Convergence to equilibrium for rough differential equations

Samy Tindel

Purdue University

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Joint work with Aurélien Deya (Nancy) and Fabien Panloup (Angers)
Outline

1. Setting and main result

2. Convergence to equilibrium for diffusion processes
   - Poincaré inequality
   - Coupling method

3. Elements of proof
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3 Elements of proof
Definition of fBm

**Definition 1.**

A 1-d fBm is a continuous process \( X = \{X_t; \ t \in \mathbb{R}\} \) such that \( X_0 = 0 \) and for \( H \in (0, 1) \):

- \( X \) is a centered Gaussian process
- \( \mathbb{E}[X_t X_s] = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t - s|^{2H}) \)

**d-dimensional fBm:** \( X = (X^1, \ldots, X^d) \), with \( X^i \) independent 1-d fBm

**Variance of increments:**

\[
\mathbb{E}[|\delta X^j_{st}|^2] \equiv \mathbb{E}[|X^j_t - X^j_s|^2] = |t - s|^{2H}
\]
Examples of fBm paths

\[ H = 0.35 \]

\[ H = 0.5 \]

\[ H = 0.7 \]
System under consideration

Equation:

\[ dY_t = b(Y_t)dt + \sigma(Y_t)\,dX_t, \quad t \geq 0 \]  \hspace{1cm} (1)

Coefficients:

- \( x \in \mathbb{R}^d \mapsto \sigma(x) \in \mathbb{R}^{d\times d} \) smooth enough
- \( \sigma = (\sigma_1, \ldots, \sigma_d) \in \mathbb{R}^{d\times d} \) invertible
- \( \sigma^{-1}(x) \) bounded uniformly in \( x \)
- \( X = (X^1, \ldots, X^d) \) is a \( d \)-dimensional fBm, with \( H > \frac{1}{3} \)

Resolution of the equation:

- Thanks to rough paths methods
  \( \leftrightarrow \) Limit of Wong-Zakai approximations
Illustration of ergodic behavior

Equation with damping: \[ dY_t = -\lambda Y_t \, dt + dX_t \]

Simulation: For 2 values of the parameter \( \lambda \)

Figure: \( H = 0.7, \, d = 1, \, \lambda = 0.1 \)

Figure: \( H = 0.7, \, d = 1, \, \lambda = 3 \)
Coercivity assumption for $b$

**Hypothesis:** for every $v \in \mathbb{R}^d$, one has

$$\langle v, b(v) \rangle \leq C_1 - C_2 \|v\|^2$$

Interpretation of the hypothesis:
Outside of a compact $K \subset \mathbb{R}^d$, $b(v) \simeq -\lambda v$ with $\lambda > 0$
Ergodic results for equation (1)

**Brownian case:** If $X$ is a Brownian motion and $b$ coercive
- Exponential convergence of $\mathcal{L}(X_t)$ to invariant measure $\mu$
- Markov methods are crucial
- See e.g Khashminskii, Bakry-Gentil-Ledoux

**Fractional Brownian case:** If $X$ is a fBm and $b$ coercive
- Markov methods not available
- Existence and uniqueness of invariant measure $\mu$, when $H > \frac{1}{3}$
  $\Leftrightarrow$ Series of papers by Hairer et al.
- Rate of convergence to $\mu$:
  - When $\sigma \equiv \text{Id}$: Hairer
  - When $H > \frac{1}{2}$ and further restrictions on $\sigma$: Fontbona–Panloup
Main result (loose formulation)

**Theorem 2.**

Let
- \( H > \frac{1}{3} \), equation \( dY_t = b(Y_t)dt + \sigma(Y_t) \, dX_t \)
- \( Y \) unique solution with initial condition \( \mu_0 \)
- \( \mu \) unique invariant measure

Then for all \( \varepsilon > 0 \) we have:

\[
\| \mathcal{L}(Y_{t}^{\mu_0}) - \mu \|_{\text{TV}} \leq c_{\varepsilon} t^{-\left(\frac{1}{8} - \varepsilon\right)}
\]

**Remark:**
- Subexponential (non optimal) rate of convergence
- This might be due to the correlation of increments for \( X \)
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Poincaré and convergence to equilibrium

**Theorem 3.**

Let $X$ be a diffusion process. We assume:

- $\mu$ is a symmetrizing measure, with Dirichlet form $\mathcal{E}$
- Poincaré inequality: $\text{Var}_\mu(f) \leq \alpha \mathcal{E}(f)$

Then the following inequality is satisfied:

$$\text{Var}_\mu(P_t f) \leq \exp \left(-\frac{2t}{\alpha}\right) \text{Var}_\mu(f)$$
Comments on the Poincaré approach

Remarks:

1. Theorem 3 asserts that
\[ X_t \xrightarrow{(d)} \mu, \quad \text{exponentially fast} \]

2. The proof relies on identity \( \partial_t P_t = LP_t \)
   \( \rightarrow \) Hard to generalize to a non Markovian context

3. One proves Poincaré with Lyapunov type techniques
   \( \rightarrow \) Coercivity enters into the picture
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A general coupling result

Consider:

- Two processes \( \{Z_t; \ t \geq 0\} \) and \( \{Z'_t; \ t \geq 0\} \)
- A coupling \((\hat{Z}, \hat{Z}')\) of \((Z, Z')\)

We set

\[
\tau = \inf \left\{ t \geq 0; \ \hat{Z}_u = \hat{Z}'_u \text{ for all } u \geq t \right\}
\]

Then we have:

\[
\|\mathcal{L}(Z_t) - \mathcal{L}(Z'_t)\|_{tv} \leq 2 \mathbf{P}(\tau > t)
\]
Comments on the coupling method

1. Proposition 4 is general, does not assume a Markov setting → can be generalized (unlike Poincaré)

2. In a Markovian setting → Merging of paths a soon as they touch

3. In our case → We have to merge both \( Y \), \( Y' \) and the noise

![Diagram showing paths merging](image-url)
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Algorithmic view of the coupling

Merging positions of $Y^x$ and $Y^\mu$ by coupling

Success

yes

Stick solutions $Y^x$ and $Y^\mu$ by a Girsanov shift of the noise

Success

yes

Wait in order to have $Y^x$ and $Y^\mu$ back in a compact

no

Estimate for the merging time $\tau$
Merging positions (1)

Simplified setting:
We start at $t = 0$, and consider $d = 1$

Effective coupling: We wish to consider $y^0$, $y^1$ and $h$ such that

- We have

\[
\begin{align*}
  dy^0_t &= b(y^0_t) \, dt + \sigma(y^0_t) \, dX_t \\
  dy^1_t &= b(y^1_t) \, dt + \sigma(y^1_t) \, dX_t + h_t \, dt
\end{align*}
\]

- Merging condition: $y^0_0 = a_0$, $y^1_0 = a_1$ and $y^0_1 = y^1_1$

Computation of the merging probability:
Through Girsanov’s transform involving the shift $h$
Merging positions (2)

Generalization of the problem:
We wish to consider a family \( \{y^\xi, h^\xi; \xi \in [0,1]\} \) such that

- We have
  \[
  dy^\xi_t = b(y^\xi_t) \, dt + \sigma(y^\xi_t) \, dX_t + h^\xi_t \, dt
  \]

- Merging condition:
  \[
  y^\xi_0 = a_0 + \xi(a_1 - a_0), \quad y^1_0 = y^1_1, \quad h^0 \equiv 0
  \]

Remark:
Here \( y \) has to be considered as a function of 2 variables \( t \) and \( \xi \)
Merging positions (3)

Solution of the problem: Consider a system with tangent process

\[
\begin{align*}
    dy_t^\xi &= \left[ b(y_t^\xi) - \int_0^\xi d\eta j_t^n \right] dt + \sigma(y_t^\xi) dX_t \\
    dj_t^\xi &= b'(y_t^\xi)j_t^\xi dt + \sigma'(y_t^\xi)j_t^\xi dX_t
\end{align*}
\]

and initial condition \( y_0^\xi = a_0 + \xi(a_1 - a_0), j_0^\xi = a_1 - a_0 \)

Heuristics: A simple integrating factor argument shows that \( \partial_\xi y_t^\xi = j_t^\xi(1 - t), \) and thus \( \partial_\xi y_1^\xi = 0 \)

Hence \( y^\xi \) solves the merging problem
Merging positions (4)

Rough system under consideration: for \( t, \xi \in [0,1] \)

\[
\begin{align*}
\{ & dy_t^\xi = \left[ b(y_t^\xi) - \int_0^\xi d\eta^\eta_j^\eta \right] dt + \sigma(y_t^\xi) \, dX_t \\
& d\eta^\eta_j^\eta = b'(y_t^\xi) \eta^\eta_j^\eta \, dt + \sigma'(y_t^\xi) \eta^\eta_j^\eta \, dX_t 
\end{align*}
\]

Then \( y_1^\xi \) does not depend on \( \xi \)!

Difficulties related to the system:

1. \( t \mapsto y_t \) is function-valued
2. Unbounded coefficients, thus local solution only
3. Conditioning \( \Rightarrow \) additional drift term with singularities
4. Evaluation of probability related to Girsanov’s transform