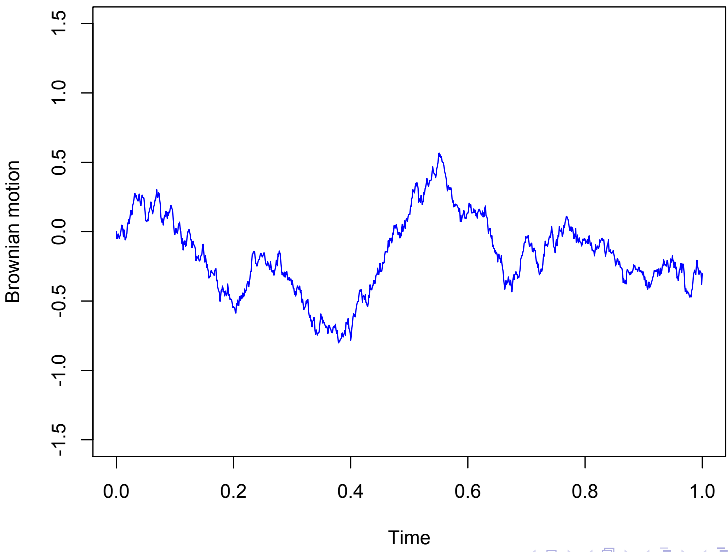


Illustration: random path

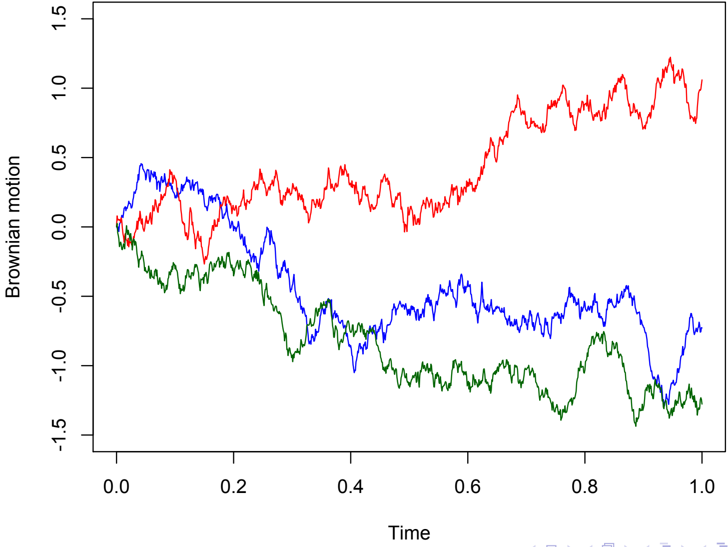
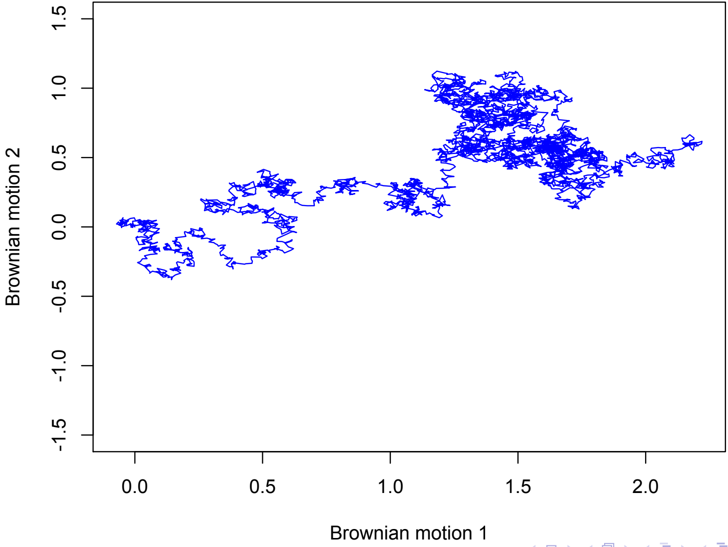


Illustration: 2-d Brownian motion



Feynman-Kac representation

Equation: Classical heat equation

$$\partial_t T(t,x) = \frac{1}{2}\Delta T(t,x), \quad \text{with } T(0,x) = T_0(x).$$

Representation: For a Brownian motion W,

$$T(t,x) = \mathbf{E}\left[T_0(x + W_t)\right]$$

A non Brownian world

A modified heat equation: Under our experimental setting

1. The data does not match the heat equation
2. Idea: replace the Brownian motion by a Levy process

Levy processes:

1. Most natural generalizations of Brownian motion
2. Involve jumps

Outline

Relativistic stable process

Definition 2.

Let $m \geq 0$, $0 < \alpha < 2$ and

- $\{X_t^m;\, t \geq 0\}$ stochastic process, $\mathbb{R}$-valued

We say that X^m is a relativistic stable process if:

1. $X_0^m = 0$ almost surely
2. Let $n \geq 1$ and $0 = t_0 < t_1 < \cdots < t_n$. The increments
$$\delta X_{t_0 t_1}^m, \delta X_{t_1 t_2}^m, \ldots, \delta X_{t_{n-1} t_n}^m \quad \text{are independent}$$
3. For $t > 0$ we have
$$\mathbf{E}\left[e^{\imath \xi X_t^m}\right] = \exp\left(-t\left(\left(|\xi|^2 + m^{2/\alpha}\right)^{\alpha/2} - m\right)\right)$$

Remarks on relativistic stable process

Case $m = \infty$: We get a Brownian motion

$$\mathbf{E}\left[e^{\imath\xi X_t^m}\right] \simeq \exp\left(-c_m t|\xi|^2\right)$$

Case $m = 0$: We get an α-stable process

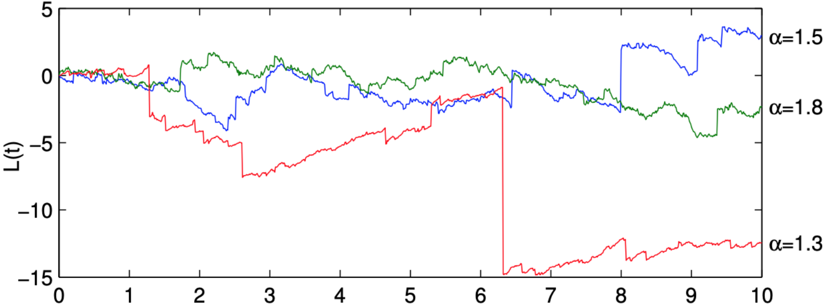
$$\mathbf{E}\left[e^{\imath\xi X_t^m}\right] = \exp\left(-t|\xi|^\alpha\right)$$

Jumps: All Levy processes have jumps

- Brownian motion is the only continuous Levy process
- Jumps can be described in terms of characteristic function
- α-stable have heavy tailed jumps
- Relativistic stable processes have light tailed jumps

Typical path of an α-stable process

Examples of paths: Different values of α, for $m = 0$



Role of parameter α: If α is small

- Larger jumps
- Less integrability: $\mathbf{E}[|X_t|^p] < \infty$ for $p < \alpha$

Transition from α-stable to Brownian

Theorem 3.

Let

- $X^m \equiv$ Relativistic stable process
- $p_t^m(x) \equiv$ Density of X_t^m

Then

$$p_t^m(x) = \begin{cases} c_{1,m}\, t^{-d/\alpha} \times \text{ decaying function}(x), & t \text{ small} \\ c_{1,m}\, t^{-d/2} \times \text{ decaying function}(x), & t \text{ large} \end{cases}$$

Interpretation:

- For t small, α-stable behavior
- For t large, Brownian behavior

Outline

Feynman-Kac representation

Model: We assume, for a relativistic stable X^m,

$$T(t,x) = \mathbf{E}\left[T_0(x + X_t^m)\right]$$

Corresponding PDE: Nonlocal, of the form

$$\partial_t T(t,x) = \mathcal{L}^m T(t,x),$$

with

$$\mathcal{L}^m = m - \left(-\Delta + m^{2/\alpha}\right)^{\alpha/2}.$$

Experimental setting (repeated)

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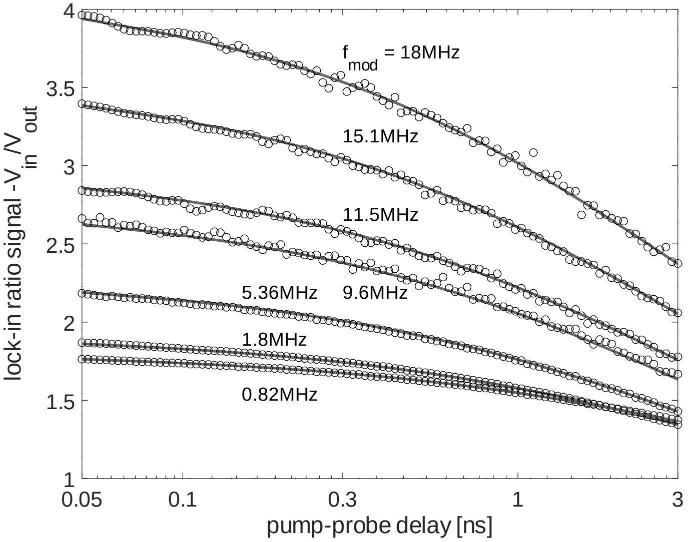
Estimation for α:

- Experimental: $\alpha = 1.695$
- Theoretical Physics: $\alpha = 1.75$

Data fit

Comparison data/theoretical: For different values of f_{mod}



Outline

Conclusions

Advantages of the Levy formalism

- Natural extension of the Brownian formalism
- Justified by theoretical Physics considerations
- Connexions to a rich mathematical literature
    - Identification of the distribution
    - Kernel estimates
- Excellent data fit

Perspectives

Multilayer setting:

- Two layers of different materials, either in 2-d or 3-d
- Coupling of 2 nonlocal PDEs
- Boundary conditions: TBD