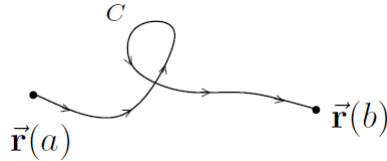


## Study Guide # 3

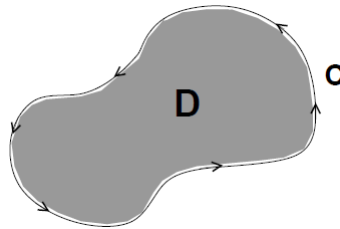
You also need Study Guides # 1 and # 2 for the Final Exam

1. FUNDAMENTAL THEOREM OF CALCULUS FOR LINE INTEGRALS:  $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$ :



2. A vector field  $\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$  is *conservative* (i.e.  $\vec{F} = \nabla f$ ) if  $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ ;  
how to determine a potential function  $f$  if  $\vec{F} = \nabla f$ .

3. GREEN'S THEOREM:  $\int_C P(x, y) dx + Q(x, y) dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$  ( $C =$  boundary of  $D$ ):



4. DEL OPERATOR:  $\nabla = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}$ ; if  $\vec{F}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$ ,  
then

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \quad \text{and} \quad \text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Properties of curl and divergence:

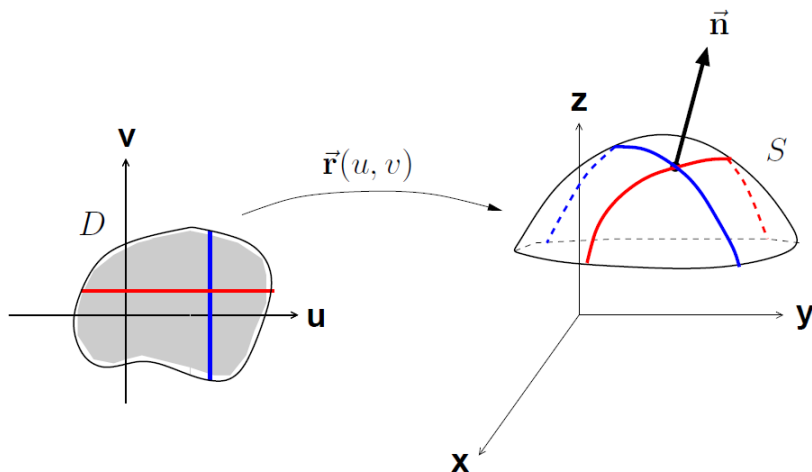
(i) If  $\text{curl } \vec{F} = \vec{0}$ , then  $\vec{F}$  is a conservative vector field ( $\vec{F} = \nabla f$ ) in a simply-connected domain.

(ii) If  $\text{curl } \vec{F} = \vec{0}$ , then  $\vec{F}$  is *irrotational*; if  $\text{div } \vec{F} = 0$ , then  $\vec{F}$  is *incompressible*.

(iii) *Laplace's Equation*:  $\nabla^2 f = \text{div}(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$ .

(iv) For functions with continuous partials,  $\text{curl}(\nabla f) = \vec{0}$ , and  $\text{div}(\text{curl } \vec{F}) = 0$ .

5. Parametric surface  $S: \vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ , where  $(u, v) \in D$ :



Normal vector to surface  $S: \vec{n} = \vec{r}_u \times \vec{r}_v$ ; tangent planes and normal lines to parametric surfaces.

6. Surface area of a surface  $S$ :

(i)  $A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| dA$

(ii) If  $S$  is the graph of  $z = h(x, y)$  above  $D$ , then  $A(S) = \iint_D \sqrt{1 + (\partial h/\partial x)^2 + (\partial h/\partial y)^2} dA$ ;

Remark:  $dS = |\vec{r}_u \times \vec{r}_v| dA =$  differential of surface area; while  $d\vec{S} = (\vec{r}_u \times \vec{r}_v) dA$

7. The surface integral of  $f$  over the surface  $S$ :

(i)  $\iint_S f(x, y, z) dS = \iint_D f(\vec{r}(u, v)) |\vec{r}_u \times \vec{r}_v| dA$ .

(ii) If  $S$  is the graph of  $z = h(x, y)$  above  $D$ , then

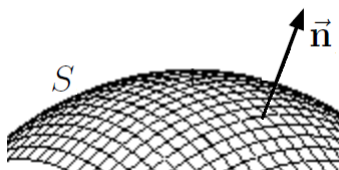
$$\iint_S f(x, y, z) dS = \iint_D f(x, y, h(x, y)) \sqrt{1 + (\partial h/\partial x)^2 + (\partial h/\partial y)^2} dA.$$

8. The surface integral of  $\vec{F}$  over the surface  $S$  (recall,  $d\vec{S} = (\vec{r}_u \times \vec{r}_v) dA$ ):

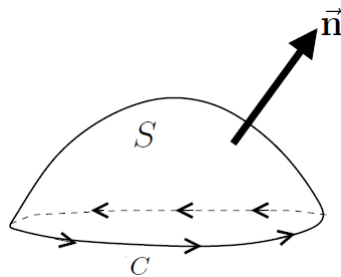
$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA = \iint_S (\vec{F} \cdot \vec{n}) dS = \text{flux of } \vec{F} \text{ across the surface } S.$$

If  $S$  is the graph of  $z = h(x, y)$  above  $D$ , with  $\vec{n}$  oriented upward, and  $\vec{F} = \langle P, Q, R \rangle$ , then

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left( -P \frac{\partial h}{\partial x} - Q \frac{\partial h}{\partial y} + R \right) dA$$



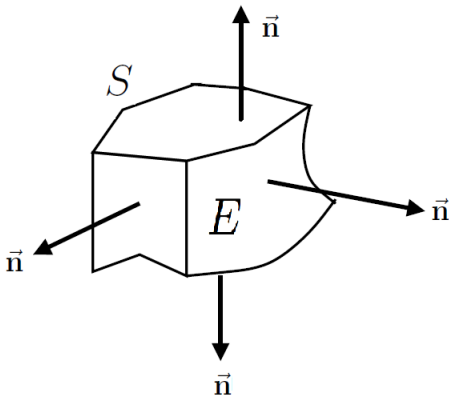
9. STOKES' THEOREM:  $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$  (recall,  $\text{curl } \vec{F} = \nabla \times \vec{F}$ ).



$\int_C \vec{F} \cdot d\vec{r} = \text{circulation}$  of  $\vec{F}$  around  $C$ .

10. THE DIVERGENCE THEOREM/GAUSS' THEOREM:  $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div } \vec{F} \, dV$

(recall,  $\text{div } \vec{F} = \nabla \cdot \vec{F}$ ).



## 11. Summary of Line Integrals and Surface Integrals:

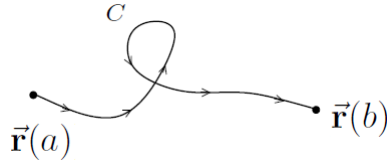
LINE INTEGRALS	SURFACE INTEGRALS
$C : \vec{\mathbf{r}}(t), \text{ where } a \leq t \leq b$	$S : \vec{\mathbf{r}}(u, v), \text{ where } (u, v) \in D$
$ds =  \vec{\mathbf{r}}'(t)  dt = \text{differential of arc length}$	$dS =  \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v  dA = \text{differential of surface area}$
$\int_C ds = \text{length of } C$	$\iint_S dS = \text{surface area of } S$
$\int_C f(x, y, z) ds = \int_a^b f(\vec{\mathbf{r}}(t))  \vec{\mathbf{r}}'(t)  dt$ (independent of orientation of $C$ )	$\iint_S f(x, y, z) dS = \iint_D f(\vec{\mathbf{r}}(u, v))  \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v  dA$ (independent of normal vector $\vec{\mathbf{n}}$ )
$d\vec{\mathbf{r}} = \vec{\mathbf{r}}'(t) dt$	$d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v) dA$
$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t) dt$ (depends on orientation of $C$ )	$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \vec{\mathbf{F}}(\vec{\mathbf{r}}(u, v)) \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v) dA$ (depends on normal vector $\vec{\mathbf{n}}$ )
$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C (\vec{\mathbf{F}} \cdot \vec{\mathbf{T}}) ds$ The <i>circulation</i> of $\vec{\mathbf{F}}$ around $C$	$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S (\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}) dS$ The <i>flux</i> of $\vec{\mathbf{F}}$ across $S$ in direction $\vec{\mathbf{n}}$

## 12. Integration Theorems:

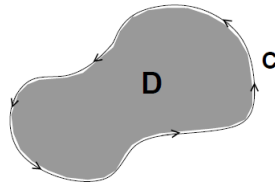
FUNDAMENTAL THEOREM OF CALCULUS:  $\int_a^b F'(x) dx = F(b) - F(a)$



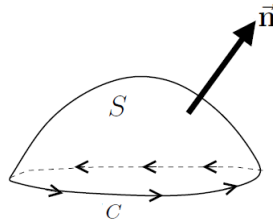
FUNDAMENTAL THEOREM OF CALCULUS FOR LINE INTEGRALS:  $\int_a^b \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$



GREEN'S THEOREM:  $\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P(x, y) dx + Q(x, y) dy$



STOKES' THEOREM:  $\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$



DIVERGENCE THEOREM:  $\iiint_E \text{div } \vec{F} dV = \iint_S \vec{F} \cdot d\vec{S}$

